Three-loop calculations in symmetric point using momentum subtraction schemes

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Outline

Definitions, history, motivation

Three-loop integrals calculation

Examples of real computations

Momentum subtraction schemes use cases

1. Gell-Mann and Low ψ-function and QED MOM scheme e.g. [Baikov et al.'12]
2. Minimal MOM scheme (asymmetric point) [Braaten,Leveille'81]

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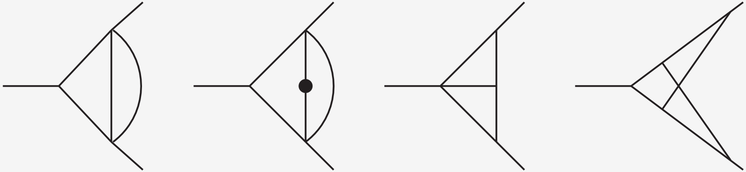
3. MOM as alternative scheme for QCD calculations [Celmaster,Gonsalves'79]
   - Redefine coupling in MOM scheme e.g. $R(s)$
   - Extract strong coupling value from lattice data
4. Various operator insertions renormalized in symmetric point [Sturm et al.'10]
   - Quark masses from lattice data
   - Moments of non-singlet operators
   - ...

Key ingredient for two-loop symmetric point calculations



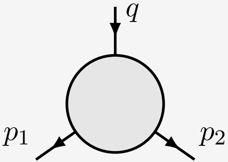
- Two-loop vertex integrals are known for arbitrary external momenta for a long time
  - From Mellin-Barnes integration [Usyukina,Davydychev'94]
  - Higher orders in terms of 2dHPLs [Birthwright,Glover,Marquard'04]
  - Parametric integration after apropriate variable change [Chavez,Duhr'12]

Two-loop applications

- Symmetric point results
  - Three loop QCD MOM beta-functions [Chetyrkin,Seidensticker'00][Gracey'11]
  - Two-loop matching factors for quark masses [Almeida,Sturm'10]
  - Amplitudes for the $n = 3$ moment of the Wilson operator [Gracey'11]
  - Two loop QCD vertices at the symmetric point [Gracey'11]
  - Flavour singlet axial vector current renormalization at two loops [Gracey'20]
- Interpolating scheme
  - Two-loop matching factors for quark masses [Gorban,Jager'10]
  - Renormalization of QCD in the interpolating MOM scheme [Gracey,Simms'18]
- General off-shell
  - Off-shell two loop QCD vertices [Gracey'14]
  - Off-shell quark bilinear operator Green's functions [Gracey'19]

Kinematics of three-point functions

We use following momenta assignment:



Kinematic configurations of interest:

- Symmetric point condition: $p_1^2 = p_2^2 = q^2 = -1$. Is a number.
- Auxiliary integrals: $p_1^2 = p_2^2 = -1$ and $q^2 = -x$. Is a function of x.
- General 3-pt functions: $p_1^2 = -1, p_2^2 = -z\bar{z}$ and $q^2 = -(1-z)(1-\bar{z})$.

Linear reducibility and direct integration

For two-loop case [Chavez,Duhr'12] and partially for three-loop case [Panzer'14] it is proven that general 3-pt integrals are linear reducible and can be calculated in terms of Generalized Polylogarithms(GPLs) $G(a_1, \ldots a_n; 1)$ with following alphabet a_i:

1,2-loop	3-loop
$z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}$	$z\bar{z}-1, z+\bar{z}-1, z\bar{z}-z-\bar{z}$

For symmetric point case alphabet reduces to sixth roots of unity $a_i = 0, \pm 1, e^{\pm i\pi/3}, e^{\pm 2i\pi/3}$. Basis of GPLs upto the transcendental weight six constructed in [Henn,Smirnov,Smirnov'15].

Our conjecture: symmetric point integrals can be expressed via more restricted basis of Harmonic Polylogrithms(HPLs) $H_{a_1,\ldots a_n}(t), a_i = 0, \pm 1$ with argument $t = e^{i\pi/3}$ constructed upto weight six in [Kniehl,AP,Veretin'17]

Symmetric point direct integration

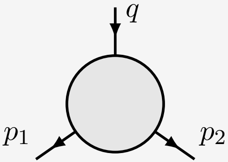
Used strategy:

1. Start with the set of master integrals at symmetric point, $p_1^2 = p_2^2 = q^2 = -1$
2. Using dimension shifts and/or increasing powers of propagators if needed we find basis of ***quasi-finite*** integrals[Manteuffel,Panzer,Schabinger'14]
3. Introduce Feynman parameters and expand in $\varepsilon = 2 - d/2$ under integral sign each member of a new basis, safe due to the absence of overlaping divergencies
4. Use slightly modified `HyperInt` package[Panzer'14] capable to work with polynomials factorized with field extension e.g.

$$z^2 - z + 1 = (z - e^{\frac{i\pi}{3}})(z - e^{-\frac{i\pi}{3}})$$

Works well for one-loop, two-loop and most of three-loop integrals. Due to the field extension works unstable. Failed on the most complicated integrals.

Differential equations for auxiliary integrals



- Consider auxiliary integrals with $p_1^2 = p_2^2 = -1$ and $q^2 = -x$
- Change variables $x = 2 - z - 1/z$ to make line reducibility explicit

Limiting cases:

1	$z \to 0$	$x \to \infty$	massless form-factor
2	$z \to 1$	$x \to 0$	massless propagators
3	$z \to -1$	$x \to 4$	threshold $q^2 = 4m^2$
4	$z \to e^{\pm \frac{i\pi}{3}}$	$x \to 1$	symmetric point

Bringing differential equations into canonical form

Original DE system

$$\frac{d\vec{f}(z)}{dz} = M(\varepsilon, z)\vec{f}(z)$$

Aplying transformation $\vec{f}(z) = T(\varepsilon, z)\vec{g}(z)$ system converted into canonical form

$$\frac{d\vec{g}(z)}{dz} = \varepsilon M'(z)\vec{g}(z)$$

ε-dependence factorized and DE system have Fuchsian form with constant A_i matrices

$$M'(z) = \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_{-1}}{z+1} + \frac{A_\lambda}{z-\lambda} + \frac{A_{\lambda^*}}{z-\lambda^*}$$

To find $T(\varepsilon, z)$ we use code `epsilon` [Prausa'17] capable to work with algebraic numbers

Solution of DE system in terms of iterated integrals

Series expansion of each integral g_i in ε starts from some finite order v_i

$$g_i(\varepsilon, z) = \sum_{k=v_i}^{\infty} \varepsilon^k g_{i,k}(z)$$

Thus for some v small enough, such that $v < \min(v_1, \ldots, v_n)$ we can put all $g_{i,v}(z) \equiv 0$

Due to the ε-form of DE system if we expand g_i in ε then DE system for $g_{i,k}(z)$ decouples and we can solve it easily order by order in ε

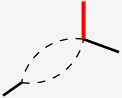
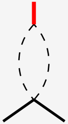
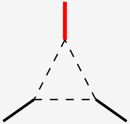
$$g_{i,k+1}(z) = \mathcal{C}_{i,k} + \int_0^z dt M'_{ij}(t) g_{j,k}(t) = \mathcal{C}_{i,k} + \int_0^z dt \sum_r \frac{(A_r)_{ij}}{t-r} g_{j,k}(t)$$

Answer expressible in terms of GPLs since we have the following integration rules

$$G(;z) = 1, \quad G(a_1;z) = \int_0^z \frac{dt}{t-a_1}, \quad G(a_1 \ldots a_n; z) = \int_0^z \frac{dt}{t-a_1} G(a_2 \ldots a_n; z)$$

One-loop example

At one-loop we have only 3 master integrals:

$f_1 =$  $f_2 =$  $f_3 =$ 

Original DE system:

$$\frac{d}{dz}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\varepsilon(1+z)}{z(1-z)} & 0 \\ \frac{2-4\varepsilon}{z^2-1} & \frac{2-4\varepsilon}{1-z^2} & \frac{\varepsilon(1-z)^2-1-z^2}{z(z^2-1)} \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

Transformation matrix T connecting original and canonical basis:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} \frac{\varepsilon}{2\varepsilon-1} & 0 & 0 \\ 0 & \frac{\varepsilon}{2\varepsilon-1} & 0 \\ 0 & 0 & \frac{z}{z^2-1} \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$$

One-loop integrals in canonical basis

$$\frac{d}{dz}\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \varepsilon \left[\frac{1}{z} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & -1 \end{pmatrix}}_{A_0} + \frac{1}{z-1} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A_1} + \frac{1}{z+1} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{A_{-1}} \right] \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$$

Several first steps of bottom up integration, we use $v = -4$ and assume that $g_{i,-4} \equiv 0$:

$$g_1 = \frac{\mathcal{C}_{1,-3}}{\varepsilon^3} + \frac{\mathcal{C}_{1,-2}}{\varepsilon^2} + \frac{\mathcal{C}_{1,-1}}{\varepsilon} + \mathcal{O}\left(\varepsilon^0\right)$$

$$g_2 = \frac{\mathcal{C}_{2,-3}}{\varepsilon^3} + \frac{1}{\varepsilon^2}\left(\mathcal{C}_{2,-3}(G(0;z) - 2G(1;z)) + \mathcal{C}_{2,-2}\right) + \mathcal{O}\left(\frac{1}{\varepsilon}\right)$$

$$g_3 = \frac{\mathcal{C}_{3,-3}}{\varepsilon^3} + \frac{1}{\varepsilon^2}\left(2\mathcal{C}_{3,-3}G(-1;z) - (2\mathcal{C}_{1,-3} - 2\mathcal{C}_{2,-3} + \mathcal{C}_{3,-3})G(0;z) + \mathcal{C}_{3,-2}\right) + \mathcal{O}\left(\frac{1}{\varepsilon}\right)$$

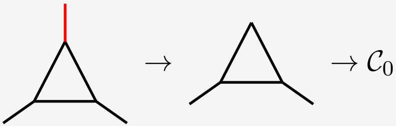
Fixing boundary conditions

Constraints on $\mathcal{C}_{i,k}$ strongly depend on the $g_i(z)$ expansion near DE singular points

1. Regularity conditions and constraints from allowed non-integer powers

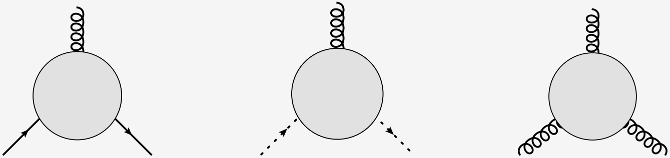
$$\vec{g} = \vec{\mathcal{C}} + \mathcal{O}(z - e^{\frac{i\pi}{3}}), \quad \vec{g} = \sum_n (1-z)^{-2n\varepsilon} \vec{\mathcal{C}}_n + \mathcal{O}(1-z)$$

2. Contributions from hard subgraphs calculated from naive expansion



3. Large momentum asymptotic expansion, all needed subgraphs generated with EXP package [Seidensticker'99] and require only massless propagator integrals treated by the MINCER package

QCD renormalization in MOM scheme from 3pt functions



Γ^{qqg} Γ^{ccg} Γ^{ggg}

Scalar form-factors relevant for renormalization:

$$\Gamma^{a}_{\mu,ij}(p_1,p_2) = g_s T^a_{ij}\left(\gamma_\mu \Gamma^{\mathrm{qqg}}(-\mu^2) + \ldots\right)$$
$$\Gamma^{abc}_{\mu}(p_1,p_2) = -i g_s f^{abc}\left(p_1^\nu g_{\nu\mu} \Gamma^{\mathrm{ccg}}(-\mu^2) + \ldots\right)$$
$$\Gamma^{abc}_{\mu\nu\rho}(p_1,p_2) = i g_s f^{abc}\left(T_{\mu\nu\rho} \Gamma^{\mathrm{ggg}}(-\mu^2) + \ldots\right)$$

For each of $V = \{qqg, ccg, ggg\}$ we define scheme MOM_V by condition $\Gamma^{\mathrm{V}}_{\mathrm{ren}}(-\mu^2) = 1$

Calculation of three-loop renormalization constants

To calculate bare three-loop form-factors we use projectors defined in [Gracey'11]

Vertex renormalization constants defined by:

$$\frac{1}{Z_{\mathrm{qqg}}} = \Gamma^{\mathrm{qqg}}_{\mathrm{bare}}(-\mu^2) \quad \frac{1}{Z_{\mathrm{ccg}}} = \Gamma^{\mathrm{ccg}}_{\mathrm{bare}}(-\mu^2) \quad \frac{1}{Z_{\mathrm{ggg}}} = \Gamma^{\mathrm{ggg}}_{\mathrm{bare}}(-\mu^2)$$

Coupling constant renormalization factors:

$$\mu^{-2\varepsilon} a_{\mathrm{bare}} = Z_{a_{\mathrm{ren}}} a_{\mathrm{ren}} = \left[\frac{Z^2_{\mathrm{ggg}}}{Z^3_{\mathrm{gg}}}\right] a_{\mathrm{ggg}} = \left[\frac{Z^2_{\mathrm{ccg}}}{Z^2_{\mathrm{cc}} Z_{\mathrm{gg}}}\right] a_{\mathrm{ccg}} = \left[\frac{Z^2_{\mathrm{qqg}}}{Z^2_{\mathrm{qq}} Z_{\mathrm{gg}}}\right] a_{\mathrm{qqg}}$$

Coupling constant conversion factors and 4-loop beta functions

Couplings a_R in different renormalization schemes are connected with $a_{\overline{\rm MS}}$:

$$a_R = \left(Z_{a_R}/Z_{a_{\overline{\rm MS}}}\right) a_{\overline{\rm MS}} \equiv a_{\overline{\rm MS}} X_R = a_{\overline{\rm MS}} \left[1 + \sum_l X_R^{(l)} a_{\overline{\rm MS}}^l\right]$$

Our main result: calculation of three-loop corrections to the conversion factors

$$X_{\rm qqg}^{(3)} \quad X_{\rm ccg}^{(3)} \quad X_{\rm ggg}^{(3)}$$

Using known L-loop conversion factor and $L+1$-loop QCD beta-function in $\overline{\rm MS}$ scheme

$$\beta_R \equiv \frac{d a_R}{d \ln \mu^2} = \frac{\partial a_R(a_{\overline{\rm MS}})}{\partial a_{\overline{\rm MS}}} \cdot \beta_{\overline{\rm MS}}(a_{\overline{\rm MS}})$$

We derive expressions for L-loop beta functions of our interest $\beta_{\rm qqg}$, $\beta_{\rm ccg}$ and $\beta_{\rm ggg}$

Numerical results and possible applications

Two-loop part is universal $\beta_{\text{uni}}(a) = -a^2(11 - 2/3\, n_f) - a^3(102 - 38/3\, n_f)$

Three-loop and four-loop parts are different in different schemes

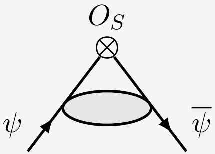
$$\beta_{\text{ccg}} = \beta_{\text{uni}}(a_{\text{ccg}}) - a_{\text{ccg}}^4 \left(2813.492952 - 617.6471546\, n_f + 21.50281811\, n_f^2\right) - a_{\text{ccg}}^5 \left(96089.34786 - 23459.32128\, n_f + 1735.992218\, n_f^2 - 33.24145137\, n_f^3\right)$$

$$\beta_{\text{qqg}} = \beta_{\text{uni}}(a_{\text{qqg}}) - a_{\text{qqg}}^4 \left(1843.652731 - 588.6548459\, n_f + 22.58781183\, n_f^2\right) - a_{\text{qqg}}^5 \left(68529.68547 - 15466.43194\, n_f + 1093.568841\, n_f^2 - 18.85323795\, n_f^3\right)$$

$$\beta_{\text{ggg}} = \beta_{\text{uni}}(a_{\text{ggg}}) - a_{\text{ggg}}^4 \left(1570.9844 + 0.56592607\, n_f - 67.089536\, n_f^2 + 2.6581155\, n_f^3\right) - a_{\text{ggg}}^5 \left(94167.261 - 27452.645\, n_f + 4152.5388\, n_f^2 - 543.68484\, n_f^3 + 20.429348\, n_f^4\right)$$

Application of conv.factor: $R(s)$ in MOM schemes, extension of [Gracey'14]

Vertex functions with operator insertions and quark masses



Quark mass conversion factor relates $m_q^{\overline{\rm MS}}$ to the the value available from lattice

$$m_q^{\overline{\rm MS}} = C_m^{\rm SMOM} m_q^{\rm SMOM} \quad C_m^{\rm SMOM} = \frac{Z_m^{\rm SMOM}}{Z_m^{\overline{\rm MS}}}$$

Bare mass and conversion between renormalization schemes

$$m_{\rm bare} = Z_m^{\rm R} m_q^{\rm R} = Z_m^{\overline{\rm MS}} m_q^{\overline{\rm MS}} = Z_m^{\rm SMOM} m_q^{\rm SMOM}$$

Mass renormalization from the scalar bilinear operator renormalization

$$O_S \equiv \bar{\psi}\psi,\, [\bar{\psi}\psi]_{\rm R} = Z_m^{\rm R} (\bar{\psi}\psi)_{\rm bare}$$

Details of calculation and results

Renormalization conditions in Landau gauge, calculated order by order

$$1 = Z_m^{\text{SMOM}} \cdot Z_\psi^{\text{SMOM}} \cdot \frac{1}{12} \cdot \text{tr}\left[\Lambda_S^{\text{bare}}\right]\Big|_{p_1^2=p_2^2=q^2=-\mu^2}$$

$$1 = Z_\psi^{\text{SMOM}} \cdot \frac{1}{12p^2} \cdot \text{tr}\left[iS_{\text{bare}}^{-1}(p)\hat{p}\right]\Big|_{p^2=-\mu^2}$$

Repeating calculation for the $\overline{\text{MS}}$ scheme and using $\overline{\text{MS}}$ scheme for $a_{\overline{\text{MS}}}$ renormalization

$$C_m^{\text{SMOM}} = 1 - 0.6455188560 a_{\overline{\text{MS}}} - (22.60768757 - 4.013539470\, n_f) a_{\overline{\text{MS}}}^2 - (860.2874030 - 164.7423004\, n_f + 2.184402262\, n_f^2) a_{\overline{\text{MS}}}^3$$

- Set of master integrals used is the same as before
- Small set of transcendental constants $\pi, \zeta_3, \zeta_5, \psi^{(1)}(1/3), \psi^{(3)}(1/3), \psi^{(5)}(1/3)$
- Two new constants HPLs of fixed transcendental weight H_5, H_6
- Result in full agreement with independent numerical calculation [Kniehl,Veretin'20]

Conclusion

- Calculated full set of three-point three-loop integrals in symmetric point. Results expressed in terms of HPLs with argument $e^{\frac{i\pi}{3}}$
- Constructed three-loop conversion factor relating couplings defined in different MOM schemes with $\overline{\mathrm{MS}}$
- Using three-loop conversion factor derived four-loop beta-functions in various MOM schemes
- Calcualted three-loop corrections to the relation between $\overline{\mathrm{MS}}$ quark masses and quark masses defined in symmetric point MOM scheme

Thank you for attention!