Calculation of master integrals in terms of elliptic multiple polylogarithms

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TEMPLE- Tools for Elliptic Multiple Polylogarithms Evaluation
(private program, for now)

M. Bezuglov, arXiv:2003.05367 [hep-th]
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Contents

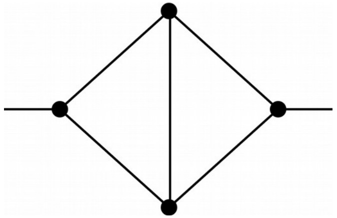
Introduction

Feynman integral: $$\int ... \int \frac{d^4k_1...d^4k_n}{D_1^{j_1}...D_l^{j_l}}, \qquad D_r = \sum_{i \geq j \geq 1} A_r^{ij} p_i p_j - m_r^2$$

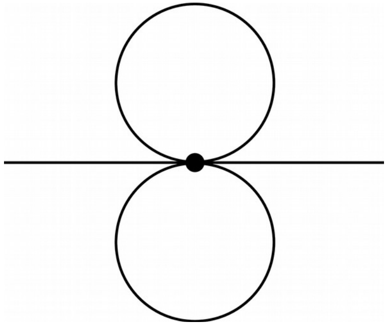
Integration by parts (IBP) $\int d^dk_1 d^dk_2 ... \frac{\partial f}{\partial k_i^\mu} = 0$ in dimensional regularization

Any integral from a given family can be represented as a linear combination of some limited basis of integrals, elements of this basis are called **master integrals**.

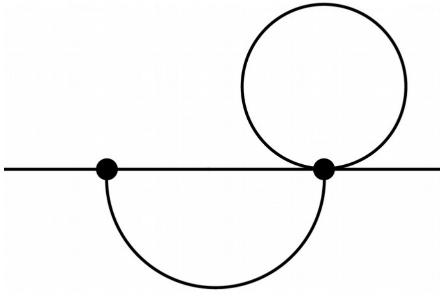
$$I_{j1,j2,j3,j4,j5} = \frac{(\mu^2)^{4-d}}{(i\pi^{\frac{d}{2}})^2} \int \frac{dk_1^d \; dk_2^d}{D_1^{j1} D_2^{j2} D_3^{j3} D_4^{j4} D_5^{j5}}$$



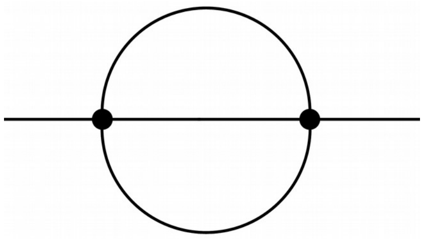
$$D_1 = k_1^2 - m^2,\; D_2 = k_2^2 - m^2,\; D_3 = (k_1 - k_2)^2 - m^2,\; D_4 = (k_1 - p)^2 - m^2,\; D_5 = (k_2 - p)^2 - m^2$$



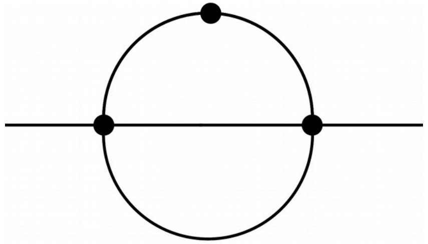
$I_{0,0,0,1,1}$



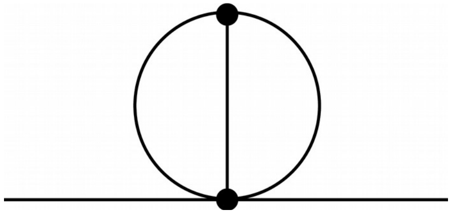
$I_{0,1,0,1,1}$



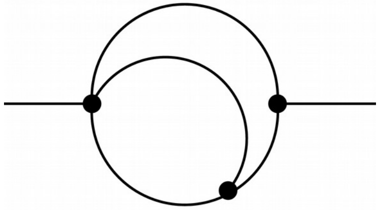
$I_{0,1,1,1,0}$



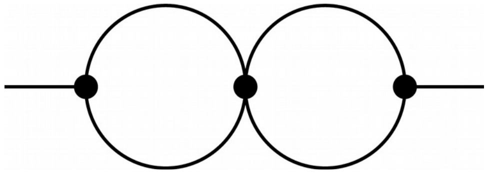
$I_{0,2,1,1,0}$



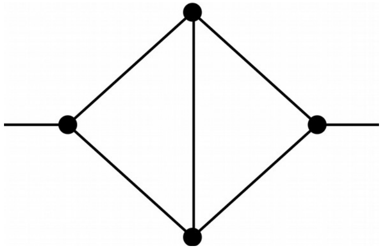
$I_{1,1,1,0,0}$



$I_{0,1,1,1,1}$



$I_{1,1,0,1,1}$



$I_{1,1,1,1,1}$

Methods for calculating loop integrals

Evaluation by differential equations

Evaluating by direct integration using some parametric representation

- ***<u>Feynman parametrisation</u>***
- Alpha parametrisation
- MB represention
- et al.

$$I_{\nu_1,\ldots,\nu_2} = \int \ldots \int \frac{d^D k_1 \ldots d^D k_n}{D_1^{\nu_1} \ldots D_l^{\nu_l}} = \frac{\Gamma(\nu - \frac{lD}{2})}{\prod\limits_{j=1}^{n} \Gamma(\nu_j)} \int_\Delta \left(\prod_{j=1}^{n} x_j^{\nu_j - 1} dx_j \right) \frac{U^{\nu - \frac{(l+1)D}{2}}}{F^{\nu - \frac{lD}{2}}}, \qquad \nu = \sum \nu_l$$

U and F - Symanzik polynomials

$$\Delta \in \left\{ \vec{x} \mid x_i > 0, \ \sum_{i=1}^{n} x_i = 1 \right\}$$

Multiple polylogarithms(MPLs)

Definition:

$$G(a_1, ..., a_n; x) = \int_0^x \frac{G(a_2, ..., a_n; x')}{x' - a_1} dx', \qquad n > 0,$$

Regularization:

$$G(;x) = 1, \qquad G(\vec{0}_n; x) = \frac{\log^n x}{n!}$$

A. B. Goncharov, Mathematical Research Letters 5, 497 (1998).
A. B. Goncharov, arXiv preprint math/0103059 (2001).

MPLs include usual logs $G(a; b) = \log\left(1 - \frac{b}{a}\right), \qquad a \neq 0$

MPLs include "classical" polylogs $\mathrm{Li}_n(x) = -G\left(\vec{0}_{n-1}, \frac{1}{x}; 1\right) = \int_0^x \frac{dx'}{x'} \mathrm{Li}_{n-1}(x')$

Closed space under derivatives $dG(a_1, ..., a_n; x) = \sum_{i=1}^{n} G(a_1, ..., a_{i-1}, a_{i+1}, ..., a_n; x) d\log\left(\frac{a_{i-1} - a_i}{a_{i+1} - a_i}\right)$

Closed space under primitives $R(x)G(\vec{a}; x)$

MPLs form a shuffle algebra: $G(\vec{v}, x)G(\vec{u}, x) = \sum_{\vec{c} = \vec{v} \sqcup\!\sqcup \vec{u}} G(\vec{c}, x)$

MPLs form a Hopf algebra

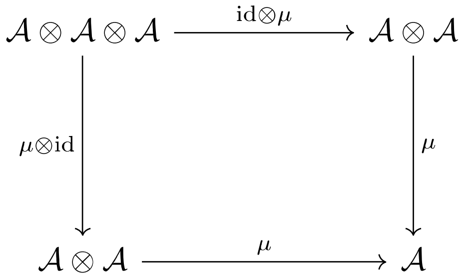
Algebras, coalgebras and Hopf algebras

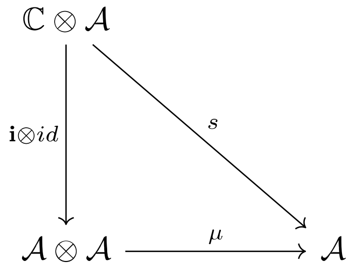
An **algebra** over a field $\mathbb{C}$ is a complex-vector space $\mathcal{A}$ with basis elements $\{e_i\}, e_i \in \mathcal{A}$

unit element

mapping $\mu : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$

mapping $\mathbf{i} : \mathbb{C} \longrightarrow \mathcal{A}$



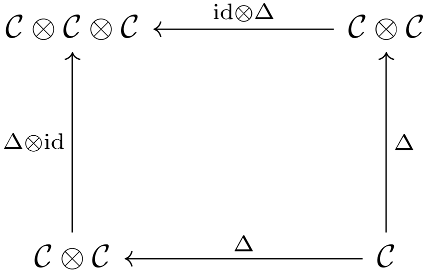


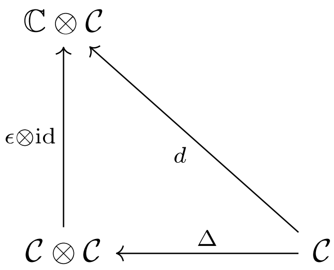
A **coalgebra** over a field $\mathbb{C}$ is a complex-vector space $\mathcal{C}$ with basis elements $\{e_i\}, e_i \in \mathcal{A}$

unit element

mapping $\Delta : \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$

mapping $\epsilon : \mathcal{C} \longrightarrow \mathbb{C}$





Hopf algebra $\underbrace{\mathcal{H} = \text{ mapping } \mu + \text{ mapping } \Delta}_{} + \underbrace{\text{antipode } S : \mathcal{H} \longrightarrow \mathcal{H}}_{}$

$$\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b),$$
$$(a \otimes b) \cdot (c \otimes d) = (a \cdot c) \otimes (b \cdot d).$$

$$S(a \cdot b) = S(a) \cdot S(b)$$
$$\mu(id \otimes S)\Delta = \mu(S \otimes id)\Delta = 0$$

Hopf algebra for MPLs

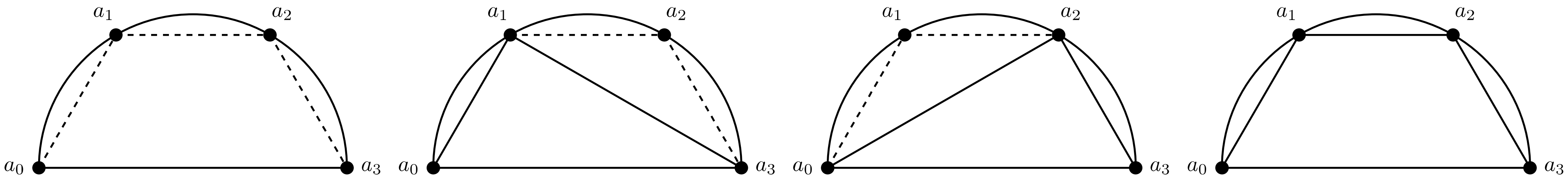
$$I(a_0; a_1, ..., a_n; a_{n+1}) = \int_{a_0}^{a_{n+1}} \frac{dx}{x - a_n} I(a_0; a_1, ..., a_{n-1}; x).$$

$$G(a_n, ..., a_1; a_{n+1}) = I(0; a_1, ..., a_n; a_{n+1}).$$

$$I(a_0; a_1; a_2) = G(a_1; a_2) - G(a_1; a_0).$$

$$\Delta I(a_0; a_1, ..., a_n; a_{n+1}) = \sum_{0=i_1<i_2<...<i_k<i_{k+1}=n} I(a_0; a_{i_1}, ..., a_{i_k}; a_{n+1}) \otimes \left(\prod_{p=0}^{k} I(a_{i_p}; a_{i_p+1}, ..., a_{i_{p+1}-1}; a_{i_{p+1}}) \right)$$

A. B. Goncharov et al., Duke Mathematical Journal 128, 209 (2005).



$1 \otimes I(a_0; a_1, a_2; a_3)$ | $I(a_0; a_1; a_3) \otimes I(a_1; a_2; a_3)$ | $I(a_0; a_2; a_3) \otimes I(a_0; a_1; a_2)$ | $I(a_0; a_1, a_2; a_3) \otimes 1$

$$\Delta I(a_0; a_1, a_2, a_3) = 1 \otimes I(a_0; a_1, a_2; a_3) + I(a_0; a_1; a_3) \otimes I(a_1; a_2; a_3) +$$
$$+ I(a_0; a_2; a_3) \otimes I(a_0; a_1; a_2) + I(a_0; a_1, a_2; a_3) \otimes 1$$

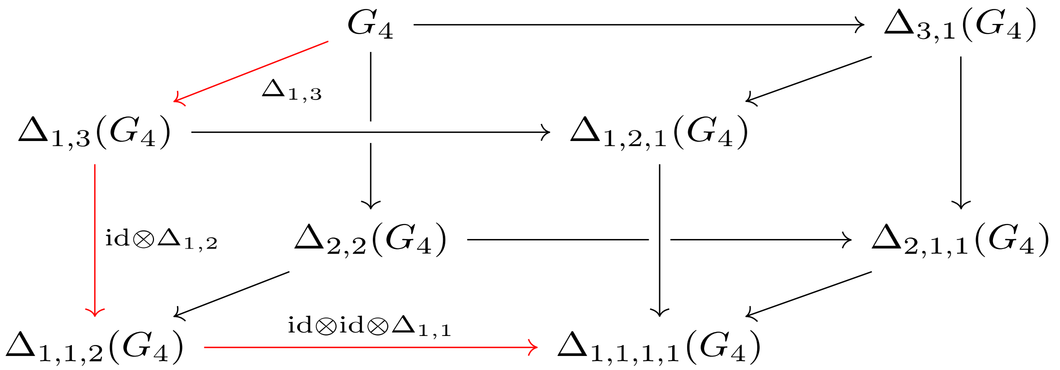
Iterated coproduct and symbol

$$\text{MPL Hopf algebra is graded}: \quad \mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

$$\Delta : \mathcal{H}_n \longrightarrow \bigoplus_{p+q=n} \mathcal{H}_p \otimes \mathcal{H}_q$$

$$\text{iterated coproduct } \Delta = \sum_{p+q=n} \Delta_{p,q}$$

$$(\mathrm{id} \otimes \Delta)\Delta : \mathcal{H}_n \longrightarrow \bigoplus_{p+q+w=n} \mathcal{H}_p \otimes \mathcal{H}_q \otimes \mathcal{H}_w$$



$$(\mathrm{id} \otimes \Delta)\Delta = \sum_{p+q+w=n} \Delta_{p,q,w}$$

maximum iteration of the coproduct $\Delta_{1,1,\ldots,1}$

$$\textbf{Symbol} \text{ definition: } \mathcal{S}(G) \equiv \Delta_{1,1,\ldots,1}(G) \mod i\pi$$

$$\log a \otimes \log b \otimes \ldots \otimes \log c \Longrightarrow a \otimes b \otimes \ldots \otimes c$$

Functional relations for MPLs

$$\lim_{x\to\infty} G(a_1,...,a_n;x) =? \qquad G(a,b,x), \qquad x \to \frac{1}{\epsilon},\ \epsilon \to 0$$

$$\mathcal{S}(G\left(a,b;\frac{1}{\epsilon}\right)) = -b\otimes(a\epsilon-1)-(a\epsilon-1)\otimes b+(a\epsilon-1)\otimes(b-a)-(b\epsilon-1)\otimes(b-a)+$$
$$+(b\epsilon-1)\otimes(a\epsilon-1)+a\otimes b-a\otimes(b-a)+b\otimes(b-a)-\epsilon\otimes(a\epsilon-1)+$$
$$+b\otimes\epsilon+\epsilon\otimes b-(b\epsilon-1)\otimes\epsilon+\epsilon\otimes\epsilon$$

Is there some other combination of MPLs with the same symbol?

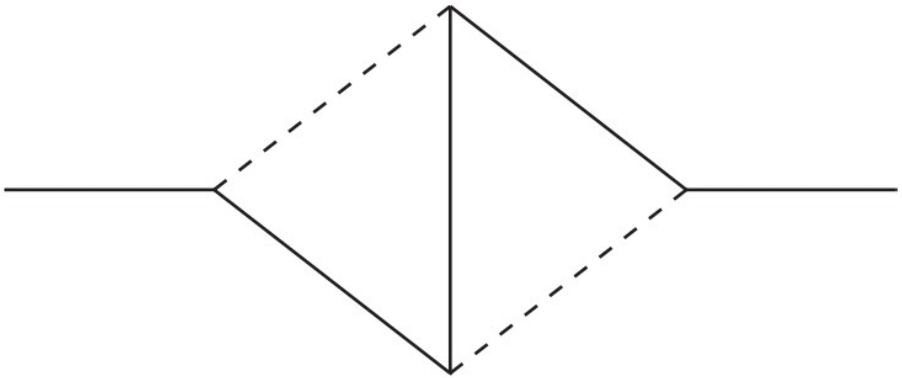
There exist a method that allows us to reduce MPLs in to the canonical form G(…,ϵ) using the symbol (assuming the symbol is linear on all variables)

C. Anastasiou, C. Duhr, F. Dulat and B. Mistlberger, Journal of High Energy Physics 2013, 3 (2013).

$$\lim_{\epsilon\to 0} G\left(a,b;\frac{1}{\epsilon}\right) = -\frac{\pi^2}{3} - i\pi(G(0;0)+G(0;b))+G(0;0)G(0;b)+$$
$$+(G(0;b)-G(0;a))G(b;a)+G(0,0;0)+G(0,0;b)+G(0,b;a)$$

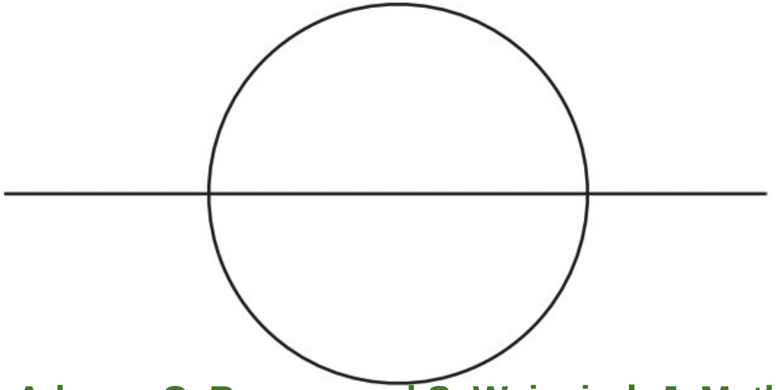
Motivation for elliptic extension of MPLs

Kite Integral



A. Sabry, Nuclear Physics 33, 401 (1962).

Sunset Integral



L. Adams, C. Bogner and S. Weinzierl, J. Math. Phys. 55, 102301 (2014)

S. Bloch and P. Vanhove, J. Number Theor. 148, 328 (2015)

O. Tarasov, Phys. Lett. B 638, 195 (2006)

$$\int R(x,y)\cdot \mathrm{MPL}(x,y)\cdot dx, \qquad y=\sqrt{P_4(x)}, \qquad P_4(x)\ -\ \text{4-th order polinomial}$$

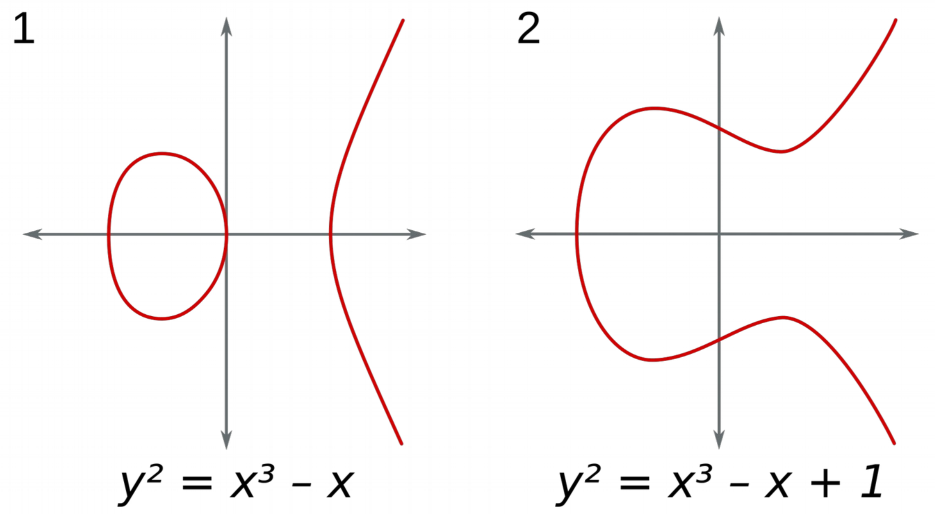
Elliptic curves

$$y^2 = (x-a_1)(x-a_2)(x-a_3)(x-a_4)$$

$$\omega_1 = 2c_4 \int_{a_2}^{a_3} \frac{dx}{y} = 2\mathrm{K}(\lambda), \quad \omega_2 = 2c_4 \int_{a_1}^{a_2} \frac{dx}{y} = 2i\mathrm{K}(1-\lambda),$$

$$\lambda = \frac{a_{14}a_{23}}{a_{13}a_{24}}, \; c_4 = \sqrt{a_{13}a_{24}}, \; a_{ij} = a_i - a_j$$

module of the elliptic curve $\tau = \dfrac{\omega_2}{\omega_1}, \; \mathrm{Im}(\tau) \neq 0$



modular group: $x \to \dfrac{ax-b}{cx-d}, \qquad y \to \dfrac{y}{(cx-d)^2}, \qquad ad-bc=1.$

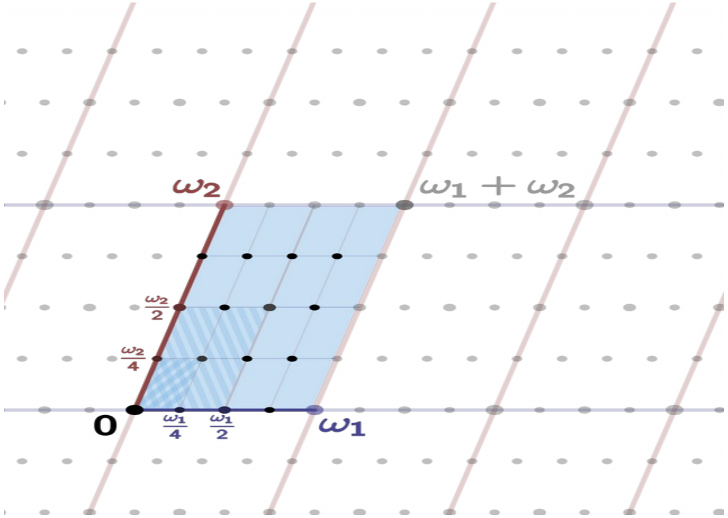
Weierstrass canonical form $y^2 = 4x^3 - g_2(\tau)x - g_3(\tau) \qquad j(\tau) = 1728\dfrac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2}.$

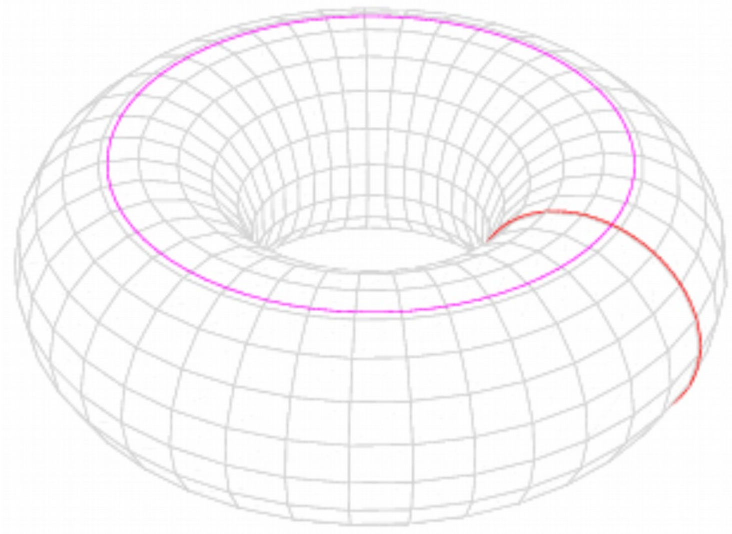
Elliptic functions

$$\text{Weierstrass } \wp \text{ function: } \wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z+m\omega_1+n\omega_2)^2} - \frac{1}{(m\omega_1+n\omega_2)^2} \right)$$

$$\wp(z + i\omega_1 + j\omega_2) = \wp(z), \qquad \wp'(z + i\omega_1 + j\omega_2) = \wp'(z), \qquad i, l \in \mathbb{Z}$$

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau)$$





eMPLs as iterated integrals on a torus

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1,\ldots,n_k \\ z_1,\ldots,z_k \end{smallmatrix}; z; \tau\right) = \int_0^z dz' g^{(n_1)}(z'-z_1,\tau)\tilde{\Gamma}\left(\begin{smallmatrix} n_2,\ldots,n_k \\ z_2,\ldots,z_k \end{smallmatrix}; z'; \tau\right), \quad k - \text{length and } \sum_i n_i - \text{the weight}$$

Broedel, J., Mafra, C.R., Matthes, N. et al. J. High Energ. Phys. 2015, 112 (2015).

$$\text{Eisenstein-Kronecker series: } F(z,\alpha,\tau) = \frac{1}{\alpha}\sum_{n\geq 0} g^{(n)}(z,\tau)\alpha^n = \frac{\theta_1'(0,\tau)\theta_1(z+\alpha,\tau)}{\theta_1(z,\tau)\theta_1(\alpha,\tau)}$$

θ_1 - odd Jacobi theta function

$$g^{(n)}(-z,\tau) = (-1)^n g^{(n)}(z,\tau),$$

$$g^{(n)}(z+1,\tau) = g^{(n)}(z,\tau), \qquad g^{(n)}(z+\tau,\tau) = \sum_{k=0}^{n} \frac{(-2\pi i)^k}{k!} g^{(n-k)}(z,\tau).$$

Pure eMPLs as iterated integrals

$$\mathcal{E}_4\left(\begin{smallmatrix} n1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x; \vec{a}\right) = \int_0^x dx' \Psi_{n_1}(c_1, x', \vec{a}) \mathcal{E}_4\left(\begin{smallmatrix} n2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; x'; \vec{a}\right), \quad k - \text{length and } \sum_i |n_i| - \text{the weight}$$

J. Broedel, C. Duhr, F. Dulat, B. Penante and L. Tancredi, Journal of High Energy Physics 2019, 23 (2019).

$$\Psi_0(0, x, \vec{a}) = \frac{c_4}{\omega_1 y}$$

$$\Psi_1(c, x, \vec{a}) = \frac{1}{x - c}, \qquad \Psi_{-1}(c, x, \vec{a}) = \frac{y(c)}{y(x - c)} + Z_4(c, \vec{a}), \qquad c \neq \infty,$$

$$\Psi_1(\infty, x, \vec{a}) = -Z_4(x, \vec{a}) \frac{c_4}{y}, \qquad \Psi_{-1}(\infty, x, \vec{a}) = \frac{x}{y} - \frac{a_1 + 2c_4 G_*(\vec{a})}{y}.$$

$$Z_4(x, \vec{a}) = -\frac{1}{\omega_1}\left(g^{(1)}(z_x - z_*, \tau) + g^{(1)}(z_x + z_*, \tau)\right), \qquad G_*(\vec{a}) = \frac{g^{(1)}(z_*, \tau)}{\omega_1}$$

Properties of pure eMPLs

Purity, function is called pure if it is unipotent and its total differential involves only pure functions and one-forms with at most logarithmic singularities

Ordinary MPLs are a subset of eMPLs $\mathcal{E}_4\left(\begin{smallmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_n \end{smallmatrix}; x; \vec{a}\right) = G(a_1, ..., a_n; x)$

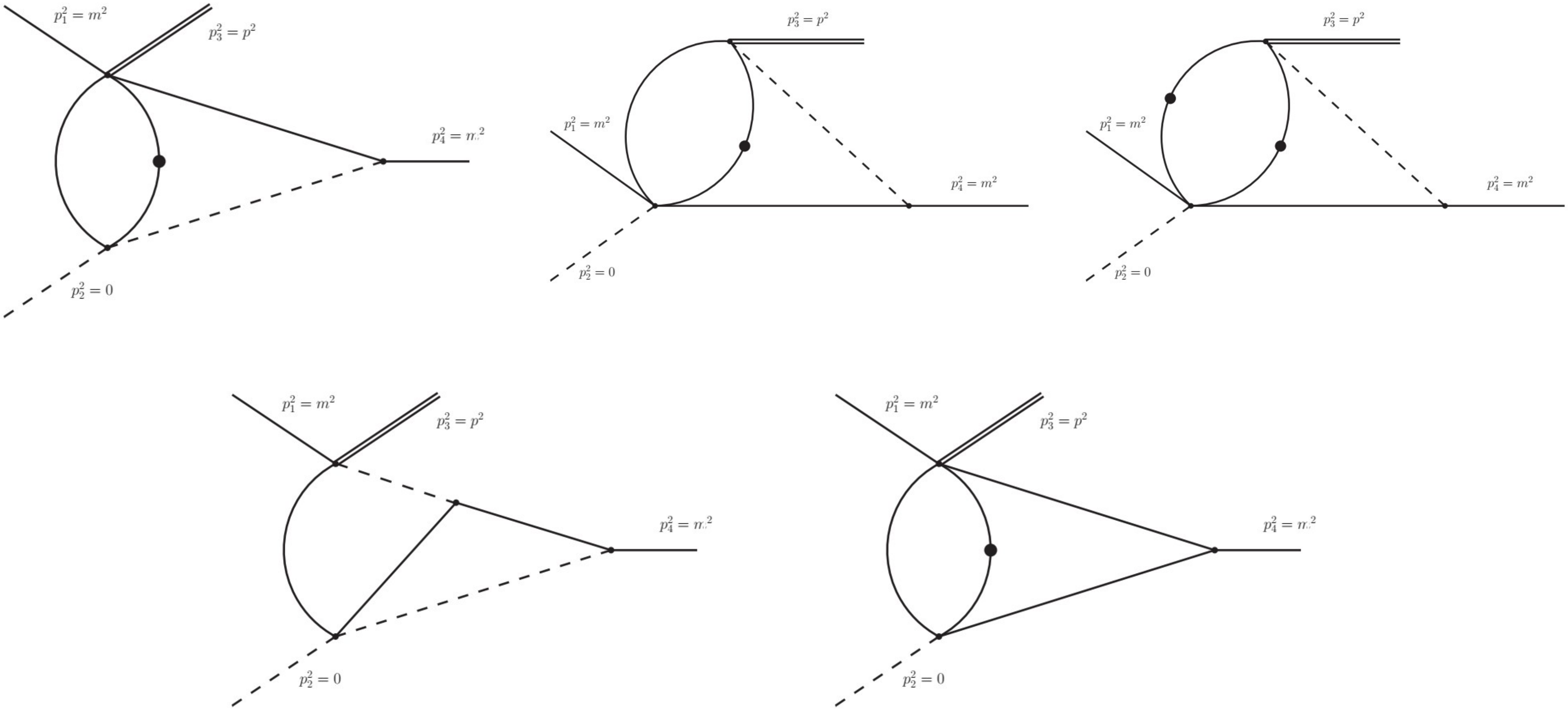
Rescaling of arguments $\mathcal{E}_4\left(\begin{smallmatrix} n_1, \ldots, n_k \\ pc_1, \ldots, pc_k \end{smallmatrix}; px; p\vec{a}\right) = \mathcal{E}_4\left(\begin{smallmatrix} n_1, \ldots, n_k \\ c_1, \ldots, c_k \end{smallmatrix}; x; \vec{a}\right)$

eMPLs form a shuffle algebra $\mathcal{E}_4(\vec{V}; x; a)\mathcal{E}_4(\vec{U}; x; a) = \sum_{\vec{C} = \vec{V} \sqcup\!\sqcup \vec{U}} \mathcal{E}_4(\vec{C}; x; a).$

eMPLs form a Hopf algebra

J. Broedel, C. Duhr, F. Dulat, B. Penante and L. Tancredi, Journal of High Energy Physics 2019, 23 (2019).

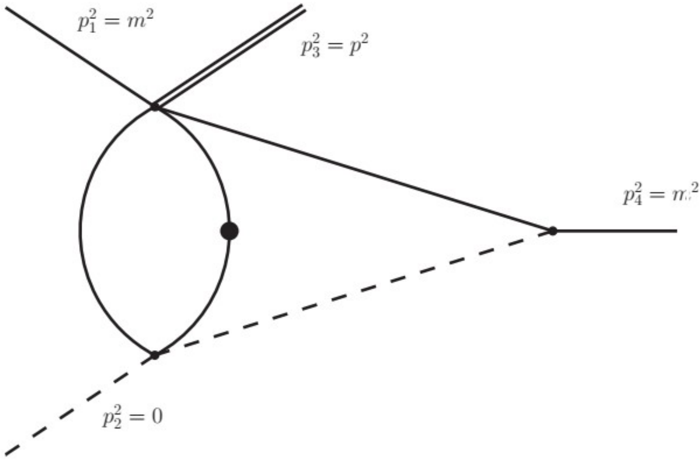
Practical examples



M. Bezuglov, arXiv:2003.05367 [hep-th]
accepted in Int. J. Mod. Phys. A

Triangle with one massless line and massive loop

$$I_1 = \frac{e^{2\gamma_E\varepsilon}(\mu^{2\varepsilon})}{(i\pi^{d/2})^2}\int \frac{d^dk_1 d^dk_2}{(k_1^2-m^2)^2((k_1-k_2)^2-m^2)((p_1-p_3+k_2)^2-m^2)(p_2-k_2)^2}$$



$$I_1 = e^{2\gamma_E\varepsilon}(\mu^{2\varepsilon})\Gamma(1+2\varepsilon)\int\limits_{\Delta}\left(\prod_{i=1}^{4} dx_i\right) x_1 \frac{U^{5-\frac{3d}{2}}}{F^{5-d}} =$$

$$= \left(-\frac{\mu^2}{m^2}\right)^{2\varepsilon}\left[I_1^{(0)} + \varepsilon I_1^{(1)} + \mathcal{O}(\varepsilon^2)\right]$$

U and F - Symanzik polynomials

$$U = x_2x_3 + x_2x_4 + x_1x_2 + x_1x_3 + x_1x_4$$

$$F = -tx_1x_2x_3 - m^2\left((x_1+x_2+x_3)(x_2x_3+x_1x_2+x_1x_3) + x_4(x_1+x_2)^2\right) \qquad t = -(p_1-p_3)^2 > 0$$

$$I_1^{(0)} = \int\limits_0^\infty \frac{x_1 dx_1 dx_2 dx_3 dx_4}{UF}\delta(1-x_1) = \int\limits_0^\infty \frac{dx_2 dx_3 dx_4}{UF|_{x_1=1}}.$$

furst we integrate by x_4

$$I_1^{(0)} = \int\limits_0^\infty dx_3 dx_2 \frac{G\left(0; -1+\frac{1}{1+x_2} - x_3\right) - G\left(0; -\frac{tx_2x_3 + m^2(1+x_2+x_3)(x_2+x_3+x_2x_3)}{m^2(1+x_2)^2}\right)}{(1+x_2)x_3(tx_2 + m^2(x_2+x_3+x_2x_3))}.$$

$$\text{Möbius transformation } x_2 = \frac{x}{1-x}, \quad J = \frac{1}{(1-x)^2}$$

$$I_1^{(0)} = \int_0^1 \frac{dx}{(m^2+t)x}\left(-G(0;-x)G\left(x;\frac{(m^2+t)x}{m^2}\right) + G\left(\Upsilon(y);-\frac{(m^2+t)x}{m^2}\right)G(0;\Upsilon(y)) + G(0;\Upsilon(-y))G\left(\Upsilon(-y);-\frac{(m^2+t)x}{m^2}\right) + \right.$$

$$+G\left(0;-\frac{(m^2+t)x}{m^2}\right)\left(G\left(x;\frac{(m^2+t)x}{m^2}\right) - G\left(\Upsilon(y);-\frac{(m^2+t)x}{m^2}\right) - G\left(\Upsilon(-y);-\frac{(m^2+t)x}{m^2}\right)\right) - G\left(0,x,\frac{(m^2+t)x}{m^2}\right) +$$

$$\left. +G\left(0,\Upsilon(y),-\frac{(m^2+t)x}{m^2}\right) + G\left(0,\Upsilon(-y),-\frac{(m^2+t)x}{m^2}\right)\right).$$

$$\Upsilon(y) = \frac{m^2(1+x-x^2) + tx(1-x) + (m^2+t)y}{2m^2(x-1)}, \qquad y = \sqrt{(x-a_1)(x-a_2)(x-a_3)(x-a_4)}.$$

$$a_1 = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4m^2}{(m+i\sqrt{t})^2}}\right), \quad a_2 = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4m^2}{(m-i\sqrt{t})^2}}\right),$$

$$a_3 = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4m^2}{(m+i\sqrt{t})^2}}\right), \quad a_4 = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4m^2}{(m-i\sqrt{t})^2}}\right).$$

$$\int R(x,y) \cdot \mathrm{MPL}(x,y) \cdot dx$$

MPLs as eMPLs and the final answer

- take the full derivative of the selected MPL of weight n with respect to the variable x, after which we get a linear combination of MPLs of weight n-1(due to the recursion all MPLs of weight n-1 are already rewritten as linear combinations of eMPLs).
- integrate the obtained combination of eMPLs using the definition.
- fix the integration constant

$$\frac{dG(0;\Upsilon(y))}{dx} = -\frac{1}{2(x-1)x} + \frac{1}{2y} - \frac{x}{y} + \frac{m^2(1-2x)}{2(m^2+t)y(x-1)x}, \qquad \Upsilon(y) = \frac{m^2(1+x-x^2)+tx(1-x)+(m^2+t)y}{2m^2(x-1)},$$

$$G(0;\Upsilon(y)) = i\pi - \frac{1}{2}\mathcal{E}_4\left(\begin{smallmatrix}-1\\0\end{smallmatrix};x;\vec{a}\right) - \frac{1}{2}\mathcal{E}_4\left(\begin{smallmatrix}-1\\1\end{smallmatrix};x;\vec{a}\right) - \mathcal{E}_4\left(\begin{smallmatrix}-1\\\infty\end{smallmatrix};x;\vec{a}\right) + \frac{1}{2}\mathcal{E}_4\left(\begin{smallmatrix}1\\0\end{smallmatrix};x;\vec{a}\right) -$$
$$- \frac{1}{2}\mathcal{E}_4\left(\begin{smallmatrix}1\\1\end{smallmatrix};x;\vec{a}\right) + \frac{\omega_1}{2}\mathcal{E}_4\left(\begin{smallmatrix}0\\0\end{smallmatrix};x;\vec{a}\right)\underbrace{\left\{\frac{1-2a_1}{c_4} - 4G_*(\vec{a}) + Z_4(0,\vec{a}) + Z(1,\vec{a})\right\}}_{=0}.$$

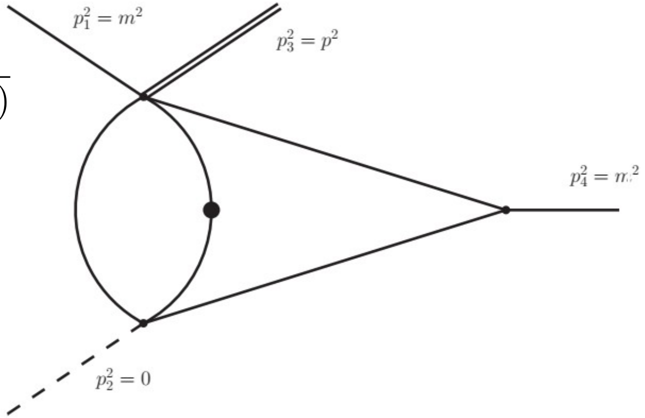
integration constant

The final answer for the whole diagram :

$$I_1^{(0)} = -\frac{2}{(m^2+t)}\left(\mathcal{E}_4\left(\begin{smallmatrix}1 & -1 & -1\\0 & \infty & 0\end{smallmatrix};x;\vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix}1 & -1 & -1\\0 & \infty & 1\end{smallmatrix};x;\vec{a}\right) + 2\mathcal{E}_4\left(\begin{smallmatrix}1 & -1 & -1\\0 & \infty & \infty\end{smallmatrix};x;\vec{a}\right) + \right.$$
$$\left. + i\pi\left(\mathcal{E}_4\left(\begin{smallmatrix}1 & 0 & -1\\0 & 0 & 0\end{smallmatrix};x;\vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix}1 & 0 & -1\\0 & 0 & 1\end{smallmatrix};x;\vec{a}\right) + 2\mathcal{E}_4\left(\begin{smallmatrix}1 & 0 & -1\\0 & 0 & \infty\end{smallmatrix};x;\vec{a}\right)\right)\right).$$

Triangle with all massive lines and massive loop

$$I_m = \frac{e^{2\gamma_E \varepsilon}(\mu^{2\varepsilon})}{(i\pi^{d/2})^2} \int \frac{d^d k_1 d^d k_2}{(k_1^2 - m^2)^2((k_1-k_2)^2 - m^2)((p_1 - p_3 + k_2)^2 - m^2)((p_2 - k_2)^2 - m^2)}$$



$$I_m^{(0)} = \int_0^\infty \frac{x_1 dx_1 dx_2 dx_3 dx_4}{UF} \delta(1 - x_1),$$

$$U = x_2 x_3 + x_2 x_4 + x_1 x_2 + x_1 x_3 + x_1 x_4,$$

$$F = -t x_1 x_2 x_3 + m^2 (x_1 + x_2) x_3 x_4 - m^2 \left(x_1 + x_2 + x_3 + x_4\right) \left(x_2(x_3 + x_4) + x_1(x_2 + x_3 + x_4)\right)$$

furst we integrate by x_4

$$I_m^{(0)} = \int_0^\infty \int_0^\infty \frac{dx_2 dx_3}{y} \left(\frac{-G\left(-1; x_2\right) - G\left(0; q(-y)\right) + G\left(0; -x_2\right) + G\left(-1 + \frac{1}{1+x_2}; x_3\right)}{q(-y)(1 + x_2) + x_2 + x_3 + x_2 x_3} - \right.$$
$$\left. - \frac{-G\left(-1; x_2\right) - G\left(0; q(y)\right) + G\left(0; -x_2\right) + G\left(-1 + \frac{1}{1+x_2}; x_3\right)}{q(y)(1 + x_2) + x_2 + x_3 + x_2 x_3} \right)$$

$$y = \sqrt{-4t x_2 (1 + x_2) x_3 + + m^2 \left((1 + x_2 + x_2^2)^2 - 2(1 + x_2)(1 + 3x_2 + x_2^2) x_3 - 3(1 + x_2)^2 x_3^2\right)}$$

Rationalizing roots

$$F(x_2,x_3,y) = -4tx_2(1+x_2)x_3 + m^2\left((1+x_2+x_2^2)^2 - 2(1+x_2)(1+3x_2+x_2^2)x_3 - 3(1+x_2)^2x_3^2\right) - y^2 = 0$$

$$F(x_2,x_3,y) = a(x_2)x_3^2 + b(x_2)x_3 + c(x_2)^2 - y^2 = 0$$

where a, b and c are polynomials in the variable x_2

Rational parametrization of surface$F(x_2,x_3,y)$

$$x_3 \to \frac{z(2c(x_2)-b(x_2)z)}{-1+a(x_2)z^2}, \qquad y \to c(x_2) + \frac{2c(x_2)-b(x_2)z}{-1+a(x_2)z^2}, \qquad x_2 \to x_2$$

$$x_3(z) \to -\frac{2z\left(m(1+x_2+x_2^2)+2tx_2(1+x_2)z+m^2(1+x_2)(1+3x_2+x_2^2)z\right)}{1+3m^2(1+x_2)^2z^2},$$

$$y \to \frac{-m(1+x_2+x_2^2)-2(1+x_2)(2tx_2+m^2(1+x_2(3+x_2)))z+3m^3(1+x_2)^2(1+x_2+x_2^2)z^2}{1+3m^2(1+x_2)^2z^2}$$

$$I_m^{(0)} = \int_0^1 \frac{dx}{(m^2+t)x} \Big\{ G(0, \Theta(-y_2); \frac{i}{\sqrt{3}}) + G(0, \Theta(y_2); \frac{i}{\sqrt{3}}) - G(0, \bar{\Omega}(-y_1); \frac{i}{\sqrt{3}}) - G(0, \bar{\Omega}(y_1); \frac{i}{\sqrt{3}}) + ...$$

Θ and $\bar{\Omega}$ are rational functions from variables x and y

The first elliptical structure, the same as in the previous example $y_1 = \sqrt{(x-a_1)(x-a_2)(x-a_3)(x-a_4)}$

The second elliptical structure, different from the first one $y_2 = \sqrt{(x-b_1)(x-b_2)(x-b_3)(x-b_4)}$

$$b_1 = \frac{1}{2}(1 + \sqrt{1 - \frac{2m}{\sqrt{-t}}}), \qquad b_2 = \frac{1}{2}(1 - \sqrt{1 + \frac{2m}{\sqrt{-t}}}), \qquad b_3 = \frac{1}{2}(1 - \sqrt{1 - \frac{2m}{\sqrt{-t}}}), \qquad b_4 = \frac{1}{2}(1 + \sqrt{1 + \frac{2m}{\sqrt{-t}}}).$$

$$j_1 = \frac{(3m^2+t)^3(3m^6+3m^4t+9m^2t^2+t^3)^3}{m^{12}t^2(m^2+t)^3(9m^2+t)}, \qquad j_2 = \frac{256(3m^2+t)^3}{m^4(4m^2+t)}.$$

two elliptic structures y_1 and y_2 do not "mix" with each other

The final answer

$$I_m^{(0)} = \frac{1}{2(t+m^2)}\left[I_+ + I_- + I_+^* + I_-^* - 4\pi i I_0 + 2I_G\right]$$

were $I_\pm$, $I_\pm^*$ and I_0 are elliptic parts and I_G is the MPL part

$$I_\pm = \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & \alpha_\pm & 0 \end{smallmatrix}; 1; \vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & \alpha_\pm & 1 \end{smallmatrix}; 1; \vec{a}\right) + 2\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & \alpha_\pm & \infty \end{smallmatrix}; 1; \vec{a}\right),$$

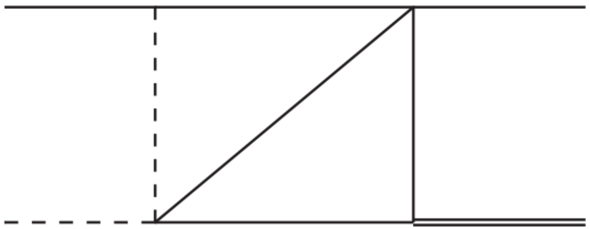
$$I_\pm^* = \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & \alpha_\pm^* & 0 \end{smallmatrix}; 1; \vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & \alpha_\pm^* & 1 \end{smallmatrix}; 1; \vec{a}\right) + 2\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & \alpha_\pm^* & \infty \end{smallmatrix}; 1; \vec{a}\right),$$

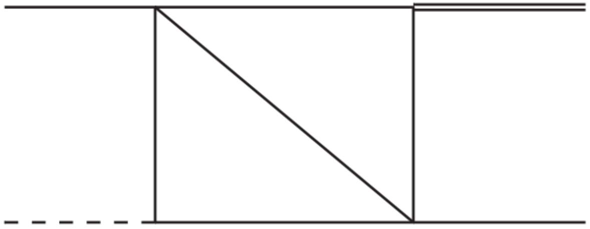
$$I_0 = \mathcal{E}_4\left(\begin{smallmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{smallmatrix}; 1; \vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \end{smallmatrix}; 1; \vec{a}\right) + 2\mathcal{E}_4\left(\begin{smallmatrix} 1 & 0 & -1 \\ 0 & 0 & \infty \end{smallmatrix}; 1; \vec{a}\right)$$

$$I_G = \frac{1}{3}i\pi G\left(0, \alpha_-, 1\right) + \frac{1}{3}i\pi G\left(0, \alpha_+, 1\right) - \frac{1}{3}i\pi G\left(0, \alpha_-^*, 1\right) - \frac{1}{3}i\pi G\left(0, \alpha_+^*, 1\right) + \ldots$$

$$\alpha_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{-m^4 + 3m^2 t + 2i\sqrt{3}m^2\left(m^2+t\right) + t^2}{m^4 + m^2 t + t^2}}, \qquad \alpha_\pm^* = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{-m^4 + 3m^2 t - 2i\sqrt{3}m^2\left(m^2+t\right) + t^2}{m^4 + m^2 t + t^2}}.$$

Problems and future plans





1) we need to develop new methods for rationalizing the roots of complicated polynomials.

$$F(x_1,x_2,y)=\sum_{i+j\leq 4} C_{ij}x_1^i x_2^j - y^2 = 0 \xrightarrow{\text{projective plane}} \tilde{F}(x_1,x_2,y,w)=\sum_{i+j+k=4}\tilde{C}_{ijk}x_1^i x_2^j w^k - y^2w^2 = 0$$

Rational parametrization of surfaces in $\mathbf{P}^3$, in future hypersurfaces in $\mathbf{P}^n, n \geq 4$.

2) we need a new class of functions which will be iterated integrals with hyperelliptic kernels.

$$\frac{dG\left(\Theta(y_2);\Omega(y_1)\right)}{dx} = R(x,y_1,y_2) = R_1(x) + \frac{1}{y_1}R_2(x) + \frac{1}{y_2}R_3(x) + \frac{1}{y_1y_2}R_4(x),$$

Thank you for your attention!