Renormalization group investigation of critical phenomena in static and dynamic models

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Structure

- Part 1. Matrix field models
  1. Example of physical system
  2. Effective field models close to phase transitions
  3. The ε-expansion analysis
  4. The nonperturbative RG
- Part 2. Effect of turbulence on critical behaviour
  1. Stochastic Langevin equation
  2. The nonperturbative RG

Fermionic systems

- the associated thermal action for the nonrelativistic problem

$$S = \int_0^{1/T} \mathrm{d}\tau \int \mathrm{d}^d x \left(\psi_n^* \left\{ \partial_\tau - \frac{\Delta}{2m} - \mu \right\} \psi_n + \frac{\lambda}{2} (\psi_n^* \psi_n)^2 \right), \quad n = 1, \ldots, N, \quad \psi_n(0, x) = -\psi_n(1/T, x)$$

- the symmetry of the action: $\psi_n \to V_{nk}\psi_k,\ V \in SU(N)$
- BCS – model: $N = 2$
- Large spin cold fermions: $N > 2$

Gorshkov, Nature Phys (2014)
Wu, Physics (2012)

Atom Species	Nuclear spin	Symmetry	Scatt. Length
^{171}Yb	1/2	$SU(2)$	-0.15 nm
^{173}Yb	5/2	$SU(6)$	10.55 nm
^{87}Sr	9/2	$SU(10)$	5.09 nm

Kitagawa et al., Phys. Rev. Lett. (2008)

de Escobar et al., Phys. Rev. A (2008)

The Hubbard-Stratonovich decoupling

1. The Cooper channel:

$$\exp\left\{-\frac{\lambda}{2}\left(\psi_n^*\psi_n\right)^2\right\} = \int \mathcal{D}\chi\mathcal{D}\chi^\dagger \exp\left\{-\frac{1}{2\lambda}\,\mathrm{tr}\,\chi\chi^\dagger + \frac{1}{2}\psi_n\chi_{nm}^\dagger\psi_m + \frac{1}{2}\psi_n^*\chi_{nm}\psi_m^*\right\}$$

The Swinger equations show:

$$\langle\chi_{nm}\rangle = \lambda\langle\psi_m\psi_n\rangle$$

In two component systems $N=2$, and thus $n,m=\uparrow,\downarrow$. The nonzero value of $|\langle\chi_{\uparrow\downarrow}\rangle|^2$ determines a gap in the spectrum of electrons in the BCS model.

2. The spin-density channel:

$$\exp\left\{-\frac{\lambda}{2}\left(\psi_n^*\psi_n\right)^2\right\} = \int \mathcal{D}\varphi\mathcal{D}n\,\exp\left\{-\frac{1}{4\lambda}\varphi_A\varphi_A + \varphi_A\psi_n^* t_{nm}^A\psi_m - \frac{N}{2\lambda}n^2 + n\,\psi_n^*\psi_n\right\}$$

$SU(N)$ generators

The Swinger equations show:

$$\langle n\rangle = \lambda\left\langle\psi_m^*\psi_m\right\rangle/N, \quad \langle\varphi_A\rangle = 2\lambda\left\langle\psi_n^* t_{nm}^A\psi_m\right\rangle$$

The magnetization of the system in the case $N=2$:

$$M^z \sim \left\langle\psi_\uparrow^*\psi_\uparrow - \psi_\downarrow^*\psi_\downarrow\right\rangle$$

Integration over fermionic fields

$\omega_s = \pi T(2s+1),\ s \in \mathbb{Z}$

1. The Cooper channel (superfluidity):

$$S_\chi = \frac{1}{2\lambda}\,\mathrm{tr}\,\chi\chi^\dagger - \mathrm{tr}\ln\begin{pmatrix} -\chi^\dagger & -i\omega_s - \frac{\Delta}{2m} - \mu \\ -i\omega_s + \frac{\Delta}{2m} + \mu & -\chi \end{pmatrix}$$

Expanding the action in fields and derivatives one obtains the leading infra-red (IR) term:

$$S_\chi = \mathrm{tr}\,\chi^\dagger(\mathbf{p}^2 + m_0^2)\chi + \frac{g_{01}}{4}\,\mathrm{tr}(\chi\chi^\dagger\chi\chi^\dagger)$$

Kalagov et al., Nucl. Phys. B (2016)

2. The spin-density channel (magnetism):

$$S_\varphi = \mathrm{tr}\,\varphi(\mathbf{p}^2 + m_0^2)\varphi + h_{00}\,\mathrm{tr}\,\varphi^3 + h_{01}\,\mathrm{tr}\,\varphi^4, \quad \varphi = \varphi_A t^A$$

Mean field outcomes

For the model (1) we obtained **the second order phase transition** at $m_0^2 = 0$ for all *N*.

$$m_0^2 \sim 1 + \lambda\nu_F \ln\frac{\gamma\mu}{\pi T}$$

Attraction $\lambda < 0$ **may lead to superfluidity**

For the model (2) we obtained **the second order phase transition** at $m_0^2 = 0$ for ***N* = 2**.

$$m_0^2 \sim 1 - \lambda\nu_F$$

Repulsion $\lambda > 0$ **may lead to magnetism**

The first order phase transition

In the case $N > 2$, the action

$$S_\varphi = \mathrm{tr}\,\varphi(\mathbf{p}^2 + m_0^2)\varphi + h_{00}\,\mathrm{tr}\,\varphi^3 + h_{01}\,\mathrm{tr}\,\varphi^4, \quad \varphi = \varphi_A t^A$$

contains the cubic term, and systems manifests **the first order phase transition**.

3/2-spin particles ($N = 4$)

Atomic species $^{135}\mathrm{Ba}$, $^{137}\mathrm{Ba}$

Wu, Phys. Rev. Lett. (2006)
Burkhardt, Phys. Rev. Lett. (1991)

Symmetry breaking $SU(4) \to SU(3) \otimes U(1)$

Ruegg, Phys. Rev. D (1980)

$$\langle\varphi\rangle = \frac{\alpha}{\sqrt{12}}\begin{pmatrix} \mathrm{I}_3 & 0 \\ 0 & -3 \end{pmatrix}$$

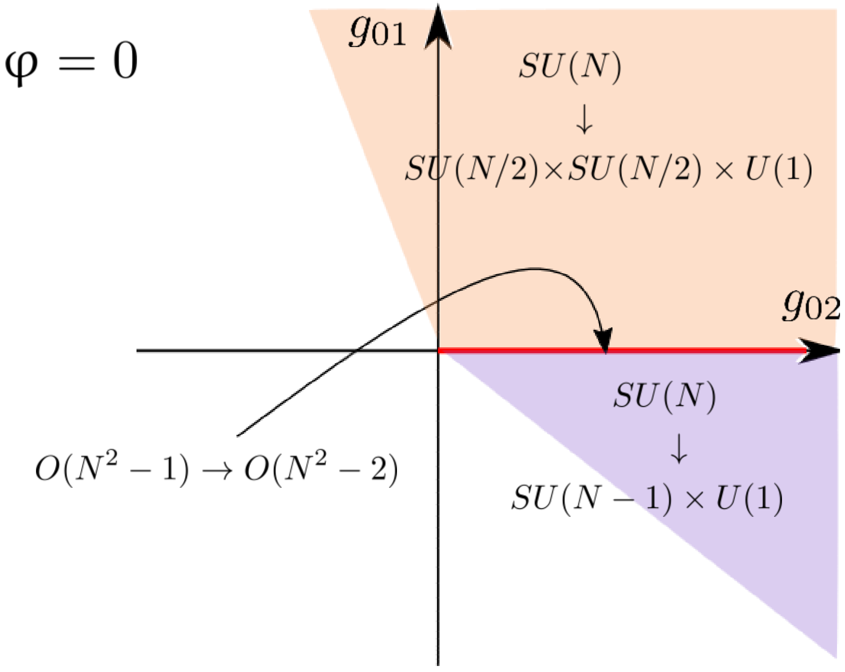
Phase transition at nonzero value of $m_0^2 \sim 1 - \lambda\nu_F$, namely $\lambda\nu_F = 21/25 = 0.84$.

The pure cubic model: 4-loop RG analysis *Gracey, Phys. Rev. D (2017)*

The field model with an adjoint field

The Euclidean action:

$$S_\varphi = \frac{1}{2}\,\mathrm{tr}\,\varphi(\mathbf{p}^2 + m_0^2)\varphi + \frac{g_{01}}{4}\,\mathrm{tr}\,\varphi^4 + \frac{g_{02}}{4}(\mathrm{tr}\,\varphi^2)^2, \quad \varphi = \varphi^\dagger, \quad \mathrm{tr}\,\varphi = 0$$



Multiplicative renormalization in $d = 4 - \varepsilon$:

$$\varphi \to Z_\varphi \varphi, \quad m_0^2 = Z_{m^2} m^2, \quad g_{0j} = g_j \mu^\varepsilon Z_{g_j}$$

RG functions:

$$\beta_1 = -\varepsilon\, g_1 + g_1\, g_k \frac{\partial}{\partial g_k} Z_{g_1}^{\{1\}}, \quad \beta_2 = -\varepsilon\, g_2 + g_2\, g_k \frac{\partial}{\partial g_k} Z_{g_2}^{\{1\}},$$

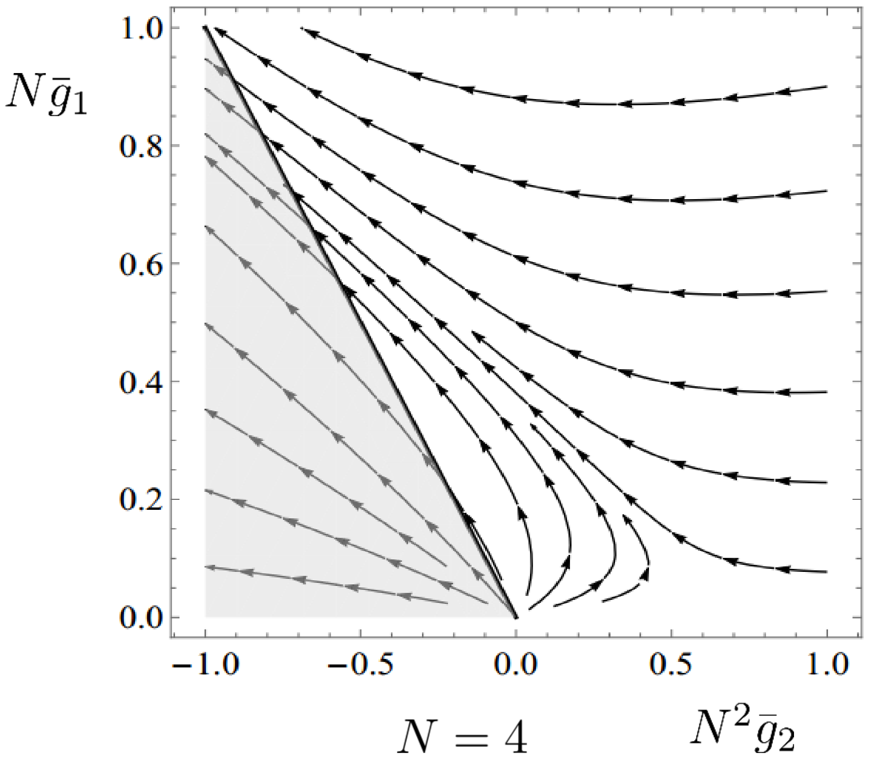
$$\gamma_\varphi = -g_k \frac{\partial}{\partial g_k} Z_\varphi^{\{1\}}, \quad \gamma_{m^2} = -g_k \frac{\partial}{\partial g_k} Z_{m^2}^{\{1\}}.$$

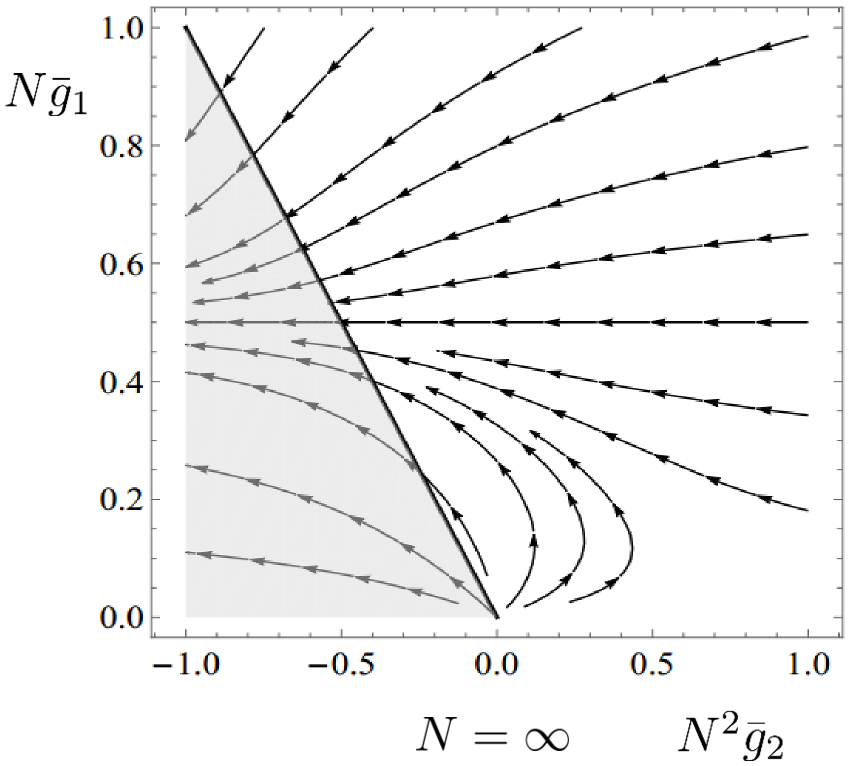
the stability conditions		
$g_{01} > 0$	$g_{01} = 0$	$g_{01} < 0$
$g_{01} + N g_{02} > 0$	$g_{02} > 0$	$g_{01} + g_{02} > 0$

Phase portrait (one-loop)

The RG flow goes beyond the domain $\bar{g}_1 + N\bar{g}_2 > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{g}_j = \frac{\beta_j}{2+\gamma_{m^2}}, \quad \bar{g}_j|_{t=0} = g_j, \quad t = \ln\frac{m^2}{\mu^2}$$





There are no IR stable fixed points. The model losses stability: the free energy is not bounded from below.

Loop corrections to the free energy

The one-loop free energy (effective action):

$$\Gamma_R(\Phi) = S_R(\Phi) + \frac{1}{2}\,\mathrm{tr}\ln\left(\frac{\delta^2 S_R(\varphi)}{\delta\varphi\,\delta\varphi}\right)\Bigg|_{\varphi=\Phi}$$

The pattern of symmentry breaking has to be chosen:

$$SU(N) \to SU(N/2)\times SU(N/2)\times U(1)$$

Background field:

$$\Phi = \alpha\begin{pmatrix} \mathrm{I}_{N/2} & 0 \\ 0 & -\mathrm{I}_{N/2}\end{pmatrix}$$

The free energy per unit volume:

$$\mathscr{F} = \frac{N}{2}m^2\alpha^2 + \frac{N}{4}(g_1 + N g_2)\,\alpha^4 + \frac{1}{8}\sum_a n_a\, M_a^4\,\ln\left(\frac{M_a^2}{\mu^2}\right)$$

$$\begin{aligned} M_1^2 &= m^2 + 3\,(g_1 + N g_2)\,\alpha^2, \quad & n_1 &= 1,\\ M_2^2 &= m^2 + (3g_1 + N g_2)\,\alpha^2, \quad & n_2 &= N^2/2 - 2,\\ M_3^2 &= m^2 + (g_1 + N g_2)\,\alpha^2, \quad & n_3 &= N^2/2. \end{aligned}$$

The free energy is defined by the Legendre transformation:

$$\Gamma(\Phi) = \sup_J\{J\Phi - W(J)\},$$

where the functional $W(J)$ is given by:

$$W(J) = \ln\int \mathcal{D}\varphi\,\exp\{-S(\varphi) + J\varphi\}.$$

Fluctuation induced first order phase transition

Let us consider the free energy near the stability boundary $g_1 + N g_2 = 0$.
Within the ε-expansion we get:

$$m^2 = \mathscr{O}(\varepsilon), \quad \alpha^2 = \mathscr{O}(1/\varepsilon), \quad g_j = \mathscr{O}(\varepsilon)$$

As a result one obtains the leading contribution in ε:

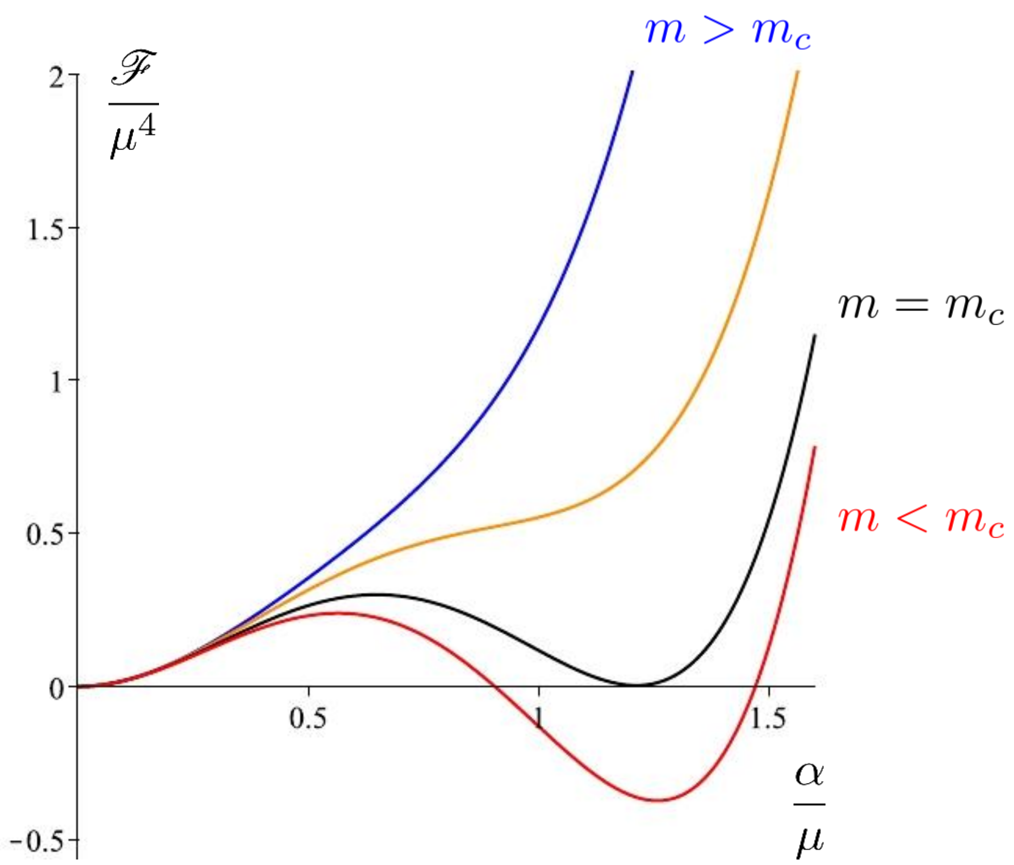
$$\mathscr{F} = \frac{N}{2} m^2 \alpha^2 + \frac{N^2 - 4}{4} g_1^2 \alpha^4 \ln\left(\frac{2 g_1 \alpha^2}{\mu^2}\right) + \mathscr{O}(\varepsilon)$$

The phase transition point:

$$\mathscr{F}(0) = \mathscr{F}(\alpha_c), \quad \mathscr{F}'|_{\alpha_c} = 0$$

Connection between a jump α_c and critical m_c:

$$m_c^2/\alpha_c^2 = g_1^2\,(N^2 - 4)/(2N)$$



The fluctuation driven first order phase transition in *SU(N > 2)* symmetric model with an adjoint field is established.

Disadvantages of perturbation expansions

- The structure of actions:

$$S = S_0 + \lambda\, S_{int}.$$

- The structure of observables:

$$f(\lambda) = \sum_{n=0}^{N} f_n \lambda^n + R_N(\lambda).$$

- We assume:

  1. $\lambda \ll 1$ – the weak coupling limit,
  2. $R_N(\lambda)$ is a small contribution.

- Typical values of expansion parameter:

  - in QED: $\lambda \sim 10^{-2}$
  - in Stat.Phys.: $\varepsilon \gtrsim 1$
  - in Turbulence: $\varepsilon \gtrsim 1 - 4$

- The Higher Order Asymptotics

$$f_n = c_0(-a)^n n^b n!$$

Lipatov (1970s)

- The Borel resummation of multi-loop expansions is required to extract physical results

The 5-loop RG analysis + Borel resummation based on the obtained HOA in the model

$$S_\chi = \operatorname{tr}\chi^\dagger(\mathbf{p}^2 + m_0^2)\chi + \frac{g_{01}}{4}\operatorname{tr}(\chi\chi^\dagger\chi\chi^\dagger)$$

did not qualitatively alter one-loop findings.

Kalagov et al., Nucl. Phys. B (2016)

The nonperturbative RG (NPRG)

The free energy is defined by the Legendre transformation:

$$\Gamma[\Phi] = \sup_j \{j\Phi - W[j]\} = J(\Phi)\Phi - W[J(\Phi)],$$

where for an appropriate sourse we get:

$$\Phi = \left. \frac{\delta W[j]}{\delta j} \right|_{j=J}.$$

The functional $W[j] = \ln Z[j]$ and

$$Z[j] = \int \mathcal{D}\varphi \exp\{-S[\varphi] + j\varphi\}.$$

Knowledge of the free energy is a solution of the problem.

Wetterich (1990s):

Construct a functional that interpolates between an action S at the UV limit and the full free energy at the IR limit.

The partition function of fast modes $p > k$:

$$Z_k[j] = \int \mathcal{D}\varphi_{p>k} \exp\{-S[\varphi] + j\varphi\}.$$

The soft cutoff procedure:

$$\int \mathcal{D}\varphi_{p>k} = \int \mathcal{D}\varphi \exp\{-\Delta S_k[\varphi]\}$$

suppresses slow modes

The cutoff term:

$$\Delta S_k[\varphi] = \frac{1}{2} \int_p \varphi(p) R_k(p) \varphi(-p)$$

The properties od $R_k(p)$

- Mass additive to slow modes:

$$R_k(p) \sim k^2, \quad p << k$$

- IR limit:

$$R_k(p) \to 0, \quad k \to 0$$

- UV limit:

$$R_k(p) \to \infty\, (\sim \Lambda^2), \quad k \to \infty\, (\Lambda)$$

The effective average action

The effective average action:

$$\Gamma_k[\Phi] = J(\Phi)\Phi - W_k[J(\Phi)] - \Delta S_k[\Phi]$$

The functional $W_k[j] = \ln Z_k[j]$ and

$$Z_k[j] = \int \mathcal{D}\varphi \exp\{-S[\varphi] - \Delta S_k[\Phi] + j\varphi\}$$

New object meets the disered conditions:

$$\Gamma_{k=0}[\Phi] = \Gamma[\Phi], \quad \Gamma_{k\to\Lambda}[\Phi] = S[\Phi]$$

Applying a k-derivative to Γ_k leads us to the equation

$$\partial_k \Gamma_k = \frac{1}{2}\,\mathrm{Tr}\left\{\left(\Gamma_k^{(2)} + R_k\right)^{-1} \partial_k R_k\right\}$$

second derivative matrix

1. The Wetterich equation is exact but not exactly solvable.
2. A wide used approximation – the field and derivative expansion.
3. The cutoff function may have an arbitrary shape that meets the necessary conditions.
4. Numerical outcomes weakly depend on the cutoff.

The field and derivative expansion

The Euclidean action:

$$S = \frac{1}{2}\,\mathrm{tr}\,\varphi(\mathbf{p}^2 + m_0^2)\varphi + \frac{g_{01}}{4}\,\mathrm{tr}\,\varphi^4 + \frac{g_{02}}{4}(\mathrm{tr}\,\varphi^2)^2, \quad \varphi = \varphi^\dagger, \quad \mathrm{tr}\,\varphi = 0$$

The leading term of the derivative expansion:

$$\Gamma_k = \frac{1}{2}\,\mathrm{tr}\,(\partial\Phi)^2 + U_k(\rho,\sigma),$$

where invariants ρ, σ are defined as

$$\rho \equiv \mathrm{tr}\,\Phi^2, \quad \sigma \equiv \mathrm{tr}\,\Phi^4 - \frac{\rho^2}{N}$$

Close to the transition $SU(N) \to SU(N/2) \otimes SU(N/2) \otimes U(1)$:

$$U_k(\rho,\sigma) = U_{1;k}(\rho) + U_{2;k}(\rho)\,\sigma + \mathscr{O}(\sigma^2)$$

The cutoff function:

$$R_k(p) = (k^2 - p^2)\Theta(k^2 - p^2)$$

Litim, Phys. Rev. D (2001)

the flow equation

$$\partial_k U_{1;k} \sim k^{d+1} \sum_a \frac{n_a}{k^2 + M_a^2}$$

$$M_1^2 = U'_{1;k} + \frac{4\,U_{2;k}}{N}\,\rho$$

$$M_2^2 = U'_{1;k}$$

$$M_3^2 = U'_{1;k} + 2\,\rho U''_{1;k}$$

The full flow equation

System of coupled partial differential equtions:

$$\partial_k U_{1;k} = k^{d+1} \left\{ \frac{1}{k^2 + U'_{1;k} + 2\rho U''_{1;k}} + \frac{N^2/2 - 2}{k^2 + U'_{1;k} + 4\rho U_{2;k}/N} + \frac{N^2/2}{k^2 + U'_{1;k}} \right\},$$

$$\partial_k U_{2;k} = k^{d+1} \left\{ \frac{N U_{2;k}^2}{(k^2 + U'_{1;k})^3} - \frac{N^2 \left(U_{2;k} + 2\rho U'_{2;k}\right)}{4\rho (k^2 + U'_{1;k})^2} + \frac{9(N^2 - 16) U_{2;k}^2}{N (k^2 + U'_{1;k} + 4\rho U_{2;k}/N)^3} + \right.$$

$$+ \frac{1}{(k^2 + U'_{1;k} + 4\rho U_{2;k}/N)^2} \left(\frac{N^2+4}{4\rho} U_{2;k} - \frac{N^2+4}{2} U'_{2;k} - \frac{N U''_{1;k}}{2\rho} + \frac{(N U''_{1;k} + 4 U'_{2;k}\rho + 4 U_{2;k})^2}{2\rho (N U''_{1;k} - 2 U_{2;k})} \right) -$$

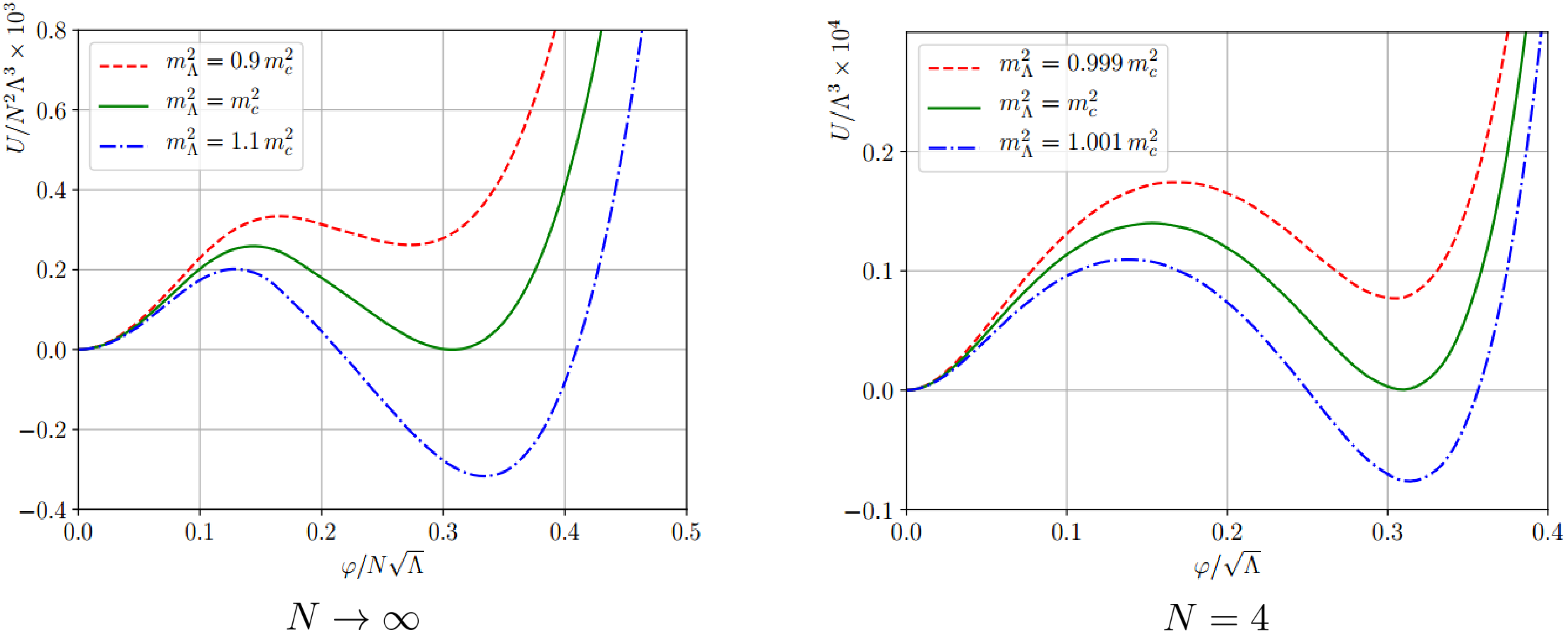
$$\left. - \frac{1}{(k^2 + U'_{1;k} + 2\rho U''_{1;k})^2} \left(2\, U''_{2;k}\, \rho + 5\, U'_{2;k} + \frac{U_{2;k}}{\rho} - \frac{N U''_{1;k}}{2\rho} + \frac{\left(N U''_{1;k} + 4\, U'_{2;k}\, \rho + 4\, U_{2;k}\right)^2}{2\rho \left(N U''_{1;k} - 2\, U_{2;k}\right)} \right) \right\}$$

must be solved subject to the initial conditions at the UV scale

$$U_{1;k=\Lambda} = m_\Lambda^2 \rho + \lambda_{1,\Lambda} \frac{\rho^2}{2}, \quad U_{2;k=\Lambda} = \lambda_{2,\Lambda}$$

Results of numerical solution

The free energy of the $d = 3$ system for the model parameters:
$\lambda_{1,\Lambda} = \Lambda,\ \lambda_{2,\Lambda} = 10\,\Lambda;\ m_c^2(N=4) = -0.272\,\Lambda^2;\ m_c^2(N\to\infty) = -0.063\,\Lambda^2.$



The model considered exhibits **the first order phase transition** for $N > 2$, in agreement with the one-loop ε expansion results.

Preliminary conclusions

- The matrix models considered undergo the fluctuation induced first-order phase transitions which are revealed by means

  1. the Borel resummation of ε-expansion
  2. the nonperturbative RG

  Note! One-loop approximation properly predicts qualitatively picture.

- In particular, these results can be employed to describe large spin fermi systems.

Dynamical critical behaviour

A typical feature of critical dynamics – critical slowing down:

$$t_{relaxation} \sim \xi^{z} \sim |T - T_c|^{-\nu z} \to \infty$$

z – dynamical critical exponent

The Landau model of a critical point:

$$S = \frac{1}{2}(\nabla\varphi)^2 + \frac{\tau_0}{2}\varphi^2 + \frac{g_0}{4!}\varphi^4, \quad \tau_0 \sim T - T_c$$

1. Ising model

2. liquid –vapor critical point

3. binary mixture etc.

Dynamics of order parameter $\varphi = \varphi(t, x)$:

$$\lambda\,\partial_t\varphi = -\frac{\delta S}{\delta\varphi} + \eta$$

noize

$$\langle\eta\rangle = 0$$

$$\langle\eta(t,x)\eta(t',x')\rangle = 2\lambda\,\delta(t-t')\delta(x-x')$$

Aims:

- Investigate IR asymptotics of Green (or thermodinamical functions)
- Obtain possible scaling regimes
- Calculate critical exponents - ν, η, z

Turbulent motion

Fully developed turbulence:

$$Re = \frac{VL}{\nu} \to \infty$$

Universal spectrum of fluctuations (Kolmogorov):

$$E(k) \sim k^{-5/3}$$

The Kraichnan model: velocity $v_i(t,x)$ is a random field

$$D_{ij} = \langle v_i(t,x) v_j(t,x) \rangle => \frac{1}{k^{d+\zeta}} \left[\delta_{ij} - \frac{k_i k_j}{k^2} + \alpha \frac{k_i k_j}{k^2} \right]$$

$\alpha = 0$ – incompressible fluid

$\xi = 4/3$ – physical value

Coupling with the velocity field:

$$\partial_t \varphi \to \nabla_t \varphi \equiv \partial_t \varphi + (v_i \partial_i)\, \varphi$$

Effective model

The Martin-Siggia-Rose action

$$S_{MSR} = \lambda\, \varphi' \nabla_t \varphi + \varphi' \frac{\delta S}{\delta \varphi} - \lambda\, \varphi' \varphi' + \frac{1}{2} v_i D^{-1}_{ij} v_j$$

Vasiliev , The Field Theoretic Renormalization Group in Critical Behavior Theory and Stochastic Dynamics (2004)

Green functions:

$$\langle \varphi \dots \varphi \rangle \sim \int \mathcal{D}\varphi\, \mathcal{D}\varphi' \mathcal{D}v \varphi \dots \varphi\, e^{-S_{MSR}}$$

Ansatz for the Wetterich equation:

$$\Gamma_k = X_k\, \varphi' \left\{ \nabla_t + A_k\, (\partial_i v_i) \right\} \varphi + \varphi' \frac{\delta S_k}{\delta \varphi} - Y_k\, \varphi' \varphi' + \frac{1}{2} v_i D^{-1}_{ij} v_j, \quad \Gamma_{k=\Lambda} = S_{MSR}$$

where:

$$S_k = \frac{1}{2} Z_k\, (\nabla \varphi)^2 + U_k(\varphi)$$

Flowing “anomalous dimensions”:

$$\gamma^X_k = -k\partial_k \ln X_k, \quad \gamma^Y_k = -k\partial_k \ln Y_k, \quad \eta_k = -k\partial_k \ln Z_k.$$

$$z = 2 - \eta_{k=0} + \gamma^X_{k=0}$$

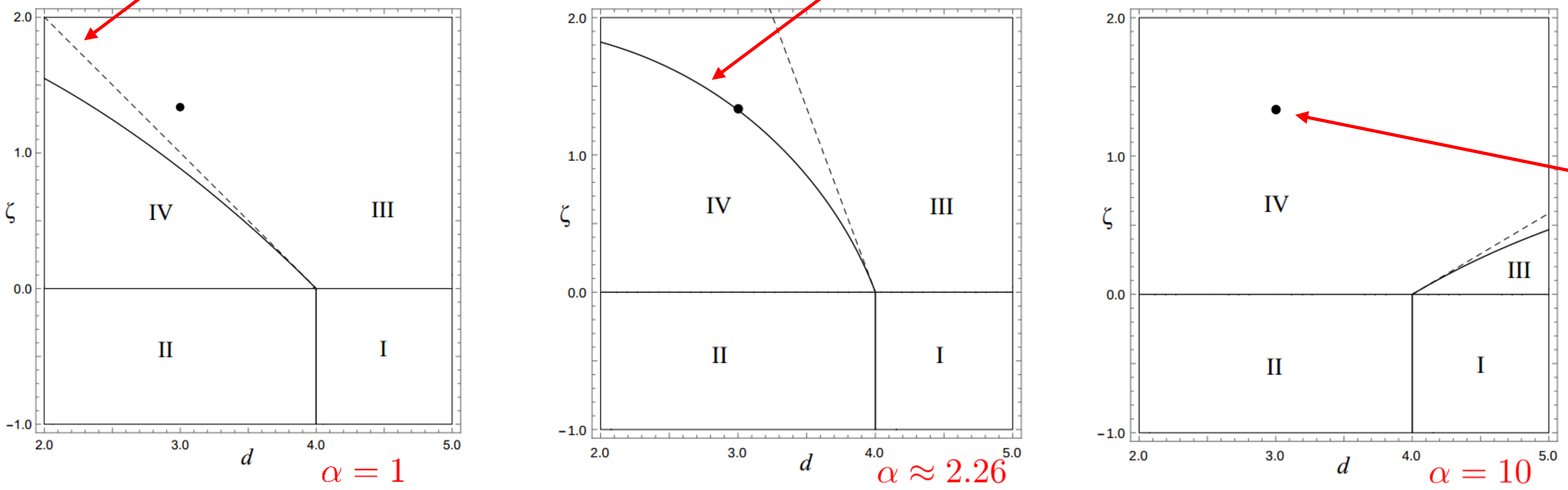
Results

The model contains three parameters: d - space dimensionality, ζ and "compressibility" α. We will consider the possible scaling regimes in (d, ζ) plane at given α. **There are 4 scaling regimes.**

$$D_{ij} = \frac{1}{k^{d+\zeta}}\left[\delta_{ij} - \frac{k_i k_j}{k^2} + \alpha\frac{k_i k_j}{k^2}\right]$$

one-loop ε expansion *Antonov, Phys. Rev. E (2009)*

NPRG



physical point:
$d = 3$
$\zeta = 4/3$

$t_{relaxation} \sim \xi^z$

I. Gaussian fixed point.
II. Pure A-model (turbulence is not relevant).
III. Pure turbulence (critical fluct. are not relevant).

IV. NEW regime. Due to competition between turbulence and critical fluctuations the system may show a new stable scaling regime, where $z_{NEW} = 2/3$.

$z_{without\,turbul.} \approx 2.036 > z_{NEW}$

Conclusion

- We employed the NPRG to analyse the impact of fully developed turbulence on the scaling behaviour of critical (compressible) liquids.
- The new scaling regime was established.
- Numerical values of respective critical exponents are estimated.

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thank you for your time