How to construct a symmetric surface?

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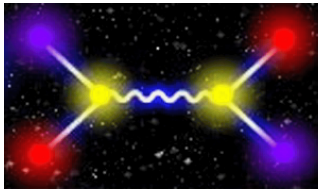

Isometric embedding

Friedman theorem (1961)
An arbitrary D-dimensional pseudo-Riemannian spacetime can be locally isometrically embedded in a N-dimensional pseudo-Riemannian space of suitable signature,
$N \geq D(D+1)/2$.

Embedding class: $p = N - D$.

Main object:
embedding function $y^a(x^\mu)$.

Induced metric:

$$g_{\mu\nu} = \partial_\mu y^a \partial_\nu y^b \eta_{ab},$$

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Example: sphere embedding

$$y^1 = x = R\cos\theta,$$
$$y^2 = y = R\sin\theta\cos\phi,$$
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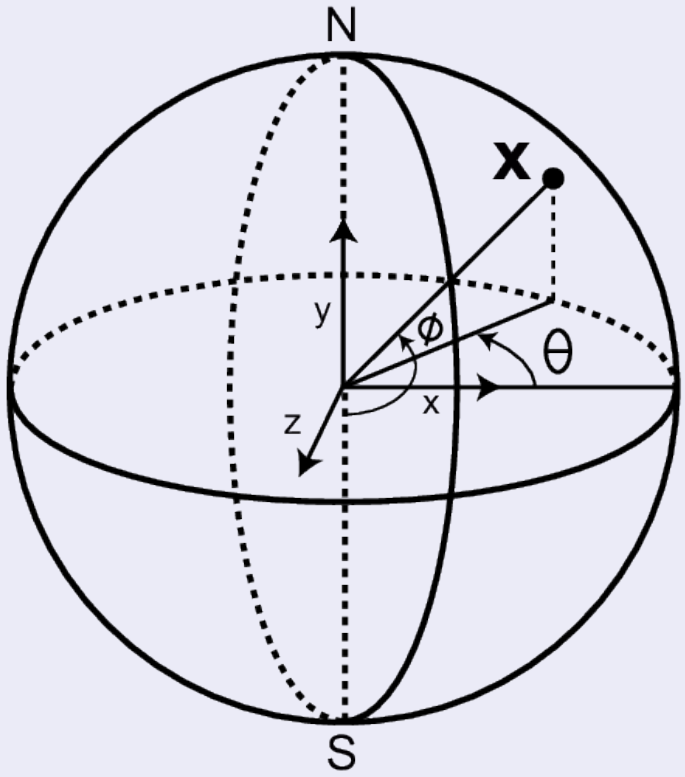

Method of exact embeddings construction

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How to separate the variables?

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A surface is G-symmetric,
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- Transform it using the matrices of $\tilde{G}$ to obtain the embedding function:

$$\begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ 1 \end{pmatrix}$$

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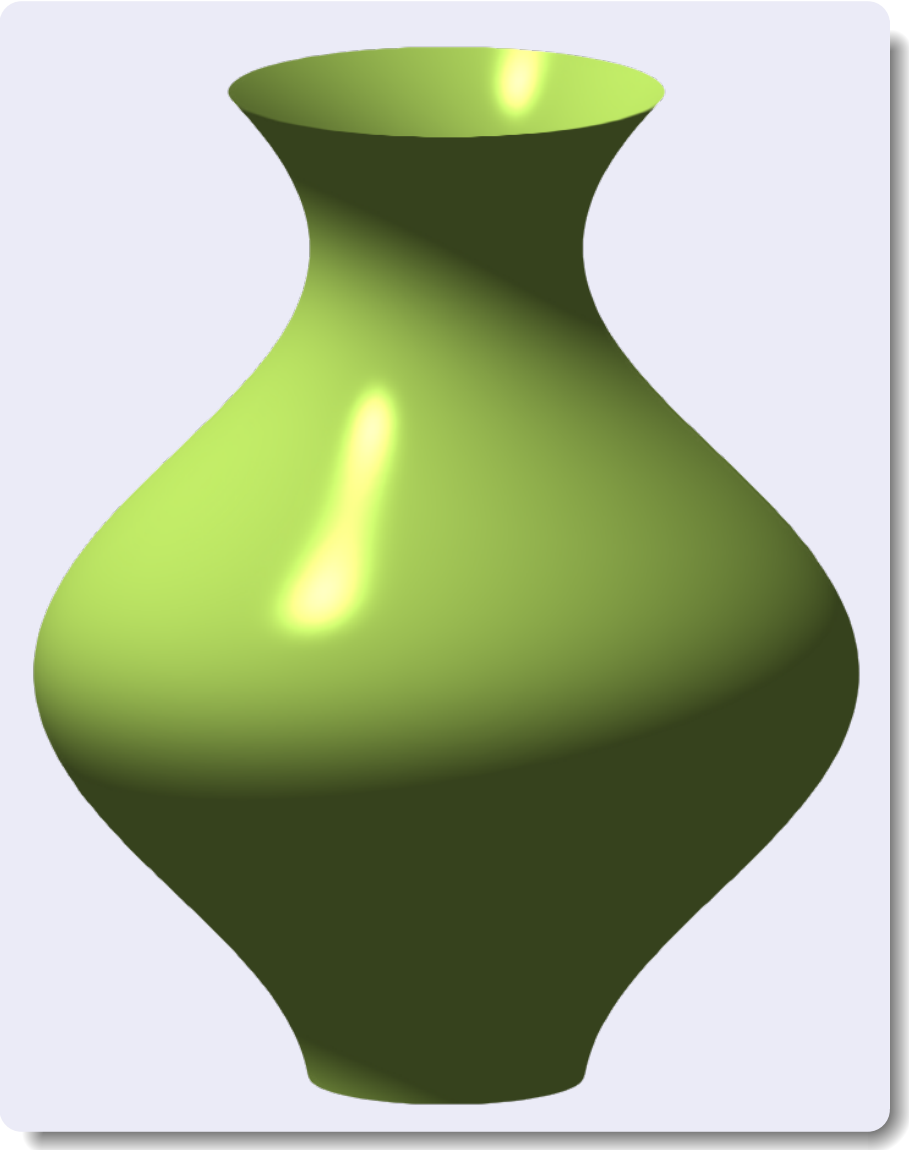

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Main idea: use not the whole symmetry group G,
but rather its abelian subgroups.
Abelian transformation w.r.t. parameter t can be represented by (pseudo)-rotation in an ambient spacetime:

$$y^1 = \frac{f(r)}{\alpha}\sqrt{\varepsilon}\sin(\sqrt{\varepsilon}(\alpha t + w(r))), \tag{1}$$

$$y^2 = \frac{f(r)}{\alpha}\cos(\sqrt{\varepsilon}(\alpha t + w(r))) \tag{2}$$

where $\varepsilon = \pm 1$ and the signature of $\{y^1, y^2\}$ is $(\pm\varepsilon, \pm 1)$.

An example: $SO(4)$

Initial vector: $y_0 = (R, 0, 0, 0)$,
$V(g) = SO(4)$:

$$\begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} O_{ik} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} y^1 &= R\cos\theta, \\ y^2 &= R\sin\theta\ \cos\phi, \\ y^3 &= R\sin\theta\ \sin\phi\cos\chi, \\ y^3 &= R\sin\theta\ \sin\phi\sin\chi. \end{aligned}$$

The interval: $ds^2 =$
$R^2(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2))$

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(Hopf coordinates).
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N. Vilenkin, Polyspherical and orispherical functions (1965)

Another example: Godel universe (2004.05882)

$$ds^2 = dt^2 + 2\mu \sinh^2 \chi dt d\phi - d\chi^2 - (\sinh^2 \chi - (1-\mu^2)\sinh^4 \chi) d\phi^2 - dz^2 \tag{3}$$

The symmetry is $SO(2,1) \otimes SO(2) \otimes \mathbb{R}$.

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Ansatz:

$$\begin{aligned} y^0 &= \sqrt{\varepsilon}\frac{A(\chi)}{\alpha}\sin\left(\sqrt{\varepsilon}\alpha t\right),\ y^2 = B(\chi)\sin(m\phi - \beta t),\ , \\ y^1 &= \xi\frac{A(\chi)}{\alpha}\cos\left(\sqrt{\varepsilon}\alpha t\right),\ y^3 = B(\chi)\cos(m\phi - \beta t), \\ y^4 &= \frac{C(\chi)}{n}\sin(n\phi),\ y^5 = \frac{C(\chi)}{n}\cos(n\phi),\ y^6 = f(\chi),\ y^7 = z. \end{aligned} \tag{4}$$

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Solution:

$$\begin{aligned} A(\chi) &= \cosh\chi, & B(\chi) &= \mu\sinh\chi, \\ C(\chi) &= \frac{\sqrt{|\mu^2-1|}}{2}\sinh 2\chi, & f(\chi) &= \frac{\sqrt{|\mu^2-1|}}{2}\cosh 2\chi, \end{aligned} \tag{5}$$

Rotating BTZ black hole (2107.00752)

$$ds^2 = \left(-M + \frac{r^2}{l^2}\right) dv^2 + J dv d\theta - \frac{4r^2}{J^2} dr^2 - \frac{4r^2}{J} dr d\theta - r^2 d\theta^2. \qquad (6)$$

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$$\begin{aligned}
y^1 &= \frac{J}{2\alpha} \sin\left(\varphi + \frac{2}{J}(\alpha^2 v - r)\right), \; y^2 = \frac{J}{2\alpha} \cos\left(\varphi + \frac{2}{J}(\alpha^2 v - r)\right), \\
y^3 &= \sqrt{r^2 + \frac{J^2}{4\alpha^2}} \sin\left(\varphi - \frac{1}{\alpha} \arctan\left(\frac{2\alpha r}{J}\right)\right), \\
y^4 &= \sqrt{r^2 + \frac{J^2}{4\alpha^2}} \cos\left(\varphi - \frac{1}{\alpha} \arctan\left(\frac{2\alpha r}{J}\right)\right), \\
y^5 &= v\sqrt{\alpha^2 + M + \frac{J^2}{4\alpha^2 l^2}}, \\
y^6 &= \frac{1}{\alpha}\sqrt{(\alpha^2 - 1)\left(r^2 + \frac{J^2}{4\alpha^2}\right)} \sin\left(\frac{\alpha v}{l\sqrt{\alpha^2 - 1}}\right), \\
y^7 &= \frac{1}{\alpha}\sqrt{(\alpha^2 - 1)\left(r^2 + \frac{J^2}{4\alpha^2}\right)} \cos\left(\frac{\alpha v}{l\sqrt{\alpha^2 - 1}}\right),
\end{aligned} \tag{7}$$

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- Their form can be found from the induced metric condition,
- The best way to solve the corresponding PDE system is to separate the variables,
- It can be done using S. A. Paston's method (through finding the representation of a full symmetry group of the metric) or its generalization (the Abelian subgroups of this group).