JINR, Dubna, July 2018

A geometrical approach to the evaluation of Feynman diagrams

CALC-2018 lectures

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Brief overview

- **LECTURE #1** — July 24, 2018, 14:30 (end of lunch) — 15:30 (coffee break)
- **LECTURE #2** — July 25, 2018, 14:30 (end of lunch) — 15:30 (coffee break)
- **LECTURE #3** — July 27, 2018, 14:30 (end of lunch) — 15:30 (coffee break)

LECTURE #1

Earlier papers: singularities, reduction, etc.*

- L.D. Landau, Nucl. Phys. **13** (1959) 181
- G. Källén and A. Wightman, Mat. Fys. Skr. Dan. Vid. Selsk. **1**, No.6 (1958) 1
- S. Mandelstam, Phys. Rev. **115** (1959) 1742
- R.E. Cutkosky, J. Math. Phys. **1** (1960) 429
- J.C. Taylor, Phys. Rev. **117** (1960) 261
- Yu.A. Simonov, Sov. Phys. JETP **16** (1963) 1599
- R. Karplus, C.M. Sommerfield, E.H. Wichmann, Phys. Rev. **114** (1959) 376
- A.C.T. Wu, Mat. Fys. Medd. Dan. Vid. Selsk. **33**, No.3 (1961) 1
- L.M. Brown, Nuovo Cim. **22** (1961) 178
- F.R. Halpern, Phys. Rev. Lett. **10** (1963) 310
- B. Petersson, J. Math. Phys. **6** (1965) 1955
- D.B. Melrose, Nuovo Cim. **40A** (1965) 181
- G. Källén and J. Toll, J. Math. Phys. **6** (1965) 299 (1965)
- B.G. Nickel, J. Math. Phys. **19** (1978) 542
- A. Denner, U. Nierste, R. Scharf, Nucl. Phys. **B367** (1991) 637
- N. Ortner and P. Wagner, Ann. Inst. Henri Poincaré (Phys. Théor.) **63** (1995) 81
- P. Wagner, Indag. Math. **7** (1996) 527
- . . .

* the list is incomplete

Dimensional regularization

One of the most powerful tools used in loop calculations is *dimensional regularization*: the idea is to use the space-time dimension n as a regulator, $n = 4 - 2\varepsilon$ $(\varepsilon \to 0)$,

$$\int \mathrm{d}^4 k \left\{ \ldots \right\} \quad \Rightarrow \quad \int \mathrm{d}^n k \left\{ \ldots \right\}$$

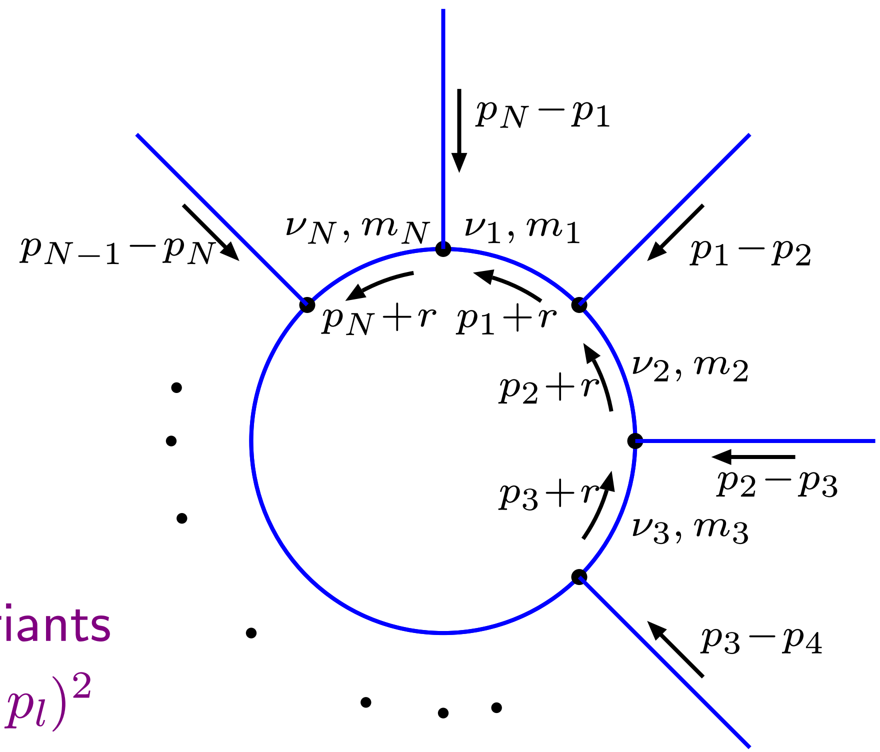
G. 'tHooft and M. Veltman, Nucl. Phys. **B44** (1972) 189;
C.G. Bollini and J.J. Giambiagi, Nuovo Cimento **12B** (1972) 20;
J.F. Ashmore, Lett. Nuovo Cim. **4** (1972) 289;
G.M. Cicuta and E. Montaldi, Lett. Nuovo Cim. **4** (1972) 329.

Then singularities appear as $1/\varepsilon$ poles. Simple example:

$$\int \frac{\mathrm{d}^n k}{[(p+k)^2 - m^2]^\nu} = \int \frac{\mathrm{d}^n k}{[k^2 - m^2]^\nu} = \pi^{n/2} \mathrm{i}^{1-2\nu} \, \frac{\Gamma(\nu - n/2)}{\Gamma(\nu)} (m^2)^{n/2-\nu}$$

For $\nu = 2$: $\Gamma(2 - n/2) = \Gamma(\varepsilon) \sim 1/\varepsilon$

One-loop N-point function $J^{(N)}(n;\nu_1,\ldots,\nu_N)$



Depends on
$\frac{1}{2}N(N-1)$ invariants
$k_{jl}^2 = (p_j - p_l)^2$
and N masses m_i

$$J^{(N)}(n;\nu_1,\ldots,\nu_N) \equiv \int \frac{\mathrm{d}^n k}{\left[(p_1+k)^2 - m_1^2\right]^{\nu_1} \cdots \left[(p_N+k)^2 - m_N^2\right]^{\nu_N}}$$

dimension powers of propagators

Feynman parameters α_i

For simplicity, let us consider unit powers of propagators, $\nu_i = 1$:

$$\frac{1}{\mathcal{A}_1 \ldots \mathcal{A}_N} = (N-1)! \int\limits_0^1 \ldots \int\limits_0^1 \frac{\mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_N \; \delta\left(\sum\limits_{i=1}^{N} \alpha_i - 1\right)}{(\alpha_1 \mathcal{A}_1 + \ldots + \alpha_N \mathcal{A}_N)^N}$$

where $\mathcal{A}_i \leftrightarrow (p_i + k)^2 - m_i^2$.

Then the momentum integrations can be easily performed, and we arrive at

$$\begin{aligned} J^{(N)}(n; 1, \ldots, 1) \quad &= \quad \mathrm{i}^{1-n} \pi^{n/2}\, \Gamma(N - n/2) \\ &\times \int\limits_0^1 \ldots \int\limits_0^1 \frac{\left(\prod \mathrm{d}\alpha_i\right) \cdot \delta\left(\sum \alpha_i - 1\right)}{\left[\sum\sum\limits_{j<l} \alpha_j \alpha_l k_{jl}^2 - \sum \alpha_i m_i^2\right]^{N-n/2}}, \end{aligned}$$

with $k_{jl}^2 = (p_j - p_l)^2$.

Geometrical approach

- The idea is to use geometrical description not only when analyzing the singularities (thresholds, etc.), but also when *calculating* dimensionally-regulated Feynman integrals.

  A.I.D. and R. Delbourgo, J. Math. Phys. **39** (1998) 4299.

- In particular, it may be used to predict types of functions (and their arguments) appearing in higher orders of ε-expansion. Examples include *all* terms of the ε-expansion for

  – one-loop 2-point function with arbitrary masses,
  – one-loop 3-point integrals with massless internal lines and arbitary (off-shell) external momenta, and
  – two-loop vacuum diagrams with arbitrary masses

  A.I.D., Phys. Rev. **D61** (2000) 087701;

  A.I.D. and M.Yu. Kalmykov, Nucl. Phys. B (PS) **89** (2000) 283; Nucl. Phys. **B605** (2001) 266

- Geometrical description of 3- and 4-point functions with arbitrary momenta and masses:

  A.I.D., AIHENP-99 Proceedings (hep-th/9908032); Nucl. Instr. Meth. **A559** (2006) 293 (3-point function);

  A.I.D., J. Phys. (CS) 762 (2016) 12068 (arXiv:1605.04828); ACAT-2017 (arXiv:1711.07351) (4-point function)

Feynman parameters

Parametric representation for the one-loop N-point function:

$$J^{(N)}(n;1,\ldots,1) = \mathrm{i}^{1-n}\pi^{n/2}\,\Gamma\left(N-\frac{n}{2}\right)\int\limits_0^1\ldots\int\limits_0^1 \frac{\left(\prod \mathrm{d}\alpha_i\right)\cdot\delta\left(\sum\alpha_i-1\right)}{\left[\sum\sum\limits_{j<l}\alpha_j\alpha_l k_{jl}^2-\sum\alpha_i m_i^2\right]^{N-n/2}}$$

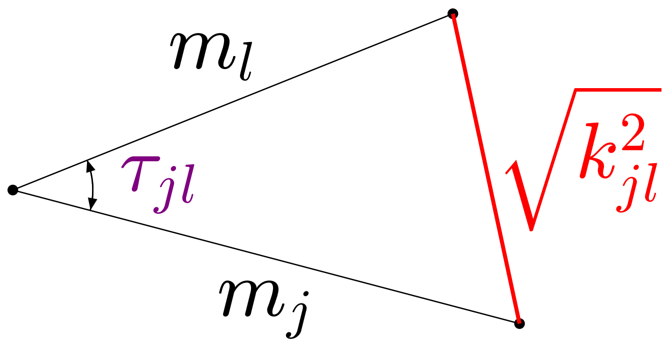
By using $\sum\alpha_i=1$ we can make the quadratic form homogeneous in α_i:

$$\left[\sum\sum_{j<l}\alpha_j\alpha_l k_{jl}^2-\left(\sum\alpha_i\right)\left(\sum\alpha_i m_i^2\right)\right] \Rightarrow -\left[\sum\alpha_i^2 m_i^2+2\sum\sum_{j<l}\alpha_j\alpha_l m_j m_l c_{jl}\right],$$

with

$$c_{jl}\equiv\frac{m_j^2+m_l^2-k_{jl}^2}{2m_j m_l}$$

Geometrical interpretation: c_{jl} as cosines



Cosine theorem:

$$k_{jl}^2 = m_j^2 + m_l^2 - 2 m_j m_l \cos\tau_{jl} \ .$$

Therefore,

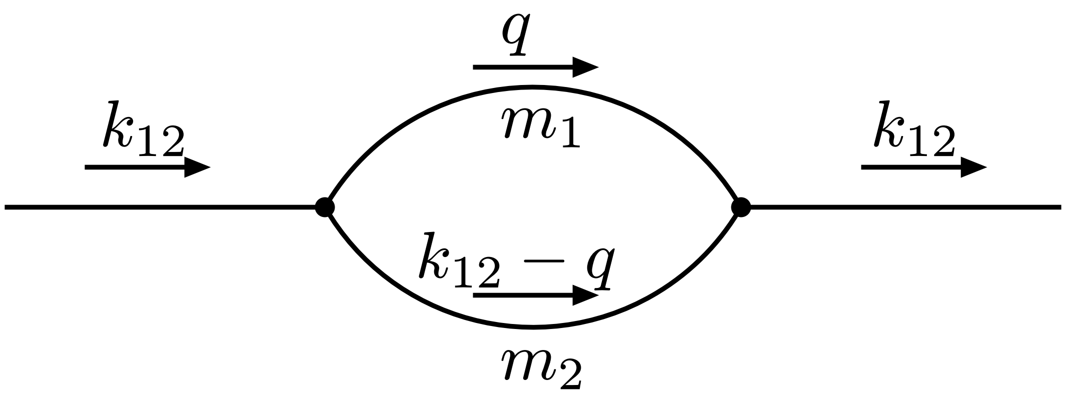
$$\cos\tau_{jl} = \frac{m_j^2 + m_l^2 - k_{jl}^2}{2 m_j m_l} = c_{jl} \ .$$

Special cases (when the triangle area vanishes):

$$c_{jl} = \cos\tau_{jl} = \begin{cases} 1, & k_{jl}^2 = (m_j - m_l)^2 \quad \text{pseudothreshold} \quad (\tau_{jl} = 0) \\ -1, & k_{jl}^2 = (m_j + m_l)^2 \quad \text{threshold} \quad (\tau_{jl} = \pi) \end{cases}$$

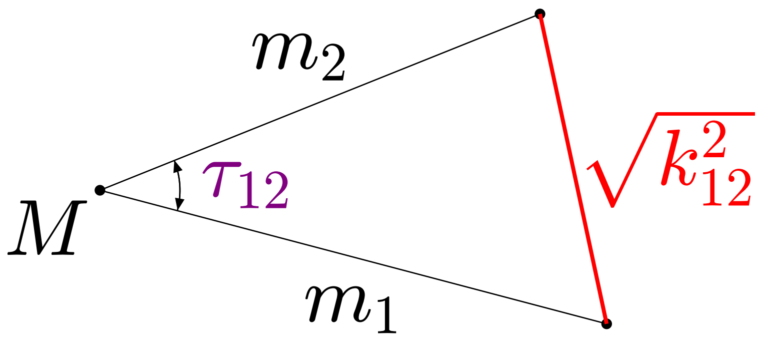
Direct geometrical interpretation: when $-1 \le c_{jl} \le 1$ (i.e., angles τ_{jl} are real)

Two-point function with arbitrary masses



$$J^{(2)}\left(n;\ 1,1\right|\, m_1, m_2\Big) \equiv \int \frac{\mathrm{d}^n r}{(q^2 - m_1^2)\big[(k_{12}-q)^2 - m_2^2\big]} \; .$$

Two-point function: the basic triangle



$$\cos\tau_{12} = c_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1m_2}$$

Gram determinant:

$$D^{(2)} = \det\|c_{jl}\| = \begin{vmatrix} 1 & c_{12} \\ c_{12} & 1 \end{vmatrix} = 1 - c_{12}^2 = \sin^2\tau_{12}$$

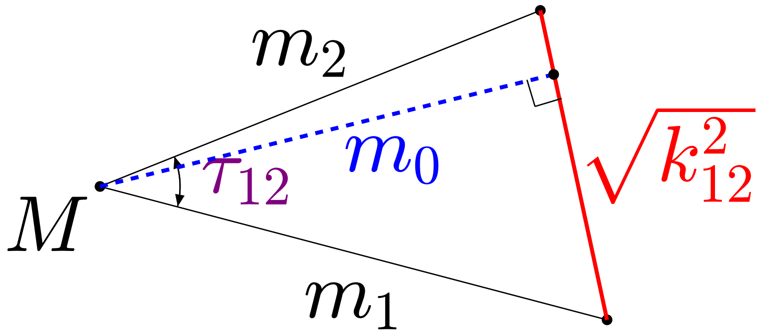
Triangle area:

$$V^{(2)} = \frac{1}{2}\,m_1m_2\sqrt{D^{(2)}} = \frac{1}{2}\,m_1m_2\sin\tau_{12}$$

Special cases (when the area vanishes):

$$c_{12} = \cos\tau_{12} = \begin{cases} 1, & k_{12}^2 = (m_1-m_2)^2 \quad \text{pseudothreshold} \quad (\tau_{12}=0) \\ -1, & k_{12}^2 = (m_1+m_2)^2 \quad \text{threshold} \quad (\tau_{12}=\pi) \end{cases}$$

Two-point function: the basic triangle



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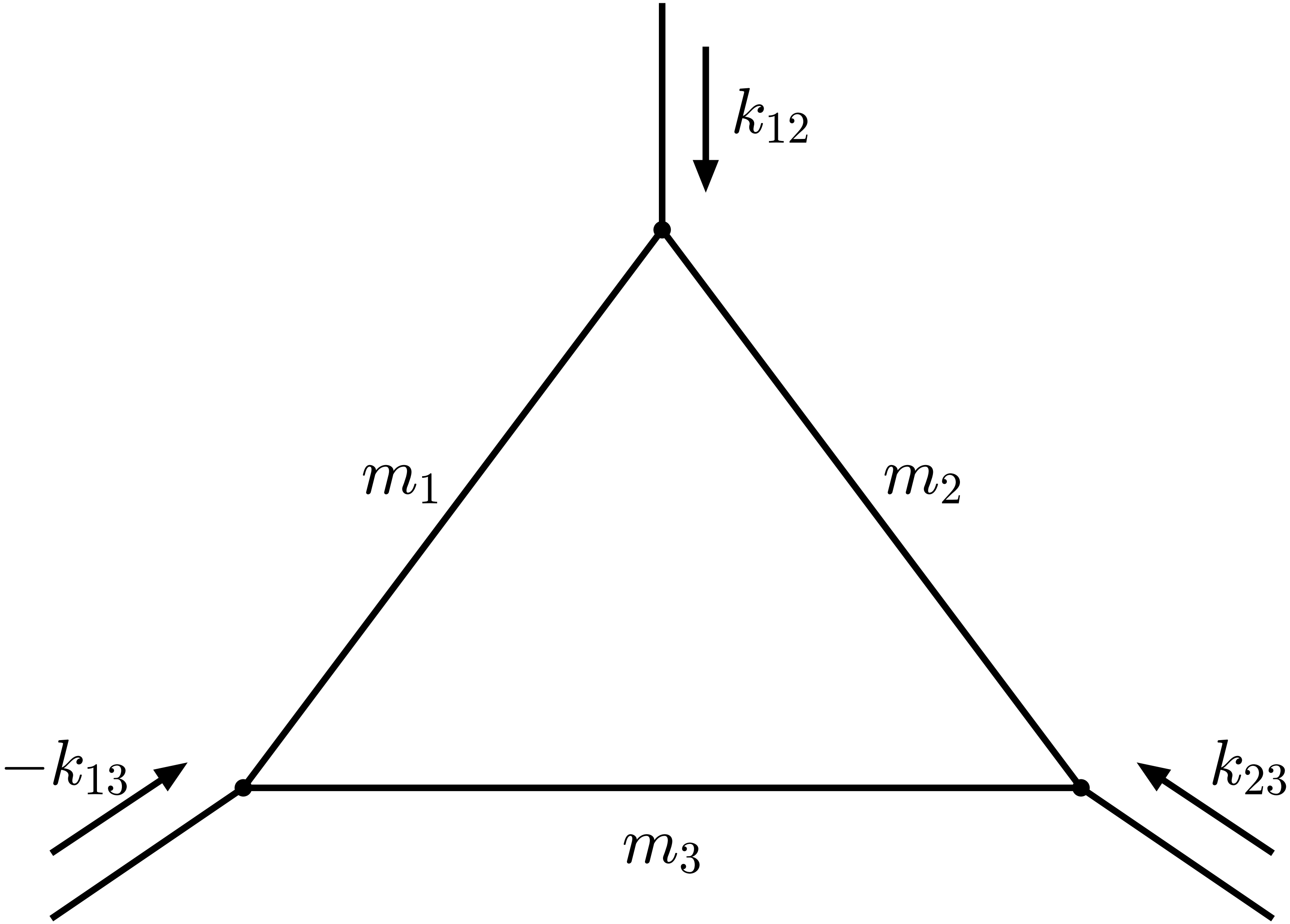
Triangle area:

$$V^{(2)} = \frac{1}{2}\, m_1 m_2 \sqrt{D^{(2)}} = \frac{1}{2}\, m_1 m_2 \sin\tau_{12} = \frac{1}{2}\, m_0 \sqrt{k_{12}^2} = \frac{1}{2}\, m_0 \overline{V}_0^{(1)} = \frac{1}{2}\, m_0 \sqrt{\Lambda^{(2)}}$$

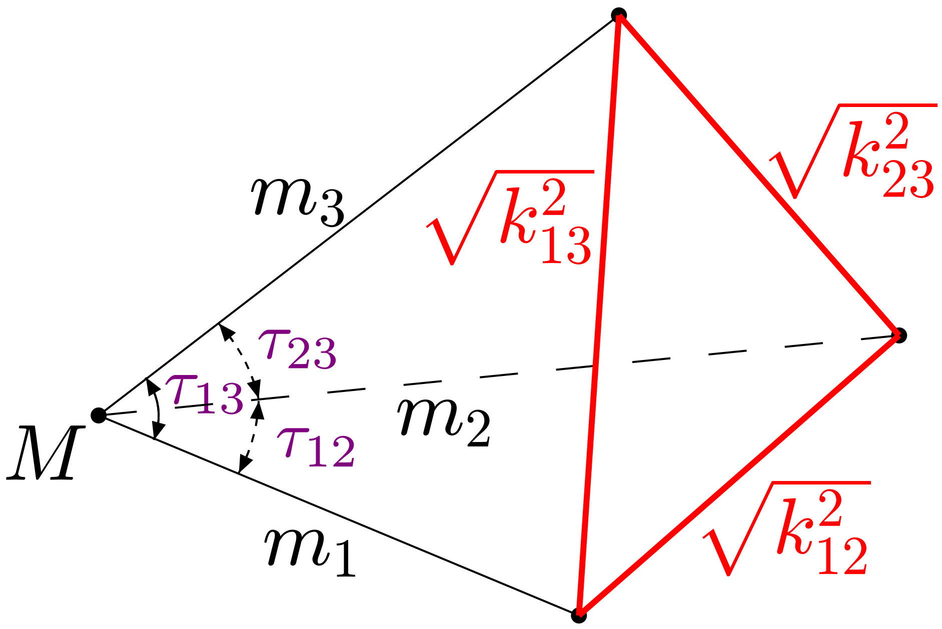
The "height":

$$m_0 = m_1 m_2 \sqrt{\frac{D^{(2)}}{\Lambda^{(2)}}} = \frac{m_1 m_2 \, \sin\tau_{12}}{\sqrt{k_{12}^2}}$$

Three-point function with arbitrary masses



Three-point function: the basic tetrahedron



$$\cos\tau_{jl} = c_{jl} = \frac{m_j^2 + m_l^2 - k_{jl}^2}{2m_j m_l}$$

Gram determinant:

$$D^{(3)} = \begin{vmatrix} 1 & c_{12} & c_{13} \\ c_{12} & 1 & c_{23} \\ c_{13} & c_{23} & 1 \end{vmatrix}$$

Tetrahedron volume:

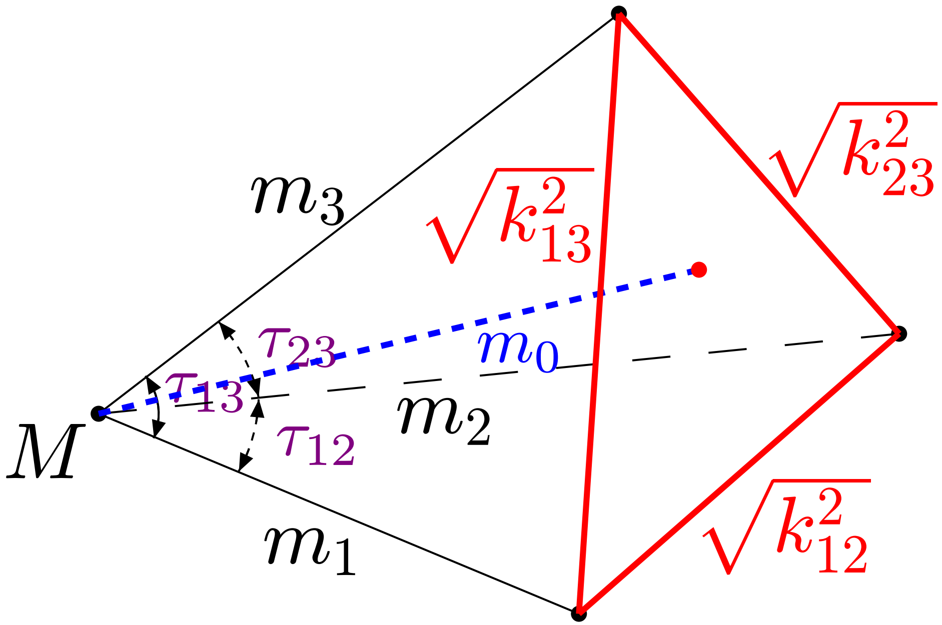
$$V^{(3)} = \tfrac{1}{6}\, m_1 m_2 m_3 \sqrt{D^{(3)}}$$

Red triangle area: $\overline{V}_0^{(2)} = \frac{1}{2}\sqrt{\Lambda^{(3)}}$,

with $\Lambda^{(3)} = \frac{1}{4}\left[2k_{12}^2k_{13}^2 + 2k_{13}^2k_{23}^2 + 2k_{23}^2k_{12}^2 - (k_{12}^2)^2 - (k_{13}^2)^2 - (k_{23}^2)^2\right] = -\frac{1}{4}\lambda(k_{12}^2, k_{13}^2, k_{23}^2)$,

where $\lambda(x,y,z)$ is the Källen function.

Three-point function: the basic tetrahedron



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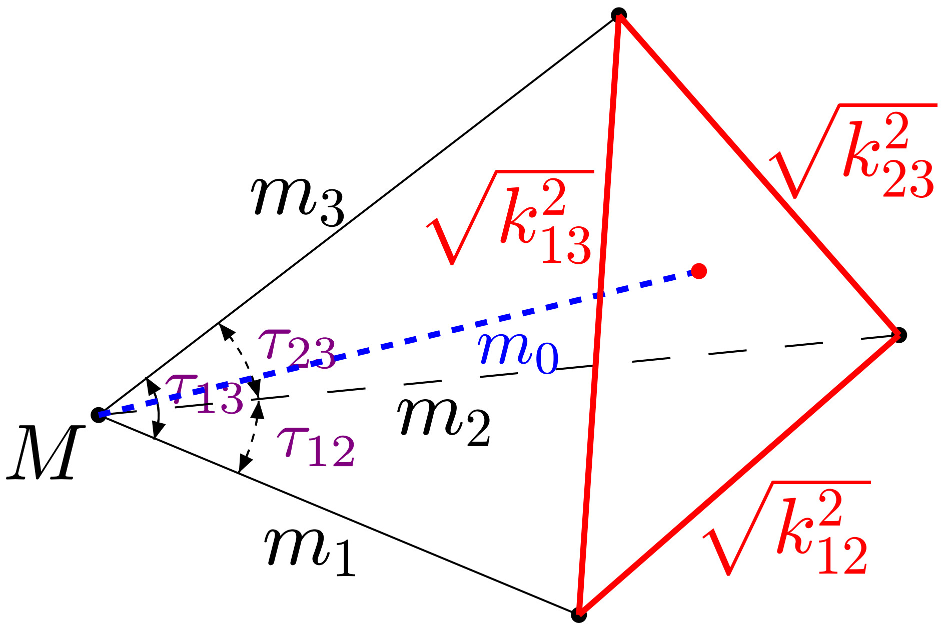
Tetrahedron volume:

$$V^{(3)} = \tfrac{1}{3}\, m_0 \overline{V}_0^{(2)} = \tfrac{1}{6}\, m_0 \sqrt{\Lambda^{(3)}} \quad\Rightarrow\quad m_0 = m_1 m_2 m_3 \sqrt{\frac{D^{(3)}}{\Lambda^{(3)}}}$$

Three-point function: the basic tetrahedron



$$\cos\tau_{jl} = c_{jl} = \frac{m_j^2 + m_l^2 - k_{jl}^2}{2m_j m_l}$$

Gram determinant:

$$D^{(3)} = \begin{vmatrix} 1 & c_{12} & c_{13} \\ c_{12} & 1 & c_{23} \\ c_{13} & c_{23} & 1 \end{vmatrix}$$

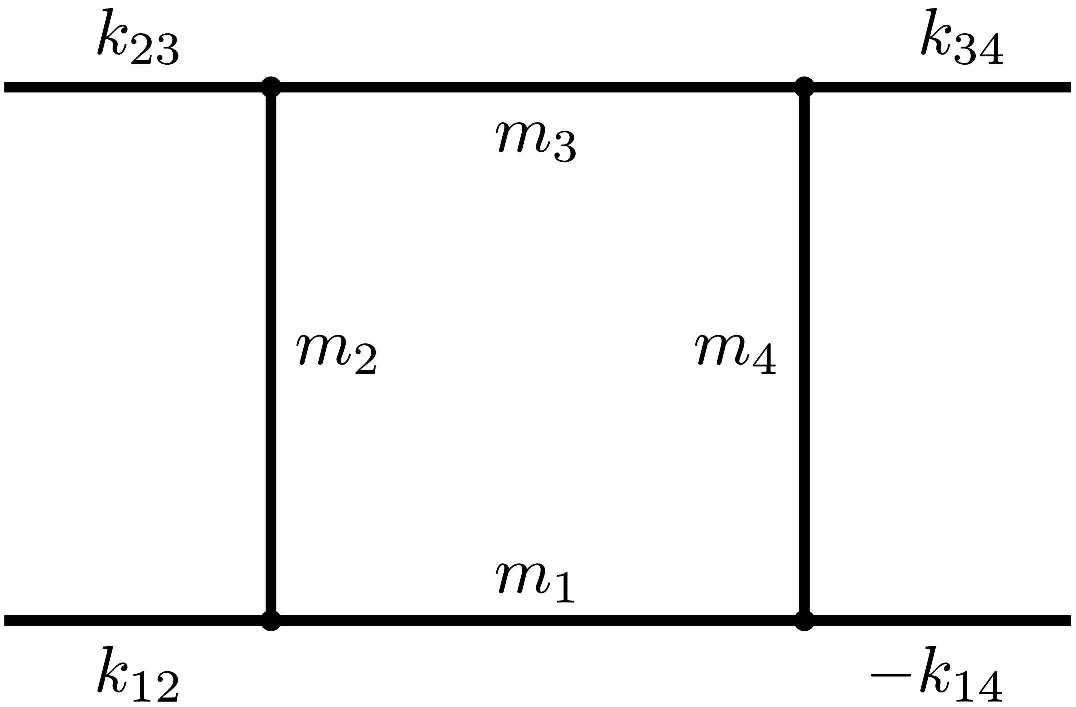
Tetrahedron volume:

$$V^{(3)} = \tfrac{1}{6}\, m_1 m_2 m_3 \sqrt{D^{(3)}}$$

Special cases (when the tetrahedron volume vanishes):

- When at least one of the areas of the faces $(m_1, m_2, \sqrt{k_{12}^2})$, $(m_1, m_3, \sqrt{k_{13}^2})$ or $(m_2, m_3, \sqrt{k_{23}^2})$ vanishes $\leftrightarrow$ two-particle thresholds and pseudo-thresholds
- When the area of the red face ("momentum face") vanishes $\leftrightarrow$ collinear singularities $(\Lambda^{(3)} = 0)$
- When $m_0 = 0$ (none of the face areas vanishes, but $D^{(3)} = 0$) $\leftrightarrow$ anomalous threshold

Four-point function

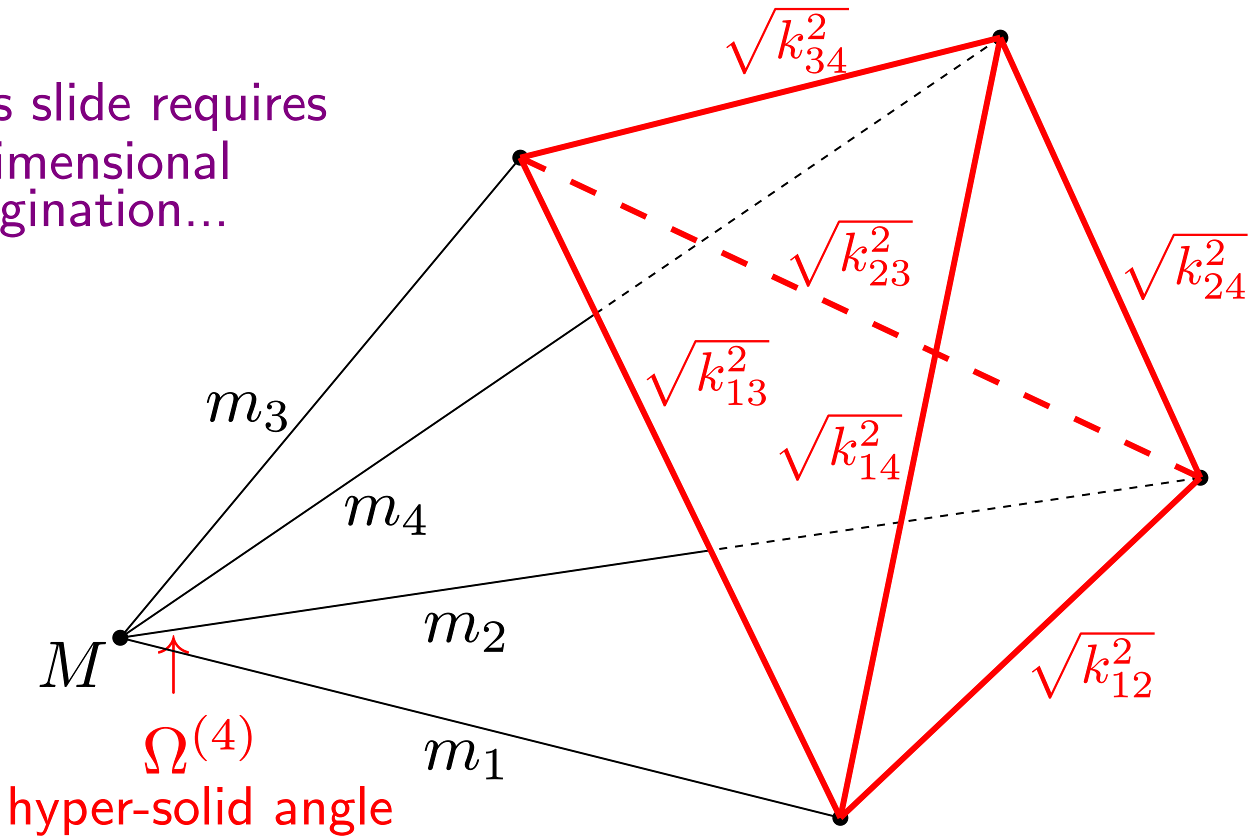


$k_{12}^2, k_{23}^2, k_{34}^2, k_{14}^2$ – external momenta squared

k_{13}^2, k_{24}^2 – Mandelstam variables s and t

The basic simplex for $N = 4$

This slide requires 4-dimensional imagination...



Gram determinants : $D^{(4)} = \det \|c_{jl}\|$, $\Lambda^{(4)} = \det \|(k_{j4} \cdot k_{l4})\|$,

Hypervolume (“content”) : $V^{(4)} = \dfrac{(\Pi m_i)}{4!}\sqrt{D^{(4)}}$, $\overline{V}_0^{(3)} = \dfrac{1}{3!}\sqrt{\Lambda^{(4)}}$

The basic simplex for $N = 4$

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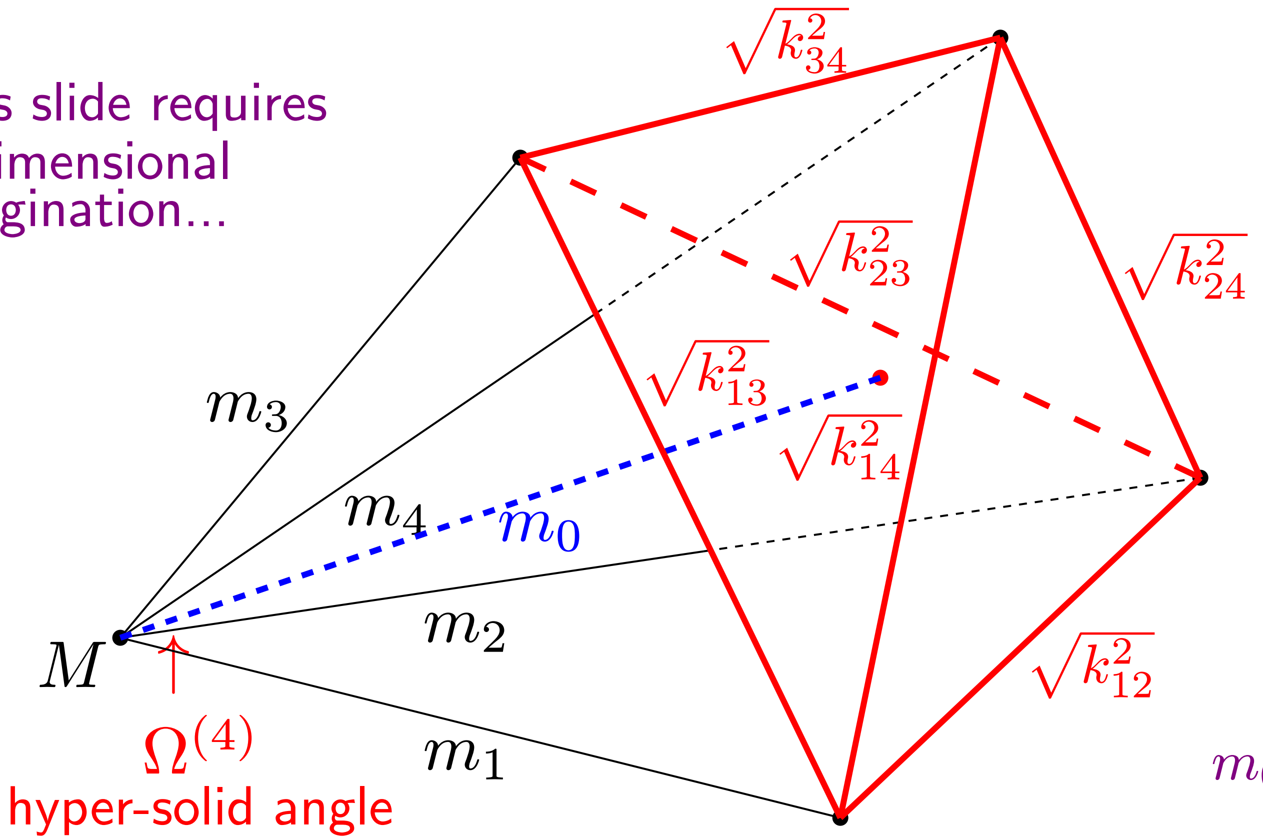

$$m_0 = (\Pi m_i)\sqrt{\frac{D^{(N)}}{\Lambda^{(N)}}}$$

Gram determinants : $D^{(4)} = \det \|c_{jl}\|$, $\Lambda^{(4)} = \det \|(k_{j4} \cdot k_{l4})\|$,

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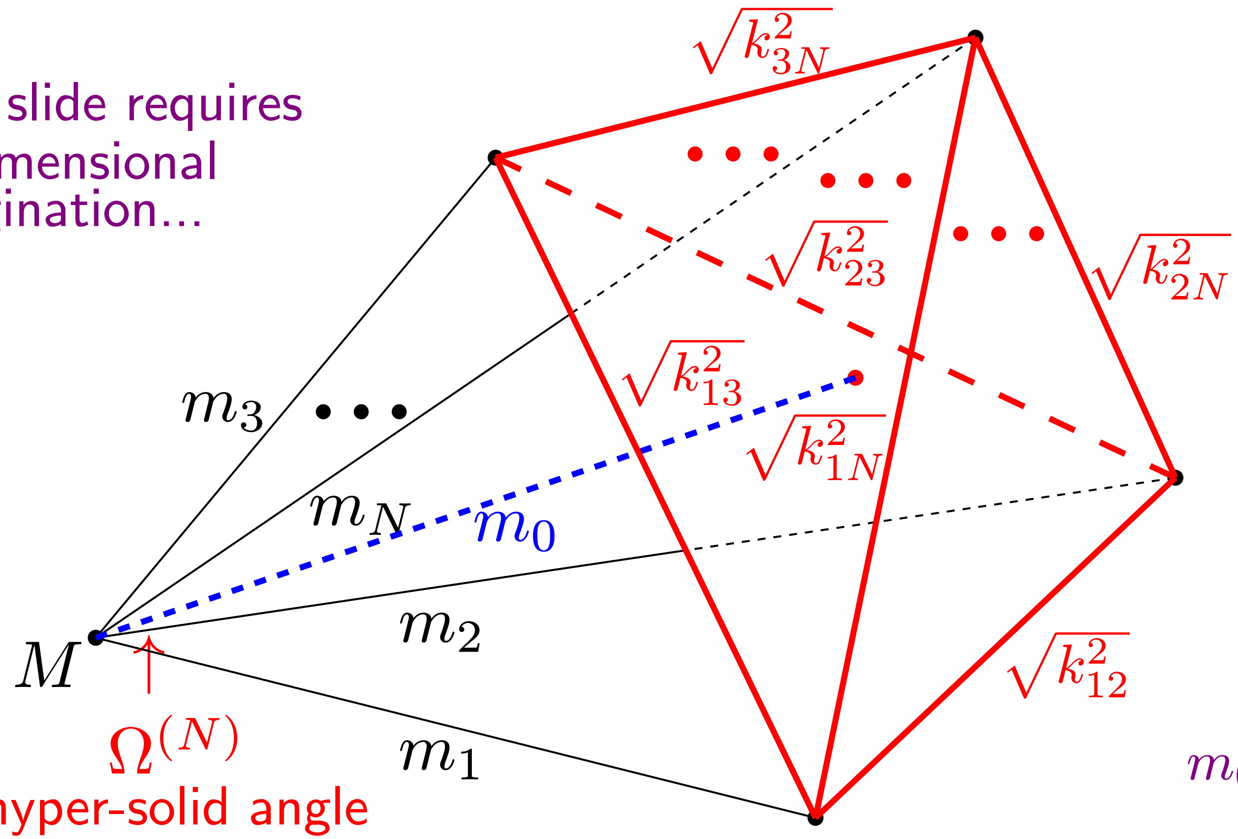
When we wish to proceed from three to more dimensions, however, we run up against a conceptual barrier so difficult that many people cannot surmount it. Just how far certain gifted individuals can visulaize, say, a space of four dimensions is an arguable matter which it would be wise to leave to the psychologists. My own opinion is that nobody can. $< \ldots >$ But these difficulties of "seeing the situation" do not prevent us from setting up a geometry of n dimensions in the classical sense and of reasoning about it.

M.G. Kendall, *A course in the geometry of n dimensions*
(Griffin, London, 1961), p. 2

The basic simplex for an arbitrary N

This slide requires N-dimensional imagination...



$$m_0 = (\Pi m_i)\sqrt{\frac{D^{(N)}}{\Lambda^{(N)}}}$$

Gram determinants : $D^{(N)} = \det \|c_{jl}\|$, $\Lambda^{(N)} = \det \|(k_{jN} \cdot k_{lN})\|$,

Hypervolume (“content”) : $V^{(N)} = \frac{(\Pi m_i)}{N!}\sqrt{D^{(N)}}$, $\overline{V}_0^{(N-1)} = \frac{1}{(N-1)!}\sqrt{\Lambda^{(N)}}$

A bit more on N-dimensional linear algebra

$$D^{(N)} = \det \|c_{jl}\| = \begin{vmatrix} 1 & c_{12} & \ldots & c_{1,l-1} & c_{1l} & c_{1,l+1} & \ldots & c_{1N} \\ c_{12} & 1 & \ldots & c_{2,l-1} & c_{2l} & c_{2,l+1} & \ldots & c_{2N} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{1N} & c_{2N} & \ldots & c_{l-1,N} & c_{l,N} & c_{l+1,N} & \ldots & 1 \end{vmatrix}$$

$$F_l^{(N)} = \frac{\partial}{\partial m_l^2}\left(m_l^2 D^{(N)}\right) = \begin{vmatrix} 1 & c_{12} & \ldots & c_{1,l-1} & m_l/m_1 & c_{1,l+1} & \ldots & c_{1N} \\ c_{12} & 1 & \ldots & c_{2,l-1} & m_l/m_2 & c_{2,l+1} & \ldots & c_{2N} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{1N} & c_{2N} & \ldots & c_{l-1,N} & m_l/m_N & c_{l+1,N} & \ldots & 1 \end{vmatrix}$$

Using notation

$D_{jl}^{(N-1)} = \{$minor of $D^{(N)}$ obtained by eliminating the jth row and the lth column$\}$

we get:

$$F_l^{(N)} = \sum_{j=1}^{N} (-1)^{j+l} D_{jl}^{(N-1)} \frac{m_l}{m_j}$$

$F_l^{(N)}$ obey the following equations:

$$\sum_{l=1}^{N} c_{jl} F_l^{(N)} \frac{1}{m_l} = D^{(N)} \frac{1}{m_j}$$

A bit more on N-dimensional linear algebra (continued)

$$F_l^{(N)} = \frac{\partial}{\partial m_l^2}\left(m_l^2 D^{(N)}\right) = \sum_{j=1}^{N} (-1)^{j+l} D_{jl}^{(N-1)} \frac{m_l}{m_j}\,,$$

where

$D_{jl}^{(N-1)} = \{$minor of $D^{(N)}$ obtained by eliminating the jth row and the lth column$\}$

$F_l^{(N)}$ obey the following equations: $$\sum_{l=1}^{N} c_{jl} F_l^{(N)} \frac{1}{m_l} = D^{(N)} \frac{1}{m_j}\,.$$

If we introduce a modified matrix, $C_{jl} = \left(\sqrt{F_j^{(N)}}\, c_{jl} \sqrt{F_l^{(N)}}\right)$,

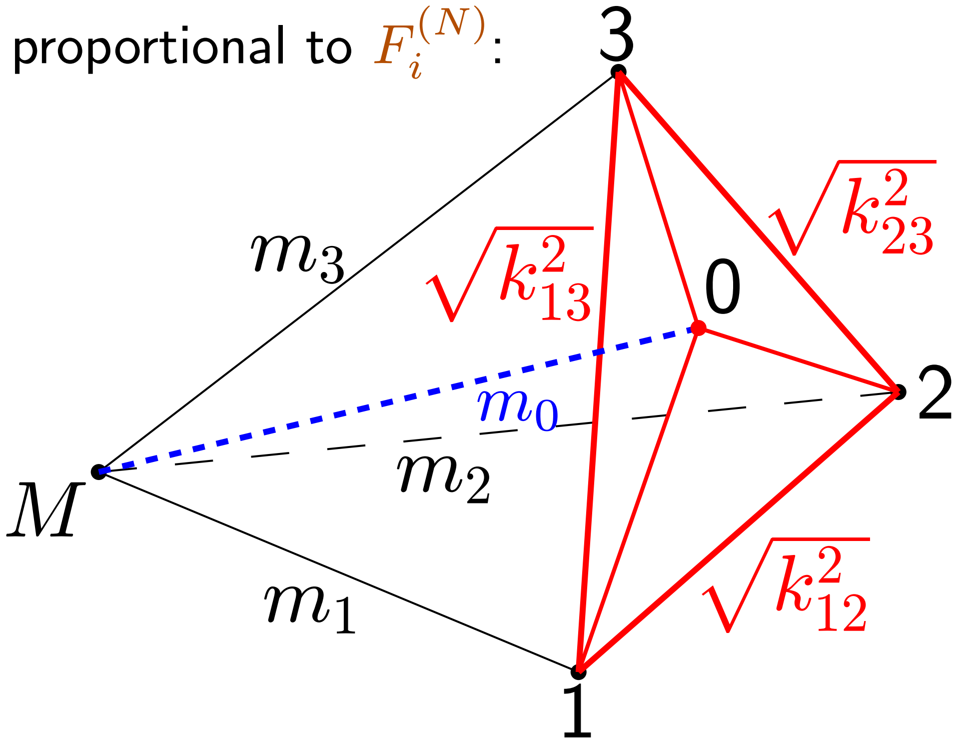
then
$$\sum_{l=1}^{N} C_{jl} \frac{\sqrt{F_l^{(N)}}}{m_l} = D^{(N)} \frac{\sqrt{F_j^{(N)}}}{m_j}$$

$\Rightarrow$ Eigenvector: $f_i = \dfrac{\sqrt{F_i^{(N)}}}{m_i}$, eigenvalue: $D^{(N)} = \det \|c_{jl}\|$,

determinant: $\det \|C_{jl}\| = D^{(N)} \prod_{i=1}^{N} F_i^{(N)}$

Geometrical meaning of $F_i^{(N)}$

The content of the ith rectangular simplex is proportional to $F_i^{(N)}$:



$$V_i^{(N)} = \frac{V^{(N)}}{\Lambda^{(N)}} \Big(\prod_{l \neq i} m_l^2 \Big) F_i^{(N)}$$

Using the condition

$$\sum_{i=1}^{N} V_i^{(N)} = V^{(N)}$$

we get

$$\sum_{i=1}^{N} \frac{F_i^{(N)}}{m_i^2} = \frac{\Lambda^{(N)}}{\prod_{j=1}^{N} m_j^2}$$

An example: $N = 3$
$V^{(3)} = V_1^{(3)} + V_2^{(3)} + V_3^{(3)}$

In particular, for the eigenvector $f_i = \frac{\sqrt{F_i^{(N)}}}{m_i}$ we get

$$|f| = \sqrt{\sum_{i=1}^{N} f_i^2} = \sqrt{\sum_{i=1}^{N} \frac{F_i^{(N)}}{m_i^2}} = \frac{\sqrt{\Lambda^{(N)}}}{\prod_{j=1}^{N} m_j} = \frac{\sqrt{D^{(N)}}}{m_0}$$

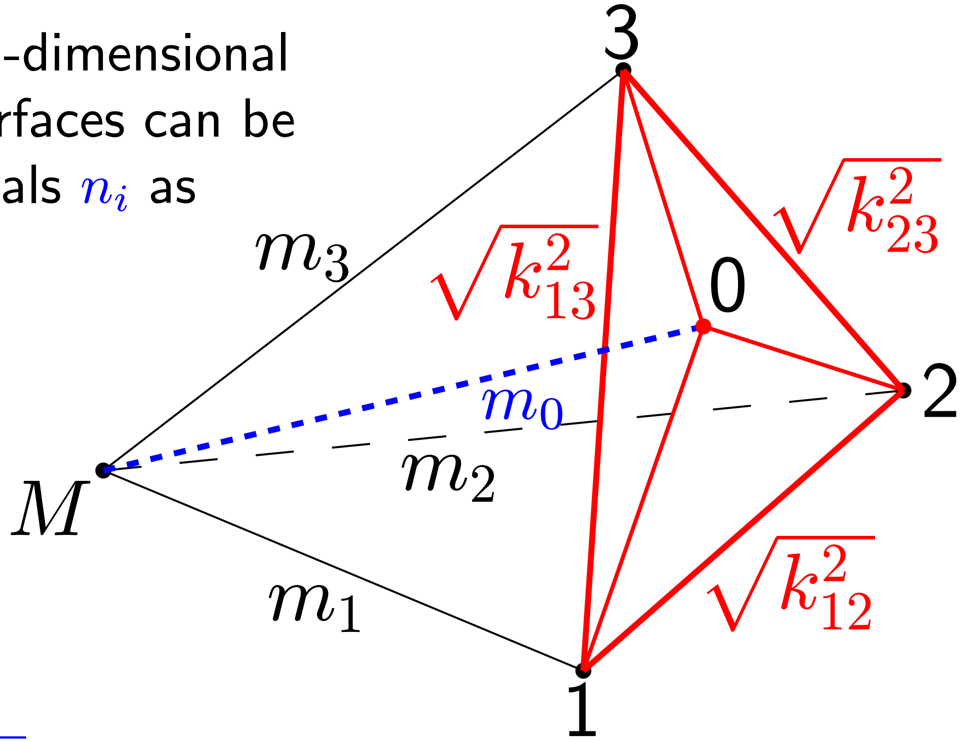
Normals, dihedral angles, and the dual matrix

The dihedral angles ψ_{jl} between the $(N-1)$-dimensional "reduced" (i.e., without one mass side) hyperfaces can be defined via the angles $\widetilde{\tau}_{jl}$ between their normals n_i as

$$\psi_{jl} = \pi - \widetilde{\tau}_{jl}, \quad \cos\psi_{jl} = -\cos\widetilde{\tau}_{jl} \ ,$$

$$(n_j \cdot n_l) = (-1)^{j+l} D_{jl}^{(N-1)} \ ,$$

$$\widetilde{c}_{jl} \equiv \cos\widetilde{\tau}_{jl} = \frac{(n_j \cdot n_l)}{\sqrt{n_j^2 n_l^2}} = \frac{(-1)^{j+l} D_{jl}^{(N-1)}}{\sqrt{D_{jj}^{(N-1)} D_{ll}^{(N-1)}}} \ ,$$



An example: $N=3$

Inverse of the Gram matrix $\|c_{jl}\|$ can be expressed through the "dual" matrix $\|\widetilde{c}_{jl}\|$,

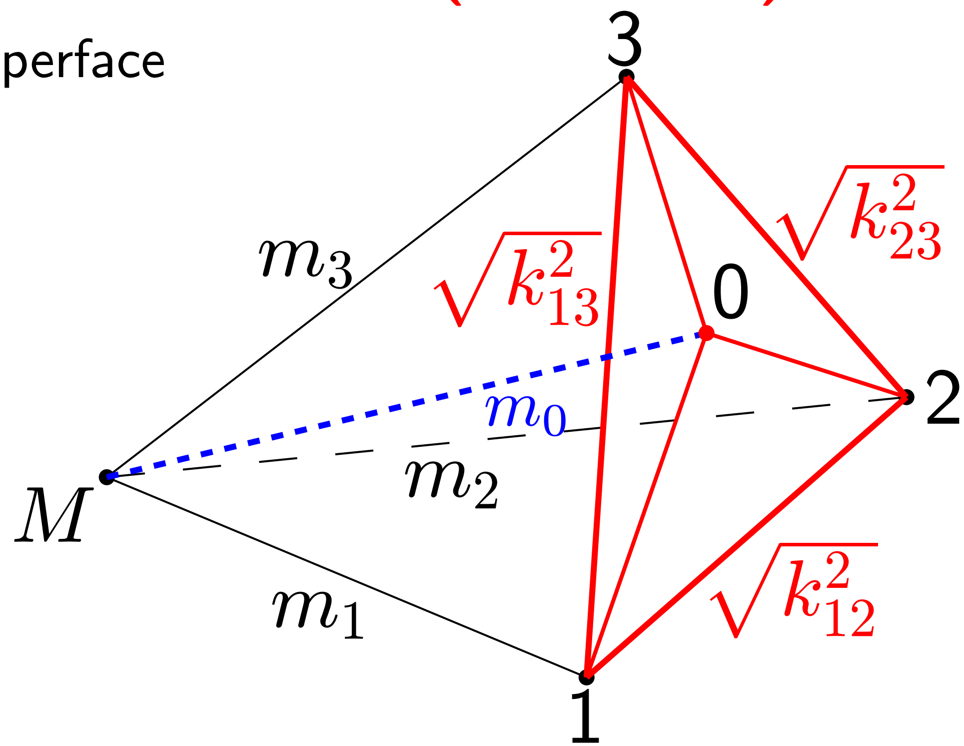
$$\|c_{jl}\|^{-1} = \frac{1}{D^{(N)}} \, \mathrm{diag}\left(\sqrt{D_{11}^{(N-1)}, \ldots, D_{NN}^{(N-1)}}\right) \ \|\widetilde{c}_{jl}\| \ \mathrm{diag}\left(\sqrt{D_{11}^{(N-1)}, \ldots, D_{NN}^{(N-1)}}\right) \ ,$$

$$\widetilde{D}^{(N)} \equiv \det\|\widetilde{c}_{jl}\| = \frac{\left(D^{(N)}\right)^{(N-1)}}{\prod_{i=1}^{N} D_{ii}^{(N-1)}} \ .$$

Normals, dihedral angles, and the dual matrix (continued)

The normal n_0 to the momentum ("red") hyperface can be expressed through the normals n_j as

$$n_0 = -\left(\prod_{i=1}^{N} m_i\right) \sum_{j=1}^{N} \frac{n_j}{m_j}$$



An example: $N=3$

Note that

$$n_0^2 = \left(\prod_{i=1}^{N} m_i^2\right) \sum_{l=1}^{N} \frac{F_l^{(N)}}{m_j^2} = \Lambda^{(N)} ,$$

$$(n_0 \cdot n_l) = -\left(\prod_{i\neq l} m_i\right) F_l^{(N)} ,$$

where

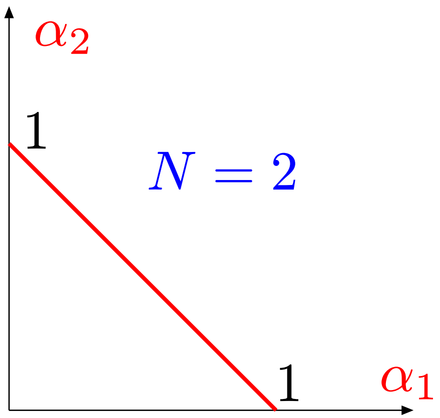
$$F_l^{(N)} = \frac{\partial}{\partial m_l^2}\left(m_l^2 D^{(N)}\right) = \sum_{j=1}^{N} (-1)^{j+l} D_{jl}^{(N-1)} \frac{m_l}{m_j} .$$

Feynman parameters: limits of integration

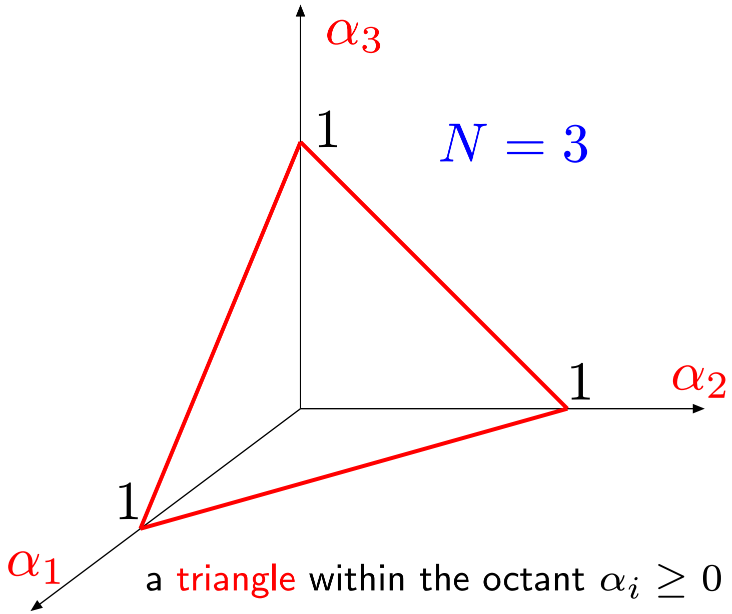
$$\int_0^1 \ldots \int_0^1 \left(\prod \mathrm{d}\alpha_i\right) \cdot \delta\left(\sum \alpha_i - 1\right) \{\ldots\} = \int_0^\infty \ldots \int_0^\infty \left(\prod \mathrm{d}\alpha_i\right) \cdot \delta\left(\sum \alpha_i - 1\right) \{\ldots\}$$

- We can extend integration limits to ∞ (in fact, to anything ≥ 1) because the true integration domain (for all $\alpha_i \geq 0$) is defined by the condition $\sum \alpha_i = 1$.

- This condition defines a line for $N = 2$, a plane for $N = 3$, and, in general, an $(N-1)$-dimensional hyperplane $\Rightarrow$ a segment for $N = 2$, a triangle for $N = 3$, and, in general, an $(N-1)$-dimensional simplex.



a segment within the quadrant $\alpha_i \geq 0$



a triangle within the octant $\alpha_i \geq 0$

$N = 4$: a tetrahedron within the sedecimant $\alpha_i \geq 0$

For an arbitrary N:
an $(N-1)$-dimensional simplex within the N-dimensional orthant $\alpha_i \geq 0$

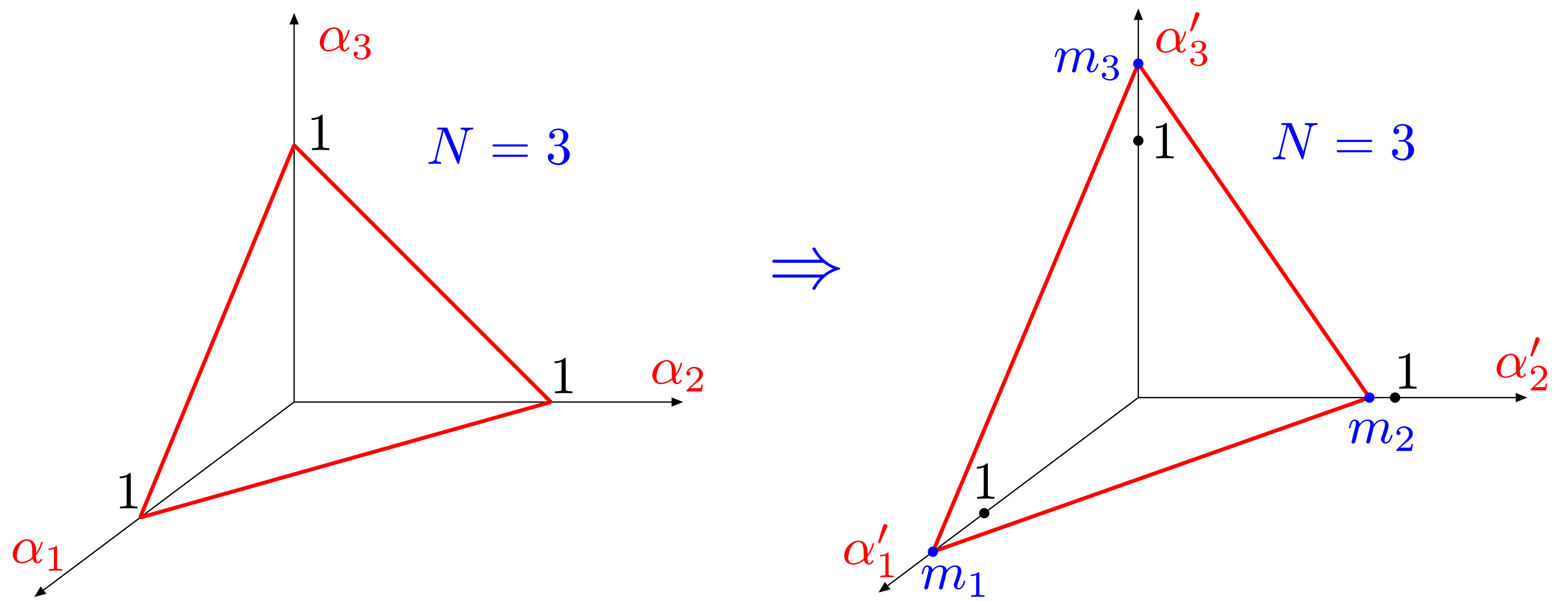
Feynman parameters: substitutions

Feynman parametric representation (with integration to ∞):

$$J^{(N)}(n;1,\ldots,1) = \mathrm{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\tfrac{n}{2}\right)\int\limits_0^\infty\!\!\ldots\!\!\int\limits_0^\infty \frac{\left(\prod \mathrm{d}\alpha_i\right)\cdot\delta\left(\sum\alpha_i-1\right)}{\left[\sum\alpha_i^2 m_i^2 + 2\sum\limits_{j<l}\sum\alpha_j\alpha_l m_j m_l c_{jl}\right]^{N-n/2}}$$

Substitute $\alpha_i = \frac{\alpha'_i}{m_i}$

$\Rightarrow$ quadratic form: $\left[\sum{\alpha'}_i^2 + 2\sum\limits_{j<l}\sum\alpha'_j\alpha'_l c_{jl}\right] \equiv \left(\alpha'^T \|c\| \alpha'\right)$, delta-function: $\delta\left(\sum\frac{\alpha'_i}{m_i}-1\right)$



Feynman parameters: substitutions (continued)

In terms of α'_i-parameters we get:

$$J^{(N)}(n;1,\ldots,1) = \mathrm{i}^{1-2N}\pi^{n/2}\,\frac{\Gamma\left(N-\frac{n}{2}\right)}{\prod m_i}\int\limits_0^\infty\ldots\int\limits_0^\infty \frac{\left(\prod \mathrm{d}\alpha'_i\right)\cdot\delta\left(\sum\frac{\alpha'_i}{m_i}-1\right)}{(\alpha'^T\|c\|\alpha')^{N-n/2}}$$

To restore the argument of the δ function in its original form, substitute:

$$\alpha'_i = \mathcal{F}(\alpha''_1,\ldots,\alpha''_N)\alpha''_i, \quad \text{with} \quad \mathcal{F}(\alpha''_1,\ldots,\alpha''_N) = \frac{\sum\alpha''_i}{\sum\frac{\alpha''_i}{m_i}}\,.$$

Note that

$$\sum\frac{\alpha'_i}{m_i}=1 \quad \Rightarrow \quad \sum\alpha''_i = 1\,, \quad \text{and the Jacobian} \quad \left|\frac{D(\{\alpha'_i\})}{D(\{\alpha''_i\})}\right| = \mathcal{F}^N$$

Therefore,

$$J^{(N)}(n;1,\ldots,1) = \mathrm{i}^{1-2N}\pi^{n/2}\,\frac{\Gamma\left(N-\frac{n}{2}\right)}{\prod m_i}\int\limits_0^\infty\ldots\int\limits_0^\infty \frac{\left(\prod \mathrm{d}\alpha''_i\right)\cdot\delta\left(\sum\alpha''_i-1\right)}{\left(\sum\frac{\alpha''_i}{m_i}\right)^{n-N}\quad(\alpha''^T\|c\|\alpha'')^{N-n/2}}$$

This is not “for free”: we get an extra linear factor from $\mathcal{F}$.

General values of ν_i and the special case $\Sigma\nu_i = n$

For general ν_i the obtained representation reads (after suppressing the primes)

$$J^{(N)}(n;\nu_1,\ldots,\nu_N) = \mathrm{i}^{1-2\Sigma\nu_i}\pi^{n/2}\,\frac{\Gamma\left(\Sigma\nu_i-\frac{n}{2}\right)}{\Pi\,\Gamma(\nu_i)}\frac{1}{\Pi m_i^{\nu_i}}\int\limits_0^\infty\!\!\ldots\!\!\int\limits_0^\infty \frac{\left(\prod \alpha_i{}^{\nu_i-1}\mathrm{d}\alpha_i\right)\cdot\delta\left(\sum\alpha_i-1\right)}{\left(\sum\frac{\alpha_i}{m_i}\right)^{n-\Sigma\nu_i}(\alpha^T\|c\|\alpha)^{\Sigma\nu_i-n/2}}$$

For the special case $\Sigma\nu_i = n$ the linear denominator disappears, and we get

$$J^{(N)}(n;\nu_1,\ldots,\nu_N)\Big|_{\Sigma\nu_i=n} = \mathrm{i}^{1-2\Sigma\nu_i}\pi^{n/2}\,\frac{\Gamma\left(\frac{n}{2}\right)}{\Pi\,\Gamma(\nu_i)}\frac{1}{\Pi m_i^{\nu_i}}\int\limits_0^\infty\!\!\ldots\!\!\int\limits_0^\infty \frac{\left(\prod \alpha_i{}^{\nu_i-1}\mathrm{d}\alpha_i\right)\cdot\delta\left(\sum\alpha_i-1\right)}{(\alpha^T\|c\|\alpha)^{n/2}}$$

In this case, except for the pre-factor $1/(\Pi m_i^{\nu_i})$, all dependence on the external momenta and masses is through

$$c_{jl} = \frac{m_j^2+m_l^2-k_{jl}^2}{2m_jm_l}$$

The case $\Sigma\nu_i = n$: reduction to the equal-mass function

$$J^{(N)}\left(n;\nu_1,\ldots,\nu_N\right)\Big|_{\Sigma\nu_i=n} = \mathrm{i}^{1-2\Sigma\nu_i}\pi^{n/2}\,\frac{\Gamma\left(\frac{n}{2}\right)}{\Pi\,\Gamma(\nu_i)}\frac{1}{\Pi m_i^{\nu_i}}\int\limits_0^\infty\ldots\int\limits_0^\infty\frac{\left(\prod \alpha_i{}^{\nu_i-1}\mathrm{d}\alpha_i\right)\cdot\delta\left(\sum\alpha_i-1\right)}{\left(\alpha^T\|c\|\alpha\right)^{n/2}}$$

We can formulate the following statement:

Let $\Sigma\nu_i = n$, and let the result for the integral with equal masses $m_i = m$ can be represented by a dimensionless function Ψ as

$$J^{(N)}(n;\nu_1,\ldots,\nu_N)\Big|_{\Sigma\nu_i=n,\ m_i=m} = \frac{1}{m^{\Sigma\nu_i}}\,\Psi\left(\{\bar{\bar{c}}_{jl}\}\right), \quad \text{with} \quad \bar{\bar{c}}_{jl} = 1-\frac{k_{jl}^2}{2m^2}\,.$$

Then the result for the integral with different masses m_i can be expressed in terms of *the same* function Ψ as

$$J^{(N)}(n;\nu_1,\ldots,\nu_N)\Big|_{\Sigma\nu_i=n} = \frac{1}{\Pi m_i^{\nu_i}}\,\Psi\left(\{c_{jl}\}\right), \quad \text{with} \quad c_{jl} = \frac{m_j^2+m_l^2-k_{jl}^2}{2m_jm_l}\,.$$

The case $\Sigma\nu_i = n$: reduction to the equal-mass function (continued)

The result for the integral with different masses m_i can be expressed in terms of *the same* function Ψ as in the equal-mass case

$$J^{(N)}(n;\nu_1,\ldots,\nu_N)\Big|_{\Sigma\nu_i=n} = \frac{1}{\Pi m_i^{\nu_i}}\,\Psi\left(\{c_{jl}\}\right), \quad \text{with} \quad c_{jl} = \frac{m_j^2+m_l^2-k_{jl}^2}{2m_jm_l}\,.$$

In practice, this means that the momentum invariants $\{\bar{\bar{k}}_{jl}^2\}$ of the equal-mass function should be adjusted in such a way that $\{\bar{\bar{c}}_{jl}\} = \{c_{jl}\}$:

$$1-\frac{\bar{\bar{k}}_{jl}^2}{2m^2} = \frac{m_j^2+m_l^2-k_{jl}^2}{2m_jm_l} \qquad \Rightarrow \qquad \bar{\bar{k}}_{jl}^2 = \frac{m^2}{m_jm_l}\left[k_{jl}^2-(m_j-m_l)^2\right]\,.$$

In particular, this applies to four-point function in four dimensions (with $\nu_i=1$),

$$J^{(4)}\left(4;1,1,1,1\Big|\{k_{jl}^2\},\{m_1,m_2,m_3,m_4\}\right) = \frac{m^4}{\Pi m_i}\,J^{(4)}\left(4;1,1,1,1\Big|\{\bar{\bar{k}}_{jl}^2\},\{m,m,m,m\}\right)$$

Feynman parameters: substitutions (continued)

Let us get back to the general case $\Sigma\nu_i \neq n$ and, for simplicity, put $\nu_i = 1$,

$$J^{(N)}(n;1,\ldots,1) = \mathrm{i}^{1-2N}\pi^{n/2}\,\frac{\Gamma\left(N-\frac{n}{2}\right)}{\prod m_i}\int\limits_0^\infty\ldots\int\limits_0^\infty \frac{\left(\prod \mathrm{d}\alpha_i\right)\cdot\delta\left(\sum\alpha_i - 1\right)}{\left(\sum\frac{\alpha_i}{m_i}\right)^{n-N}\quad(\alpha^T\|c\|\alpha)^{N-n/2}}$$

After dealing with linear combinations of α's, we are ready to try a quadratic one,

$$\alpha_i = \mathcal{G}(\alpha'_1,\ldots,\alpha'_N)\alpha'_i, \quad \text{with} \quad \mathcal{G}(\alpha'_1,\ldots,\alpha'_N) = \frac{(\alpha'^T\|c\|\alpha')}{\sum\alpha'_i}\,.$$

The Jacobian:
$$\left|\frac{D(\{\alpha_i\})}{D(\{\alpha'_i\})}\right| = 2\,\mathcal{G}^N$$

In this way, we “get rid” of the quadratic denominator, but get the quadratic structure in the argument of the δ function:

$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N}\pi^{n/2}\,\frac{\Gamma\left(N-\frac{n}{2}\right)}{\prod m_i}\int\limits_0^\infty\ldots\int\limits_0^\infty \frac{\left(\prod \mathrm{d}\alpha_i\right)\cdot\delta\left((\alpha^T\|c\|\alpha)-1\right)}{\left(\sum\frac{\alpha_i}{m_i}\right)^{n-N}}$$

Feynman parameters: substitutions (summary, so far)

- $\alpha_i = \frac{\alpha'_i}{m_i}$
- $\alpha'_i = \mathcal{F}(\alpha''_1, \ldots, \alpha''_N)\alpha''_i$, with $\mathcal{F}(\alpha''_1, \ldots, \alpha''_N) = \frac{\sum \alpha''_i}{\sum \frac{\alpha''_i}{m_i}}$
- $\alpha''_i = \mathcal{G}(\alpha'''_1, \ldots, \alpha'''_N)\alpha'''_i$, with $\mathcal{G}(\alpha'''_1, \ldots, \alpha'''_N) = \frac{(\alpha'''^T \|c\| \alpha''')}{\sum \alpha'''_i}$
- suppress the primes $\Rightarrow$

$$J^{(N)}(n; 1, \ldots, 1) = 2\mathrm{i}^{1-2N}\pi^{n/2} \frac{\Gamma\left(N - \frac{n}{2}\right)}{\prod m_i} \int\limits_0^\infty \ldots \int\limits_0^\infty \frac{\left(\prod \mathrm{d}\alpha_i\right) \cdot \delta\left((\alpha^T \|c\| \alpha) - 1\right)}{\left(\sum \frac{\alpha_i}{m_i}\right)^{n-N}}$$

Feynman parameters: substitutions (modified)

- $\alpha_i = f_i \alpha'_i$, with $f_i = \frac{\sqrt{F_i^{(N)}}}{m_i}$, where $F_i^{(N)} = \frac{\partial}{\partial m_i^2}\left(m_i^2 D^{(N)}\right)$
- $\alpha'_i = \widetilde{\mathcal{F}}(\alpha''_1, \ldots, \alpha''_N)\alpha''_i$, with $\widetilde{\mathcal{F}}(\alpha''_1, \ldots, \alpha''_N) = \frac{\sum \alpha''_i}{\sum f_i \alpha''_i}$
- $\alpha''_i = \widetilde{\mathcal{G}}(\alpha'''_1, \ldots, \alpha'''_N)\alpha'''_i$, with $\widetilde{\mathcal{G}}(\alpha'''_1, \ldots, \alpha'''_N) = \frac{(\alpha'''^T \|C\| \alpha''')}{\sum \alpha'''_i}$
- suppress the primes $\Rightarrow$ a modified representation,

$$J^{(N)}(n; 1, \ldots, 1) = 2\mathrm{i}^{1-2N}\pi^{n/2}\Gamma\left(N - \frac{n}{2}\right)(\Pi f_i)\int_0^\infty \ldots \int_0^\infty \frac{\left(\prod \mathrm{d}\alpha_i\right)\cdot\delta\left(\alpha^T\|C\|\alpha - 1\right)}{\left(\sum \alpha_i f_i\right)^{n-N}}$$

$\rightarrow$ For $n = N$ both representations (with $\|c\|$ and $\|C\|$) are equivalent.

$\rightarrow$ For the general case, the modified representation occurs to be more convenient, because we know that f_i is an eigenvector of $\|C\|$ with eigenvalue $= D^{(N)}$.

The case of general powers of propagators ν_i

Original representation, in terms of $\|c\|$:

$$J^{(N)}(n;\nu_1,\ldots,\nu_N) = 2\mathrm{i}^{1-2\Sigma\nu_i}\pi^{n/2}\,\frac{\Gamma\left(\Sigma\nu_i - \frac{n}{2}\right)}{\Pi\,\Gamma(\nu_i)}\frac{1}{\Pi m_i^{\nu_i}}\int\limits_0^\infty\!\!\ldots\!\!\int\limits_0^\infty \frac{\left(\prod {\alpha_i}^{\nu_i-1}\mathrm{d}\alpha_i\right)\cdot\delta\left((\alpha^T\|c\|\alpha)-1\right)}{\left(\sum\frac{\alpha_i}{m_i}\right)^{n-\Sigma\nu_i}}$$

Modified representation, in terms of $\|C\|$:

$$J^{(N)}(n;\nu_1,\ldots,\nu_N) = 2\mathrm{i}^{1-2\Sigma\nu_i}\pi^{n/2}\,\frac{\Gamma\left(\Sigma\nu_i - \frac{n}{2}\right)}{\Pi\,\Gamma(\nu_i)}\left(\Pi {f_i}^{\nu_i}\right)\int\limits_0^\infty\!\!\ldots\!\!\int\limits_0^\infty \frac{\left(\prod {\alpha_i}^{\nu_i-1}\mathrm{d}\alpha_i\right)\cdot\delta\left(\alpha^T\|C\|\alpha-1\right)}{\left(\sum\alpha_i f_i\right)^{n-\Sigma\nu_i}}$$

$\rightarrow$ For $\Sigma\nu_i = n$ both representations (with $\|c\|$ and $\|C\|$) are equivalent.

$\rightarrow$ For the general case, the modified representation occurs to be more convenient, because we know that f_i is an eigenvector of $\|C\|$ with eigenvalue $= D^{(N)}$.

Special case: $\Sigma\nu_i = n$

In this case, the denominator disappears, and we get

$$J^{(N)}\left(n;\nu_1,\ldots,\nu_N\right)\Big|_{\Sigma\nu_i=n} = 2\mathrm{i}^{1-2\Sigma\nu_i}\pi^{n/2}\,\frac{\Gamma\left(\frac{n}{2}\right)}{\Pi\,\Gamma(\nu_i)}\frac{1}{\Pi m_i^{\nu_i}}\int\limits_0^\infty\!\!\ldots\!\int\limits_0^\infty \left(\Pi\alpha_i{}^{\nu_i-1}\mathrm{d}\alpha_i\right)\cdot\delta\left(\left(\alpha^T\|c\|\alpha\right)-1\right)$$

When all $\nu_i = 1$ $(n = N)$, our integrand is just a δ function,

$$J^{(N)}\left(n;1,\ldots,1\right)\Big|_{N=n} = 2\mathrm{i}^{1-2N}\pi^{N/2}\,\frac{\Gamma\left(\frac{N}{2}\right)}{\Pi\,\Gamma(\nu_i)}\frac{1}{\Pi m_i}\int\limits_0^\infty\!\!\ldots\!\int\limits_0^\infty \left(\Pi\mathrm{d}\alpha_i\right)\cdot\delta\left(\left(\alpha^T\|c\|\alpha\right)-1\right)$$

In particular, in a very special situation, when $c_{jl} = \delta_{jl}$ $(\tau_{jl} = \frac{\pi}{2})$, integrating over an orthant of the hypersphere yields

$$J^{(N)}\left(n;\nu_1,\ldots,\nu_N\right)\Big|_{\Sigma\nu_i=n,\; c_{jl}=\delta_{jl}} = 2\mathrm{i}^{1-2\Sigma\nu_i}\pi^{n/2}\,\frac{\Pi\,\Gamma\left(\frac{\nu_i}{2}\right)}{2^N\Pi\,\Gamma(\nu_i)}\frac{1}{\Pi m_i^{\nu_i}}\,.$$

Feynman parameters: diagonalization

Modified representation (in terms of $\|C\|$), the case $\nu_i = 1$:

$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\frac{n}{2}\right)(\Pi f_i)\int_0^\infty\!\!\ldots\int_0^\infty \frac{\left(\prod \mathrm{d}\alpha_i\right)\cdot\delta\left(\alpha^T\|C\|\alpha-1\right)}{\left(\sum \alpha_i f_i\right)^{n-N}}$$

- To *diagonalize* the quadratic form, let us "rotate" the variables $\alpha_i \to \beta_i$ so that $\alpha^T\|C\|\alpha = \sum_{i=1}^N \lambda_i\beta_i^2$ (this is an orthogonal transformation)

- The determinants should be equal, $\lambda_1\ldots\lambda_N = \det\|C\| = D^{(N)}\prod_{i=1}^N F_i^{(N)}$

- One of the β's (say β_N) is directed along f_i (the known eigenvector), so that $\lambda_N = D^{(N)}$ and the denominator $(\sum\alpha_i f_i)$ is proportional to β_N

- The region of integration in space of β's is a piece of quadratic hypersurface $\sum\lambda_i\beta_i^2 = 1$ cut out by a rotated orthant

- If all λ_i are positive, the quadratic hypersurface is a part of an N-dimensional ellipsoid; if some of the λ's are negative, it is a part of a hyperboloid

Feynman parameters: rescaling

- Assume that all $\lambda_i > 0$ and rescale $\beta_i = \frac{\gamma_i}{\sqrt{\lambda_i}}$ $\Rightarrow$ hypersphere $\delta\left(\sum \gamma_i^2 - 1\right)$ (if some of λ_i are negative – *hyperbolic* surface $\leftrightarrow$ analytic continuation)

- This is a *non-orthogonal* transformation: the images of unit vectors $e_i^{(\alpha)}$ along α_i axes, in general, are not perpendicular to each other in γ_i space

- The transformation $\alpha_i \to \beta_i \to \gamma_i$ is equivalent to using $\|C\|$ as a new metric: $|\alpha|_C^2 = \alpha^T \|C\| \alpha$; for the images of unit vectors, $(e_j^{(\alpha)} \cdot e_l^{(\alpha)})_C = C_{jl}$; they are not unit, $|e_i^{(\alpha)}|_C^2 = C_{ii} = F_i^{(N)}$

- The cosine of the angle between the images of unit vectors:

$$\frac{(e_j^{(\alpha)} \cdot e_l^{(\alpha)})_C}{|e_j^{(\alpha)}|_C |e_l^{(\alpha)}|_C} = \frac{C_{jl}}{\sqrt{C_{jj} C_{ll}}} = \frac{\sqrt{F_j^{(N)}}\, c_{jl}\, \sqrt{F_l^{(N)}}}{\sqrt{F_j^{(N)} F_l^{(N)}}} = c_{jl} = \cos\tau_{jl}$$

$\to$ in γ_i space, the same N-dimensional solid angle $\Omega^{(N)}$ as in the *basic simplex*!

Feynman parameters: rescaling (continued)

- Moreover, in γ_i space the image of the vector f_i is directed along m_0 (the height of the basic simplex),

$$\frac{(e_i^{(\alpha)} \cdot f)_C}{|e_i^{(\alpha)}|_C |f|_C} = \frac{m_0}{m_i} = \cos\tau_{0i} = c_{0i}$$

- The linear denominator becomes: $\left(\sum_{i=1}^{N} \alpha_i f_i\right) \Rightarrow \frac{1}{\prod m_i} \sqrt{\frac{\Lambda^{(N)}}{D^{(N)}}}\, \gamma_N = \frac{1}{m_0}\, \gamma_N$

Finally, we arrive at the following **geometric representation**:

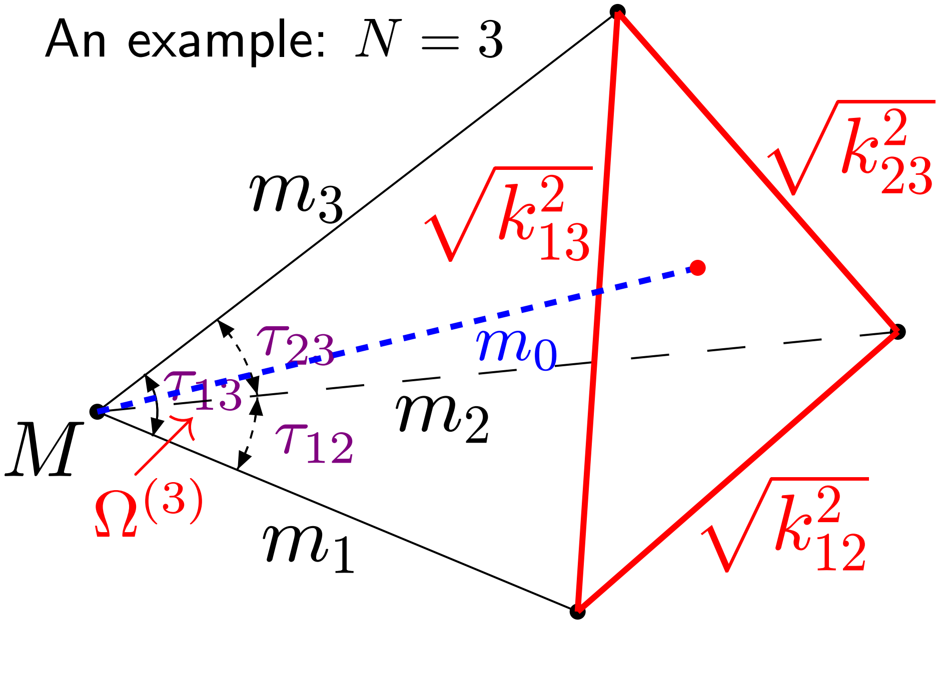
$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N} \pi^{n/2}\, \Gamma\left(N - \tfrac{n}{2}\right) \frac{{m_0}^{n-N-1}}{\sqrt{\Lambda^{(N)}}} \int\limits_{\Omega^{(N)}} \ldots \int \frac{\prod \mathrm{d}\gamma_i}{\gamma_N^{n-N}}\, \delta\left(\sum \gamma_i^2 - 1\right)$$

$\rightarrow$ Remarkably: the same N-dimensional solid angle $\Omega^{(N)}$ as in the *basic simplex*
$\rightarrow$ If some of λ_i are negative – *hyperbolic* surface $\leftrightarrow$ analytic continuation

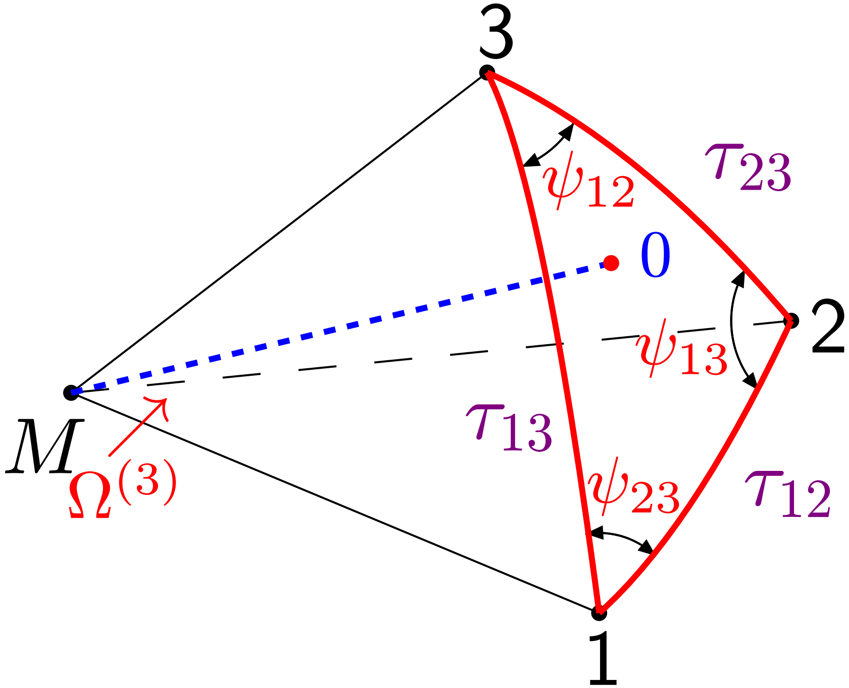
Geometric representation

$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\tfrac{n}{2}\right)\,\frac{{m_0}^{n-N-1}}{\sqrt{\Lambda^{(N)}}}\;\int_{\Omega^{(N)}}\!\!\ldots\int\,\frac{\prod \mathrm{d}\gamma_i}{\gamma_N^{\,n-N}}\,\delta\left(\sum\gamma_i^2-1\right)$$

An example: $N=3$



the basic tetrahedron



the solid angle and the integration surface

Geometric representation (continued)

$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\tfrac{n}{2}\right)\,\frac{{m_0}^{n-N-1}}{\sqrt{\Lambda^{(N)}}}\;\underset{\Omega^{(N)}}{\int\ldots\int}\;\frac{\prod \mathrm{d}\gamma_i}{\gamma_N^{\,n-N}}\,\delta\left(\sum\gamma_i^2-1\right)$$

- The integration goes over the hypersurface of the hypersphere cut out by the N-dimensional solid angle $\Omega^{(N)}$

- If we define the angle between the "running" unit vector and the Nth axis as θ, then $1/\gamma_N^{n-N}\Rightarrow 1/(\cos\theta)^{n-N}$, and we get

$$J^{(N)}(n;1,\ldots,1) = \mathrm{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\tfrac{n}{2}\right)\,\frac{{m_0}^{n-N}\Omega^{(N;n)}}{N!\;V^{(N)}}\;,$$

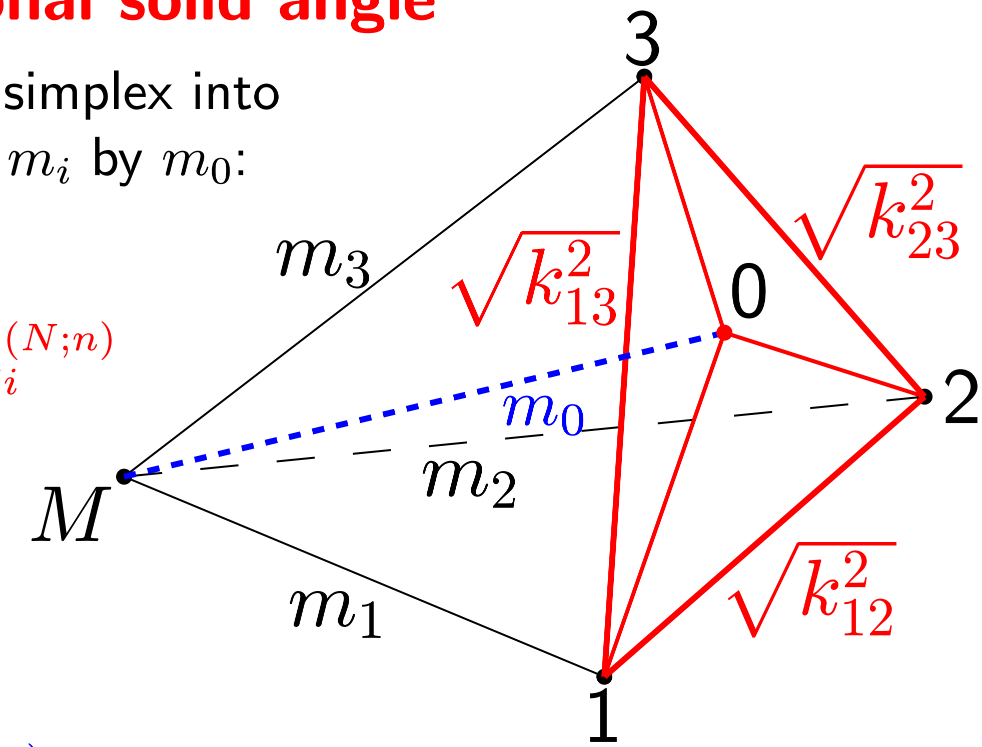
where $V^{(N)}$ is the volume of the basic simplex, m_0 – its height, and

$$\Omega^{(N;n)} \equiv \underset{\Omega^{(N)}}{\int\ldots\int}\;\frac{\mathrm{d}\Omega_N}{\cos^{n-N}\theta} \qquad \text{(obviously,}\quad \Omega^{(N;N)}=\Omega^{(N)}\text{)}$$

$\Rightarrow$ *everything is translated into the geometrical language!*

Splitting the N-dimensional solid angle

Let us use m_0 to split the basic N-dimensional simplex into N rectangular ones, each time replacing one of m_i by m_0:



An example: $N=3$

$$\Omega_i^{(N;n)} = \int_{\Omega_i^{(N)}} \ldots \int \frac{\mathrm{d}\Omega_N}{\cos^{n-N}\theta}\, , \qquad \Omega^{(N;n)} = \sum_{i=1}^{N} \Omega_i^{(N;n)}$$

Each of the resulting integrals can be associated with a one-loop N-point integral

$$J^{(N)}(n;1,\ldots,1) = \sum_{i=1}^{N} \frac{V_i^{(N)}}{V^{(N)}}\, J_i^{(N)}(n;1,\ldots,1)$$

In $J_i^{(N)}$ the internal masses are $m_1,\ldots,m_{i-1},m_0,m_{i+1},\ldots,m_N$, and the squared momenta are k_{jl}^2 (if $j\neq i$ and $l\neq i$), $m_l^2-m_0^2$ (if $j=i$), $m_j^2-m_0^2$ (if $l=i$).

In terms of $F_i^{(N)}$,

$$J^{(N)}(n;1,\ldots,1) = \frac{1}{\Lambda^{(N)}}\left(\prod m_i^2\right) \sum_{i=1}^{N} \frac{F_i^{(N)}}{m_i^2}\, J_i^{(N)}(n;1,\ldots,1)$$

See more examples below...

Feynman parameters versus geometrical approach

Feynman parametric representation (with $c_{jl} \equiv (m_j^2 + m_l^2 - k_{jl}^2)/(2m_j m_l)$):

$$J^{(N)}(n; 1, \ldots, 1) = \mathrm{i}^{1-2N} \pi^{n/2}\, \Gamma\left(N - \tfrac{n}{2}\right) \int_0^1 \ldots \int_0^1 \frac{\left(\prod \mathrm{d}\alpha_i\right) \cdot \delta\left(\sum \alpha_i - 1\right)}{\left[\sum \alpha_i^2 m_i^2 + 2 \sum\limits_{j<l}\sum \alpha_j \alpha_l m_j m_l c_{jl}\right]^{N-n/2}}$$

• depends on the masses and k_{jl}^2 through m_i and c_{jl} in the quadratic form

Geometric representation:

$$J^{(N)}(n; 1, \ldots, 1) = 2\mathrm{i}^{1-2N} \pi^{n/2}\, \Gamma\left(N - \tfrac{n}{2}\right) \frac{{m_0}^{n-N-1}}{\sqrt{\Lambda^{(N)}}} \int \ldots \int_{\Omega^{(N)}} \frac{\prod \mathrm{d}\gamma_i}{\gamma_N^{\,n-N}}\, \delta\left(\sum \gamma_i^2 - 1\right)$$

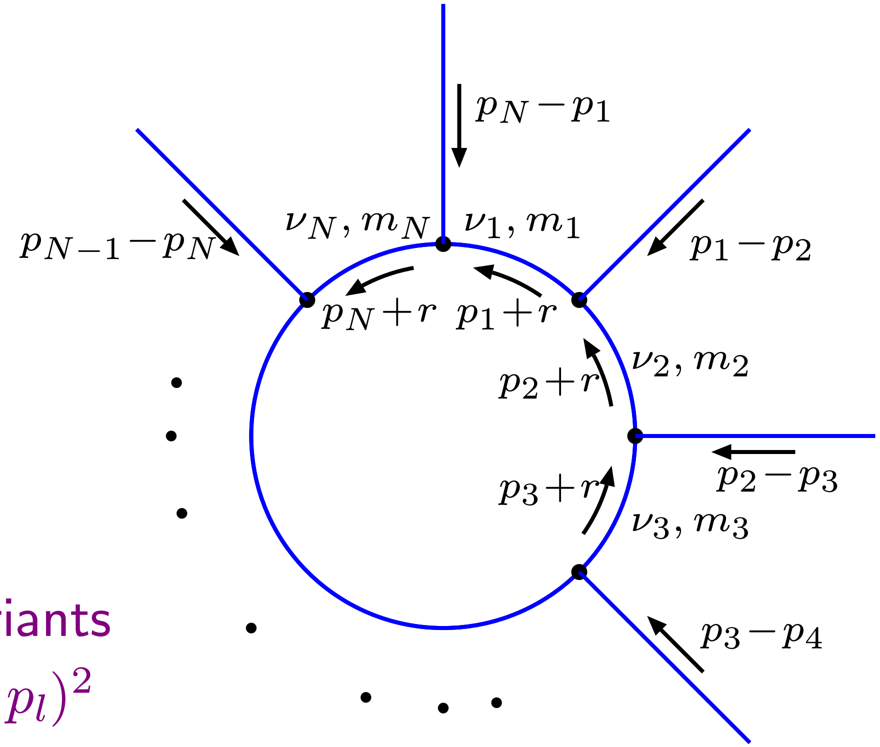
• except for the pre-factor, depends on the masses and k_{jl}^2 only through the integration limits defined by the N-dimensional solid angle $\Omega^{(N)}$

• ready for splitting!

LECTURE #2

One-loop N-point function $J^{(N)}(n; \nu_1, \ldots, \nu_N)$

from Lecture #1



Depends on
$\frac{1}{2}N(N-1)$ invariants
$k_{jl}^2 = (p_j - p_l)^2$
and N masses m_i

$$J^{(N)}(n; \nu_1, \ldots, \nu_N) \equiv \int \frac{\mathrm{d}^n k}{[(p_1+k)^2 - m_1^2]^{\nu_1} \cdots [(p_N+k)^2 - m_N^2]^{\nu_N}}$$

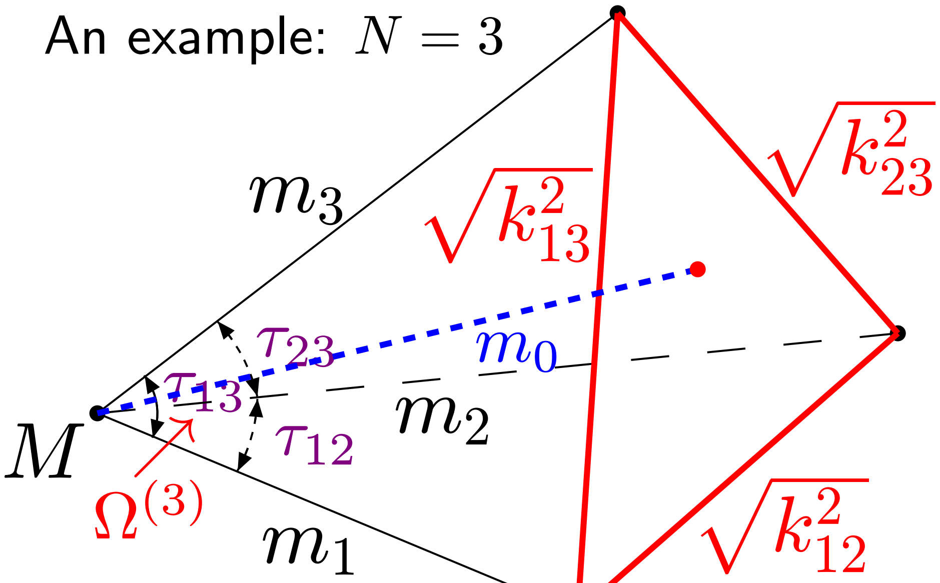
dimension powers of propagators

Geometric representation

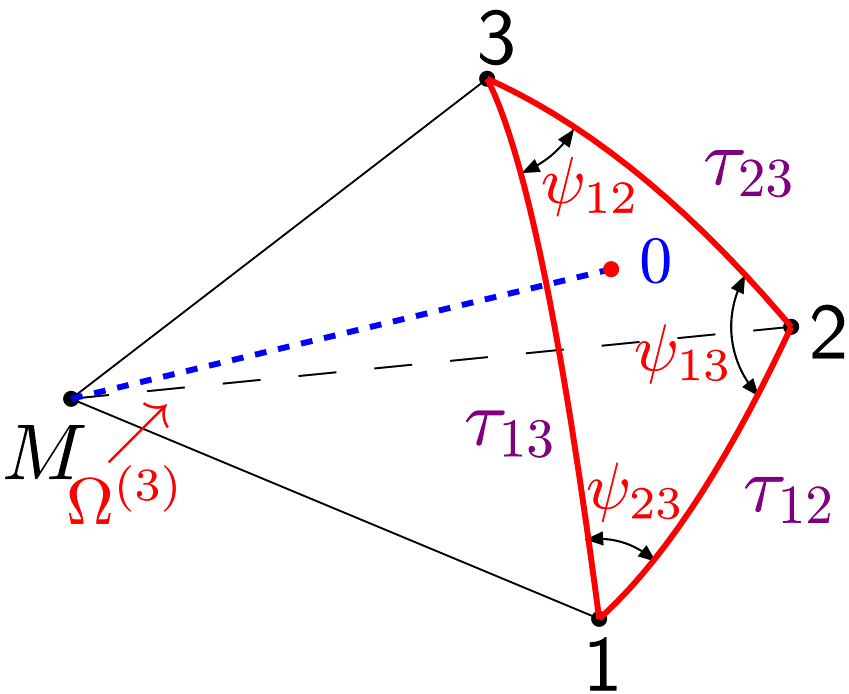
from Lecture #1

$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\tfrac{n}{2}\right)\frac{{m_0}^{n-N-1}}{\sqrt{\Lambda^{(N)}}}\int\limits_{\Omega^{(N)}}\!\!\ldots\!\int \frac{\prod \mathrm{d}\gamma_i}{\gamma_N^{\,n-N}}\,\delta\left(\sum\gamma_i^2-1\right)$$

An example: $N=3$



the basic tetrahedron



the solid angle and the integration surface

Geometric representation (continued)

from Lecture #1

$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\tfrac{n}{2}\right)\,\frac{{m_0}^{n-N-1}}{\sqrt{\Lambda^{(N)}}}\int\limits_{\Omega^{(N)}}\!\!\ldots\!\int\frac{\prod \mathrm{d}\gamma_i}{\gamma_N^{\;n-N}}\,\delta\left(\sum\gamma_i^2-1\right)$$

- The integration goes over the hypersurface of the hypersphere cut out by the N-dimensional solid angle $\Omega^{(N)}$

- If we define the angle between the "running" unit vector and the Nth axis as θ, then $1/\gamma_N^{n-N}\Rightarrow 1/(\cos\theta)^{n-N}$, and we get

$$J^{(N)}(n;1,\ldots,1) = \mathrm{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\tfrac{n}{2}\right)\,\frac{{m_0}^{n-N}\Omega^{(N;n)}}{N!\,V^{(N)}}\,,$$

where $V^{(N)}$ is the volume of the basic simplex, m_0 – its height, and

$$\Omega^{(N;n)} \equiv \int\limits_{\Omega^{(N)}}\!\!\ldots\!\int\frac{\mathrm{d}\Omega_N}{\cos^{n-N}\theta} \qquad \text{(obviously,}\quad \Omega^{(N;N)}=\Omega^{(N)}\text{)}$$

$\Rightarrow$ *everything is translated into the geometrical language!*

Feynman parameters versus geometrical approach

from Lecture #1

Feynman parametric representation (with $c_{jl} \equiv (m_j^2 + m_l^2 - k_{jl}^2)/(2m_j m_l)$):

$$J^{(N)}(n;1,\ldots,1) = \mathrm{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\tfrac{n}{2}\right)\int_0^1\!\!\ldots\!\int_0^1 \frac{\left(\prod \mathrm{d}\alpha_i\right)\cdot\delta\left(\sum\alpha_i - 1\right)}{\left[\sum \alpha_i^2 m_i^2 + 2\sum\limits_{j<l}\sum \alpha_j\alpha_l m_j m_l c_{jl}\right]^{N-n/2}}$$

• depends on the masses and k_{jl}^2 through m_i and c_{jl} in the quadratic form

Geometric representation:

$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\tfrac{n}{2}\right)\frac{{m_0}^{n-N-1}}{\sqrt{\Lambda^{(N)}}}\int\!\ldots\!\int_{\Omega^{(N)}} \frac{\prod \mathrm{d}\gamma_i}{\gamma_N^{\,n-N}}\,\delta\left(\sum\gamma_i^2 - 1\right)$$

• except for the pre-factor, depends on the masses and k_{jl}^2 only through the integration limits defined by the N-dimensional solid angle $\Omega^{(N)}$

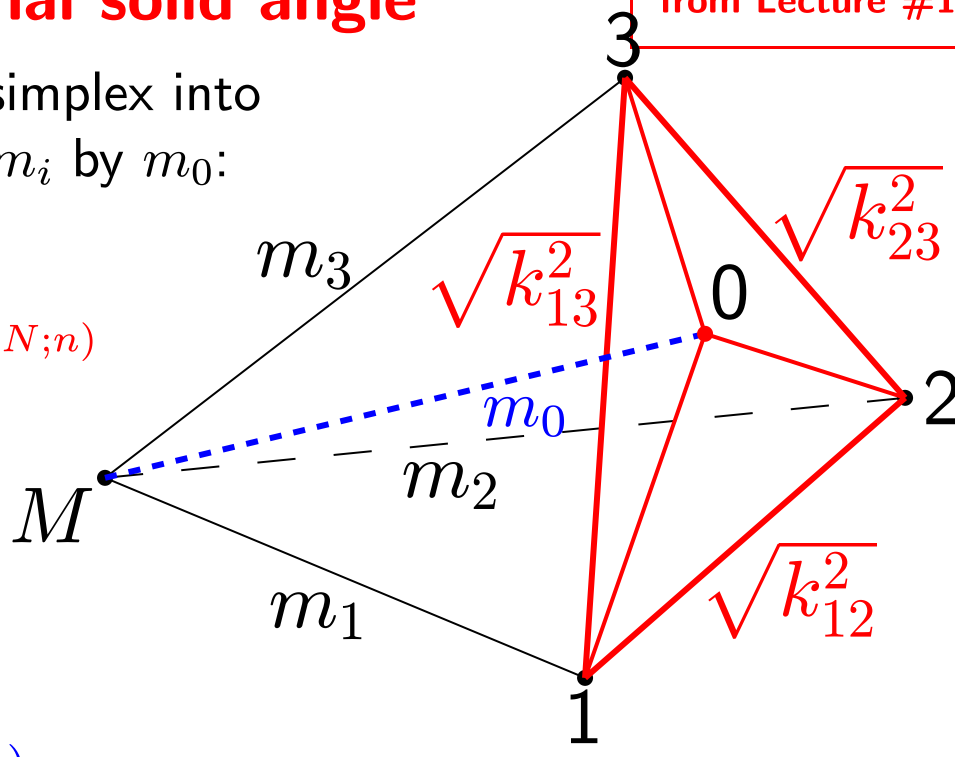
• ready for splitting!

Splitting the N-dimensional solid angle

from Lecture #1

Let us use m_0 to split the basic N-dimensional simplex into N rectangular ones, each time replacing one of m_i by m_0:

$$\Omega_i^{(N;n)} = \int_{\Omega_i^{(N)}} \!\!\ldots\!\int \frac{\mathrm{d}\Omega_N}{\cos^{n-N}\theta}\,, \qquad \Omega^{(N;n)} = \sum_{i=1}^{N} \Omega_i^{(N;n)}$$



An example: $N=3$

Each of the resulting integrals can be associated with a one-loop N-point integral

$$J^{(N)}(n;1,\ldots,1) = \sum_{i=1}^{N} \frac{V_i^{(N)}}{V^{(N)}}\, J_i^{(N)}(n;1,\ldots,1)$$

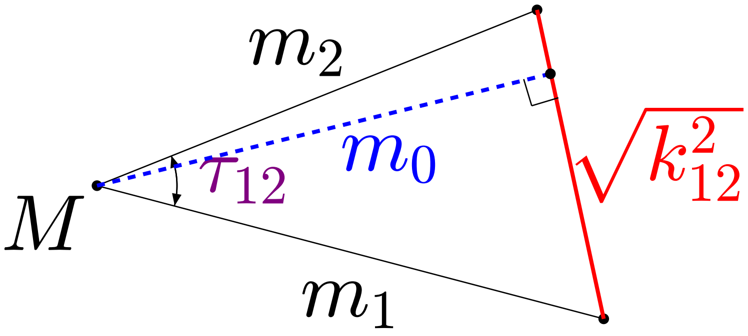
In $J_i^{(N)}$ the internal masses are $m_1,\ldots,m_{i-1},m_0,m_{i+1},\ldots,m_N$, and the squared momenta are k_{jl}^2 (if $j\neq i$ and $l\neq i$), $m_l^2-m_0^2$ (if $j=i$), $m_j^2-m_0^2$ (if $l=i$).

In terms of $F_i^{(N)}$,

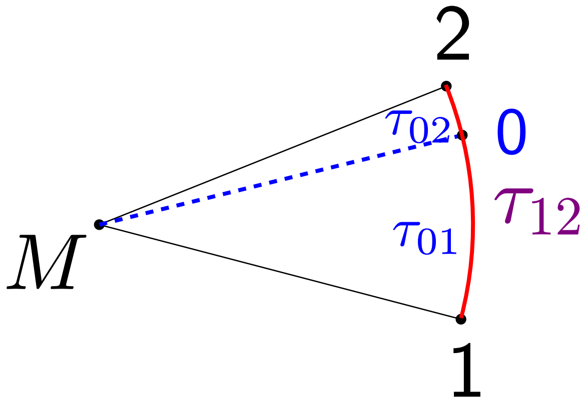
$$J^{(N)}(n;1,\ldots,1) = \frac{1}{\Lambda^{(N)}} \left(\prod m_i^2\right) \sum_{i=1}^{N} \frac{F_i^{(N)}}{m_i^2}\, J_i^{(N)}(n;1,\ldots,1)$$

See more examples below...

Two-point function, geometrical approach



the basic triangle

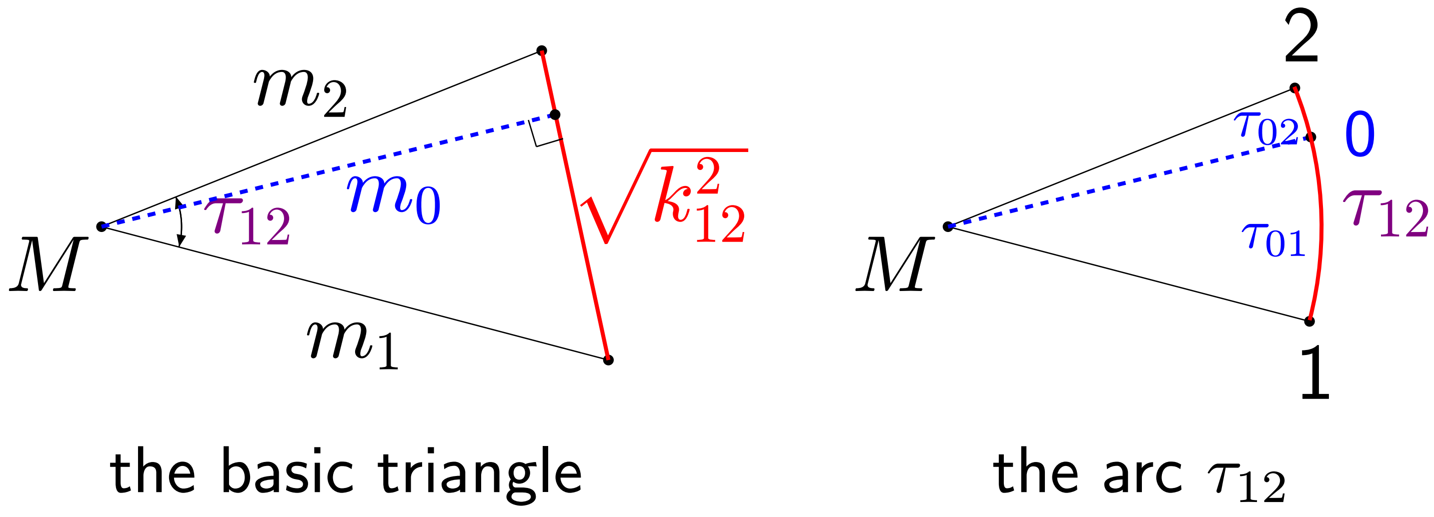


the arc τ_{12}

$$\cos\tau_{12} \equiv c_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1 m_2}, \quad D^{(2)} = 1 - c_{12}^2 = \sin^2\tau_{12}, \quad \Lambda^{(2)} = k_{12}^2,$$

$$m_0 = m_1 m_2 \sqrt{\frac{D^{(2)}}{\Lambda^{(2)}}} = \frac{m_1 m_2 \sin\tau_{12}}{\sqrt{k_{12}^2}}, \qquad \cos\tau_{0i} = \frac{m_0}{m_i}, \quad \tau_{01} + \tau_{02} = \tau_{12}.$$

Two-point function in two dimensions

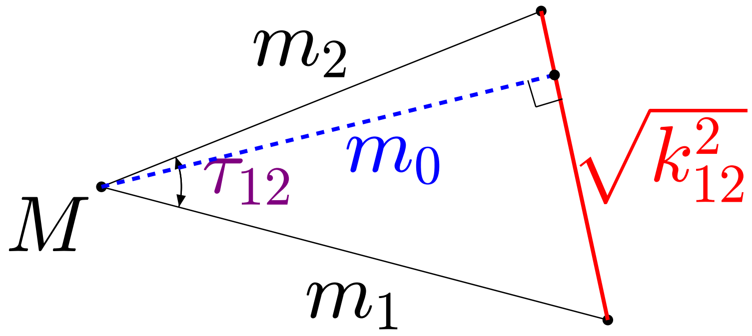


$$\cos\tau_{12} \equiv c_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1 m_2}, \quad D^{(2)} = 1 - c_{12}^2 = \sin^2\tau_{12}, \quad \Lambda^{(2)} = k_{12}^2,$$

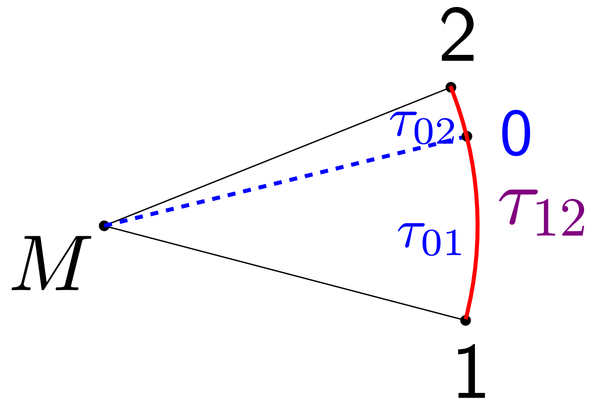
$$m_0 = m_1 m_2 \sqrt{\frac{D^{(2)}}{\Lambda^{(2)}}} = \frac{m_1 m_2 \sin\tau_{12}}{\sqrt{k_{12}^2}}, \qquad \cos\tau_{0i} = \frac{m_0}{m_i}, \quad \tau_{01} + \tau_{02} = \tau_{12}.$$

$$J^{(2)}(2;1,1) = \frac{\mathrm{i}\pi}{m_1 m_2} \frac{\Omega^{(2)}}{\sqrt{D^{(2)}}} = \frac{\mathrm{i}\pi}{m_1 m_2} \frac{\tau_{12}}{\sin\tau_{12}}$$

Two-point function in three dimensions



the basic triangle



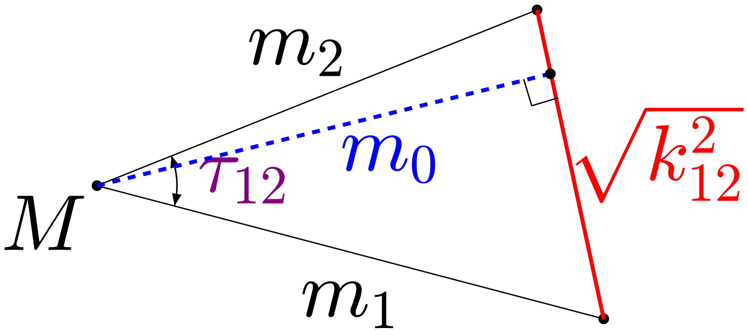
the arc τ_{12}

$$J^{(2)}(3;1,1) = \frac{\mathrm{i}\pi^2}{\sqrt{k_{12}^2}} \left\{ \Omega_1^{(2;3)} + \Omega_2^{(2;3)} \right\} , \quad \text{with} \quad \Omega_i^{(2;3)} = \int\limits_0^{\tau_{0i}} \frac{\mathrm{d}\theta}{\cos\theta} = \ln\left(\frac{1+\sin\tau_{0i}}{1-\sin\tau_{0i}} \right)$$

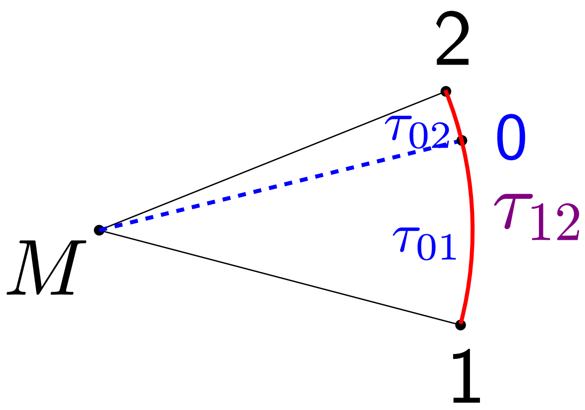
Combining the logarithms, we get

$$J^{(2)}(3;1,1) = \frac{\mathrm{i}\pi^2}{\sqrt{k_{12}^2}} \ln\left(\frac{m_1+m_2+\sqrt{k_{12}^2}}{m_1+m_2-\sqrt{k_{12}^2}} \right)$$

Two-point function in an arbitrary dimension n



the basic triangle



the arc τ_{12}

$$J^{(2)}(n;1,1) = \mathrm{i}\pi^{n/2}\Gamma\left(2-\frac{n}{2}\right) \frac{m_0^{n-3}}{\sqrt{k_{12}^2}} \left\{\Omega_1^{(2;n)} + \Omega_2^{(2;n)}\right\} ,$$

with

$$\Omega_i^{(2;n)} = \int\limits_0^{\tau_{0i}} \frac{\mathrm{d}\theta}{\cos^{n-2}\theta}$$

As expected, it has an "ultraviolet" singularity as $n \to 4$.

Two-point function, hypergeometric representation

$$J^{(2)}(n;1,1|k_{12}^2;m_1,m_2) = \mathrm{i}\pi^{n/2}\Gamma\left(2-\frac{n}{2}\right)\frac{m_0^{n-3}}{\sqrt{k_{12}^2}}\left\{\Omega_1^{(2;n)}+\Omega_2^{(2;n)}\right\}$$

with

$$\Omega_i^{(2;n)} = \int\limits_0^{\tau_{0i}} \frac{\mathrm{d}\theta}{\cos^{n-2}\theta} = \tan\tau_{0i}\ \ (\cos\tau_{0i})^{4-n}\ {}_2F_1\left(\left.\begin{array}{c}1, 2-n/2\\ 3/2\end{array}\right|\sin^2\tau_{0i}\right)$$

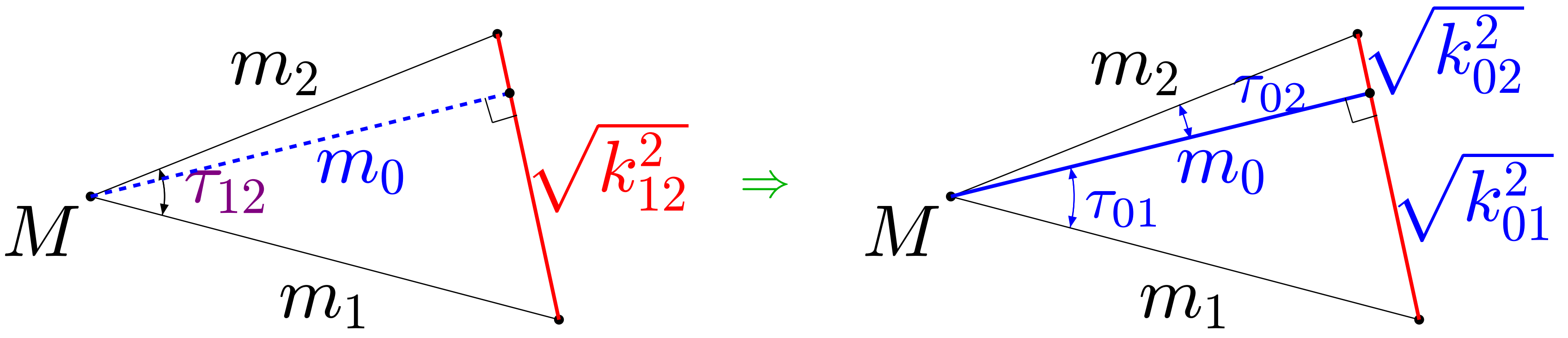
$$= \tan\tau_{0i}\ {}_2F_1\left(\left.\begin{array}{c}1/2, 2-n/2\\ 3/2\end{array}\right|-\tan^2\tau_{0i}\right),$$

where

$$c_{12} = \frac{m_1^2+m_2^2-k_{12}^2}{2m_1m_2}, \quad D^{(2)} = 1-c_{12}^2 = \sin^2\tau_{12}, \quad m_0 = m_1m_2\sqrt{\frac{D^{(2)}}{k_{12}^2}},$$

$$\cos\tau_{0i} = \frac{m_0}{m_i}, \quad \tan\tau_{0i} = \frac{\sqrt{k_{0i}^2}}{m_0}, \quad \tau_{01}+\tau_{02} = \tau_{12}.$$

Two-point function, splitting the basic triangle



$$k_{01}^2 = \frac{(k_{12}^2 + m_1^2 - m_2^2)^2}{4k_{12}^2}, \qquad k_{02}^2 = \frac{(k_{12}^2 - m_1^2 + m_2^2)^2}{4k_{12}^2} \quad \Rightarrow \quad \sqrt{k_{01}^2} + \sqrt{k_{02}^2} = \sqrt{k_{12}^2}$$

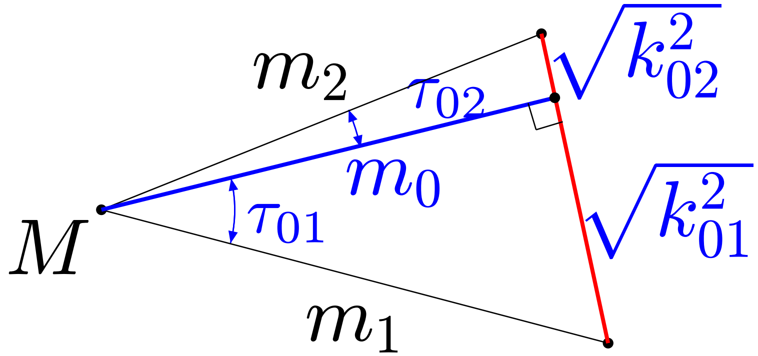
$$F_1^{(2)} = \frac{k_{12}^2 - m_1^2 + m_2^2}{2m_2^2}, \qquad F_2^{(2)} = \frac{k_{12}^2 + m_1^2 - m_2^2}{2m_1^2} \quad \Rightarrow \quad \frac{F_1^{(2)}}{m_1^2} + \frac{F_2^{(2)}}{m_2^2} = \frac{k_{12}^2}{m_1^2 m_2^2} = \frac{\Lambda^{(2)}}{m_1^2 m_2^2}$$

$$J^{(2)}\left(n;1,1|k_{12}^2;m_1,m_2\right) = \frac{1}{2k_{12}^2}\Big\{(k_{12}^2 + m_1^2 - m_2^2) J^{(2)}\left(n;1,1|k_{01}^2;m_1,m_0\right)$$
$$+(k_{12}^2 - m_1^2 + m_2^2) J^{(2)}\left(n;1,1|k_{02}^2;m_2,m_0\right)\Big\}$$

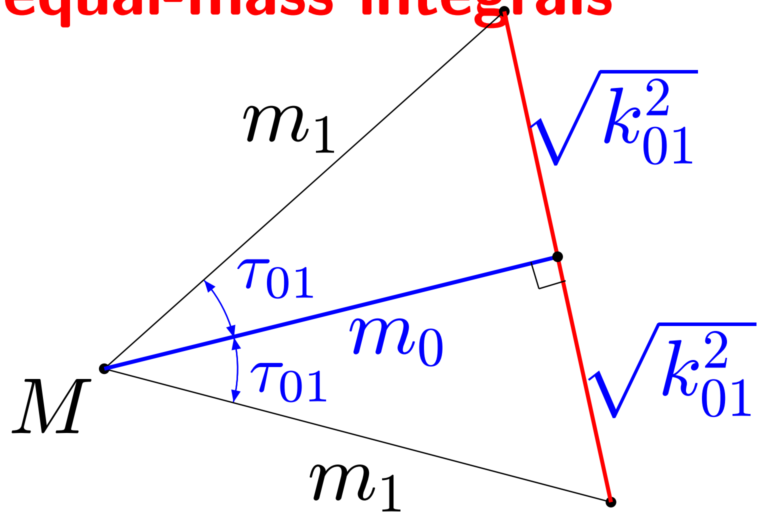
This is an example of a functional relation between integrals with different momenta and masses, similar to those described in

O.V. Tarasov, Phys.Lett. **B670** (2008) 67

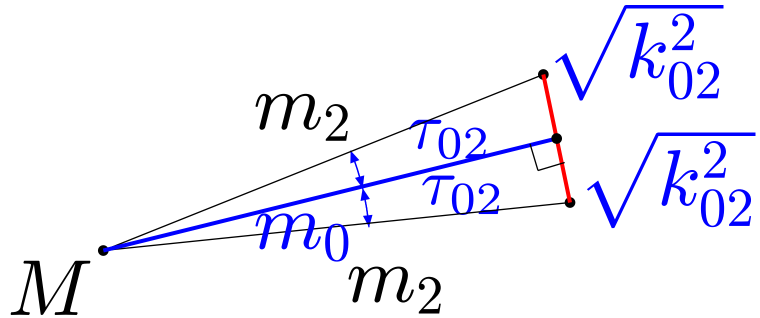
Two-point function, reduction to equal-mass integrals



$\Rightarrow \quad \frac{1}{2}$



$+ \quad \frac{1}{2}$



$\Rightarrow$ equal-mass functions

$$J^{(2)}\left(n;1,1|k_{12}^2;m_1,m_2\right) = \frac{1}{2k_{12}^2}\Big\{(k_{12}^2+m_1^2-m_2^2)J^{(2)}\left(n;1,1|4k_{01}^2;m_1,m_1\right)$$

$$+(k_{12}^2-m_1^2+m_2^2)J^{(2)}\left(n;1,1|4k_{02}^2;m_2,m_2\right)\Big\}$$

with $\quad k_{01}^2 = \dfrac{(k_{12}^2+m_1^2-m_2^2)^2}{4k_{12}^2}, \qquad k_{02}^2 = \dfrac{(k_{12}^2-m_1^2+m_2^2)^2}{4k_{12}^2}$

Two-point function: number of variables and the quadratic form

Number of dimensionless variables, before and after splitting:

in $J^{(2)}\left(n;1,1\middle|k_{12}^2;m_1,m_2\right)$: $\quad 3-1(\text{dimension})=2$

in $J^{(2)}\left(n;1,1\middle|k_{01}^2;m_1,m_0\right)$: $\quad 3-1(k_{01}^2=m_1^2-m_0^2)-1(\text{dimension})=1$

Quadratic form in Feynman parametric integral:

in $J^{(2)}\left(n;1,1\middle|k_{12}^2;m_1,m_2\right)$: $\quad [\alpha_1\alpha_2k_{12}^2-\alpha_1m_1^2-\alpha_2m_2^2]$

in $J^{(2)}\left(n;1,1\middle|k_{01}^2;m_1,m_0\right)$: $\quad [\alpha_1\alpha_2k_{01}^2-\alpha_1m_1^2-\alpha_2m_0^2]=-[\alpha_1^2k_{01}^2+m_0^2]$

Result in arbitrary dimension:

$$J^{(2)}\left(n;1,1\middle|k_{01}^2;m_1,m_0\right) = \mathrm{i}\pi^{n/2}\Gamma(2-n/2)\int\limits_0^1\int\limits_0^1\frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\delta(\alpha_1+\alpha_2-1)}{[\alpha_1^2k_{01}^2+m_0^2]^{2-n/2}}$$

$$= \mathrm{i}\pi^{n/2}\frac{\Gamma(2-n/2)}{(m_0^2)^{2-n/2}}\,{}_2F_1\left(\begin{matrix}1/2,\;2-n/2\\3/2\end{matrix}\middle|-\frac{k_{01}^2}{m_0^2}\right)$$

Two-point function in $n = 4 - 2\varepsilon$ dimensions, ε-expansion

$$J^{(2)}(4-2\varepsilon;1,1) \quad = \quad \mathrm{i}\pi^{2-\varepsilon}\, \frac{\Gamma(1+\varepsilon)}{2(1-2\varepsilon)} \left\{ \frac{m_1^{-2\varepsilon}+m_2^{-2\varepsilon}}{\varepsilon} + \frac{m_1^2-m_2^2}{\varepsilon k_{12}^2}(m_1^{-2\varepsilon}-m_2^{-2\varepsilon}) \right.$$

$$\left. -\frac{\left[\Delta(m_1^2,m_2^2,k_{12}^2)\right]^{1/2-\varepsilon}}{(k_{12}^2)^{1-\varepsilon}} \sum_{j=0}^{\infty} \frac{(2\varepsilon)^j}{j!} \left[\mathrm{Ls}_{j+1}(\pi-2\tau_{01}) + \mathrm{Ls}_{j+1}(\pi-2\tau_{02}) - 2\mathrm{Ls}_{j+1}(\pi)\right] \right\}$$

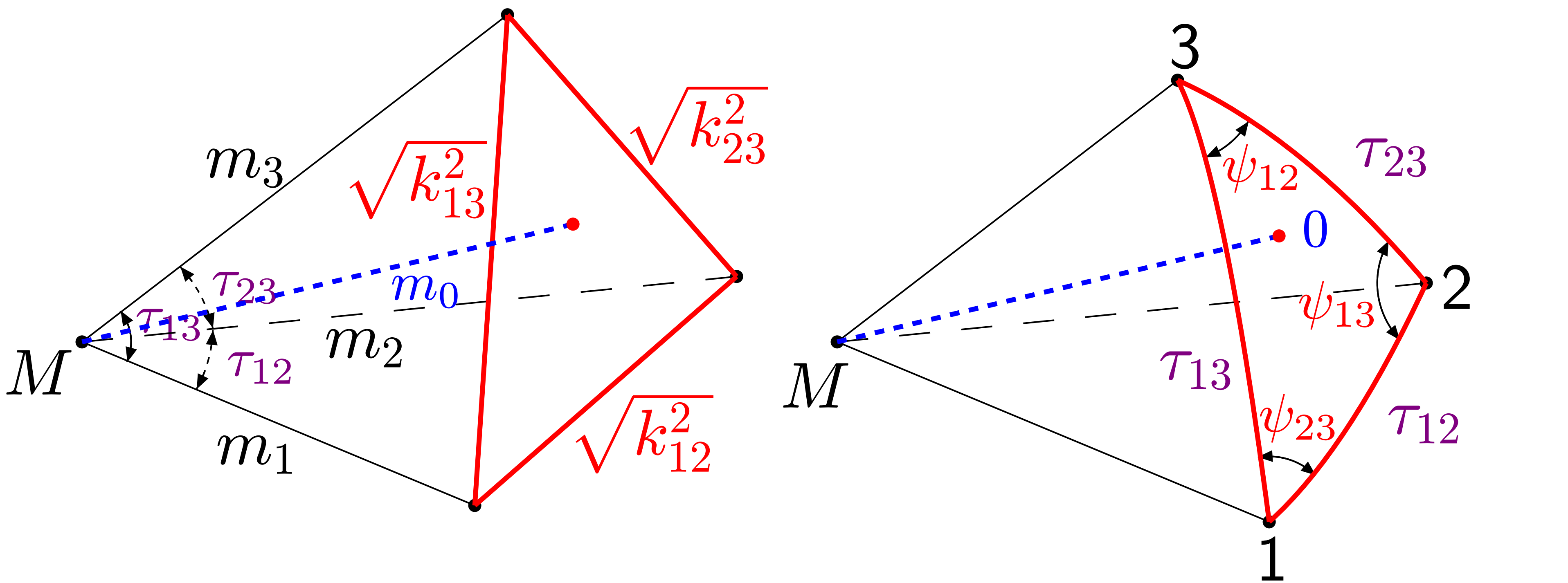
where $\Delta(m_1^2,m_2^2,k_{12}^2) = 4m_1^2m_2^2 D^{(2)} = 4m_1^2m_2^2\sin^2\tau_{12}$,
so that $\frac{1}{4}\sqrt{\Delta}$ is the triangle area.

These results are represented in terms of the log-sine integrals,

$$\mathsf{Ls}_j(\theta) = -\int_0^\theta \mathrm{d}\phi\ \ln^{j-1}\left|2\sin\frac{\phi}{2}\right| .$$

Analytic continuation $\Rightarrow$ Nielsen polylogarithms (to all orders)

Three-point function: geometrical approach



the basic tetrahedron the solid angle

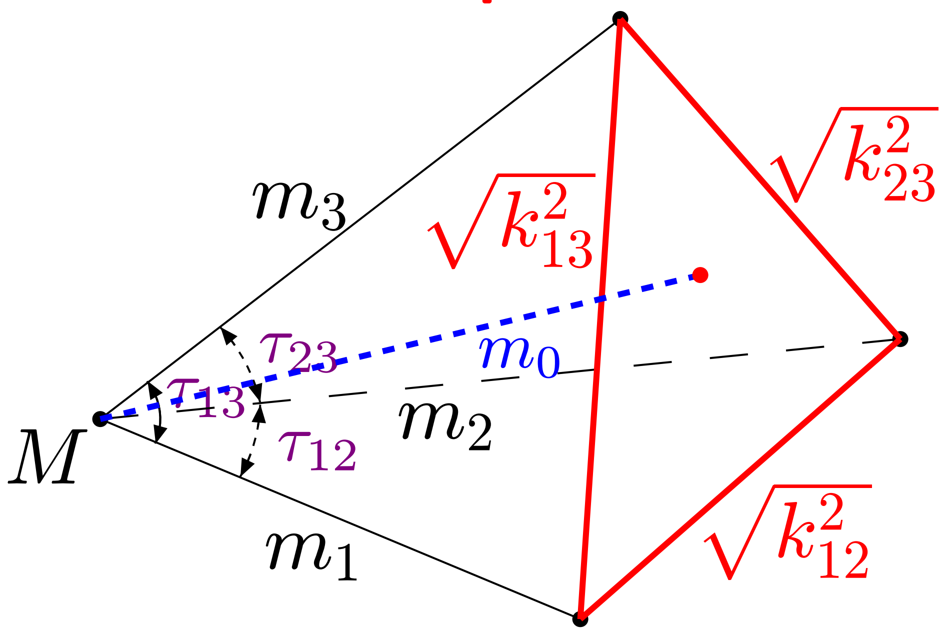
Tetrahedron volume : $V^{(3)} = \frac{1}{6}\, m_1 m_2 m_3 \sqrt{D^{(3)}}$, where $D^{(3)} = \begin{vmatrix} 1 & c_{12} & c_{13} \\ c_{12} & 1 & c_{23} \\ c_{13} & c_{23} & 1 \end{vmatrix}$

Red triangle area: $\overline{V}_0^{(2)} = \frac{1}{2}\sqrt{\Lambda^{(3)}}$,

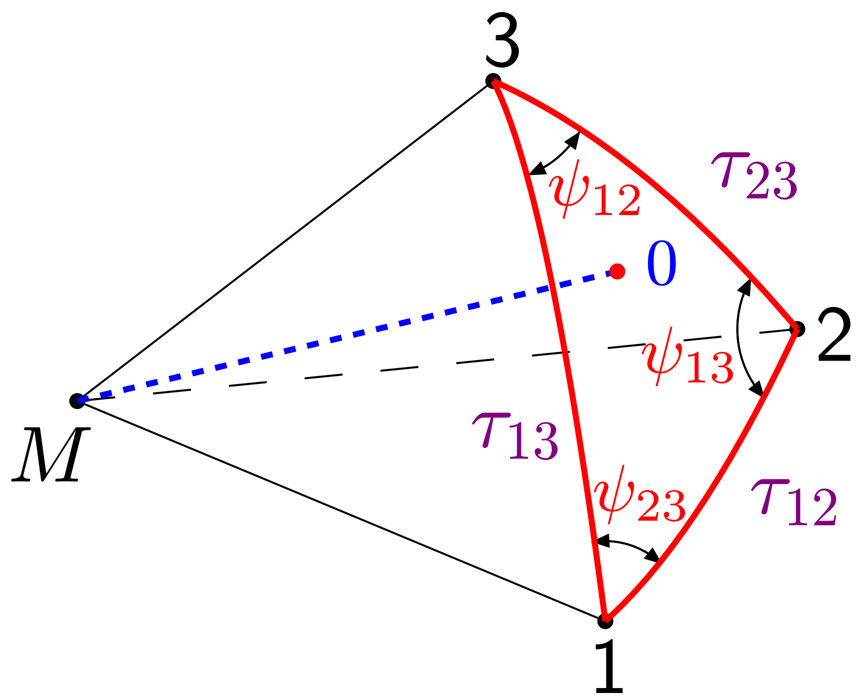
with $\Lambda^{(3)} = \frac{1}{4}\left[2k_{12}^2 k_{13}^2 + 2k_{13}^2 k_{23}^2 + 2k_{23}^2 k_{12}^2 - (k_{12}^2)^2 - (k_{13}^2)^2 - (k_{23}^2)^2\right] = -\frac{1}{4}\lambda(k_{12}^2, k_{13}^2, k_{23}^2)$,

where $\lambda(x,y,z)$ is the Källen function.

Three-point function: geometrical approach



the basic tetrahedron

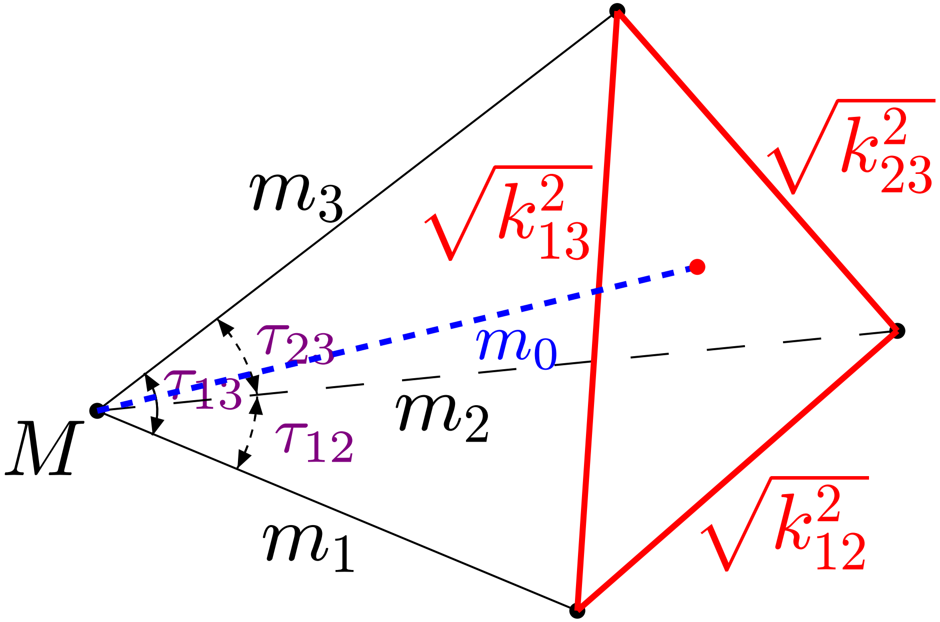


the solid angle

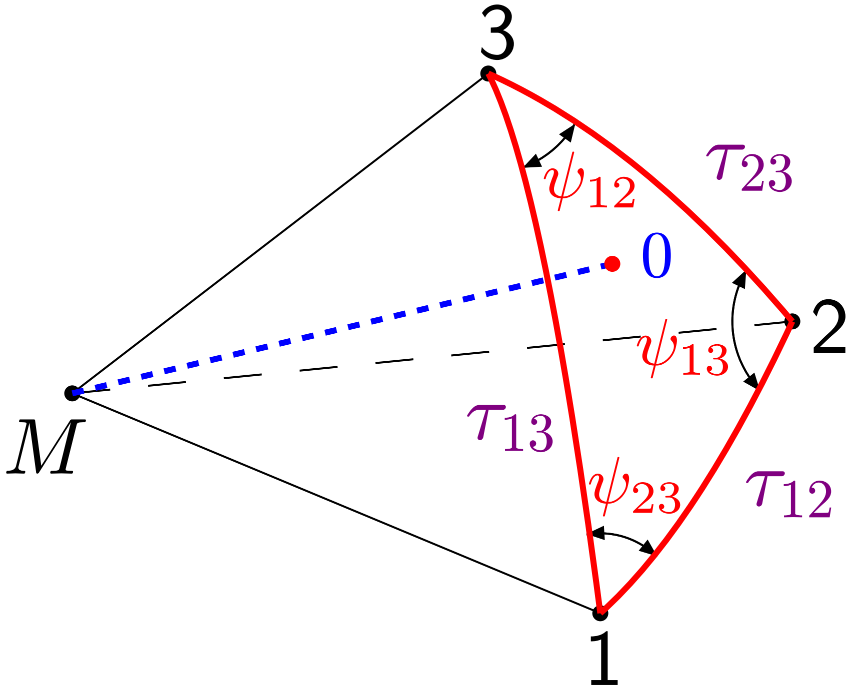
Dihedral angles ψ_{12}, ψ_{13} and ψ_{23}:

$$\cos\psi_{12} = \frac{\cos\tau_{12} - \cos\tau_{13}\cos\tau_{23}}{\sin\tau_{13}\sin\tau_{23}}, \qquad \sin\psi_{12} = \frac{\sqrt{D^{(3)}}}{\sin\tau_{13}\sin\tau_{23}}$$

Three-point function in three dimensions



the basic tetrahedron



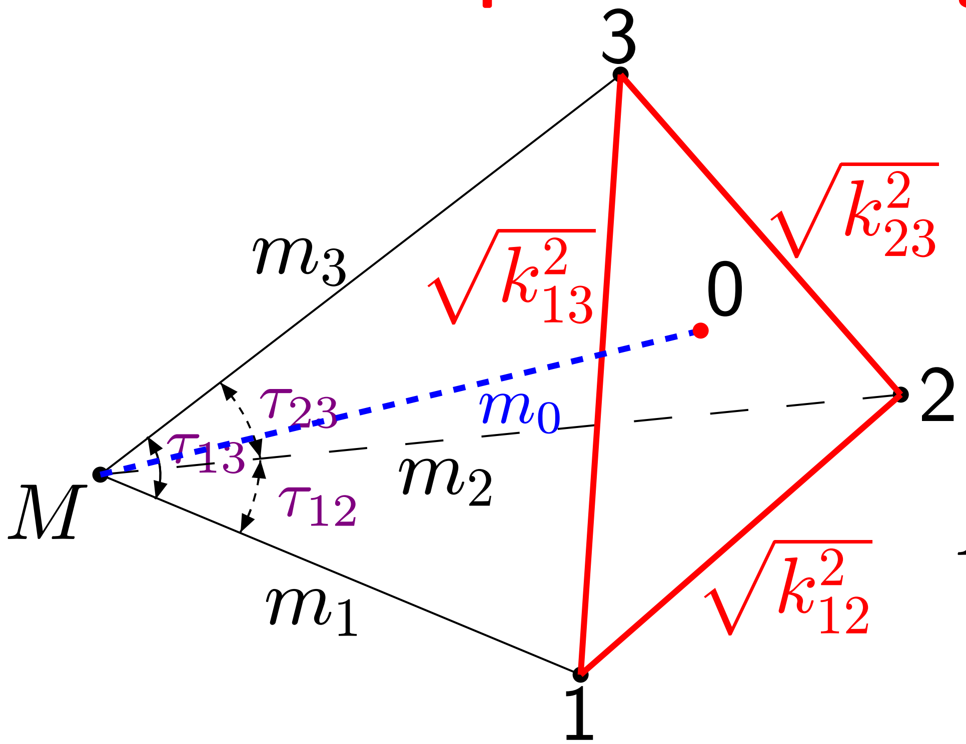
the solid angle

The area of spherical triangle ("spherical excess"): $\Omega^{(3)} = \psi_{12} + \psi_{23} + \psi_{31} - \pi$.

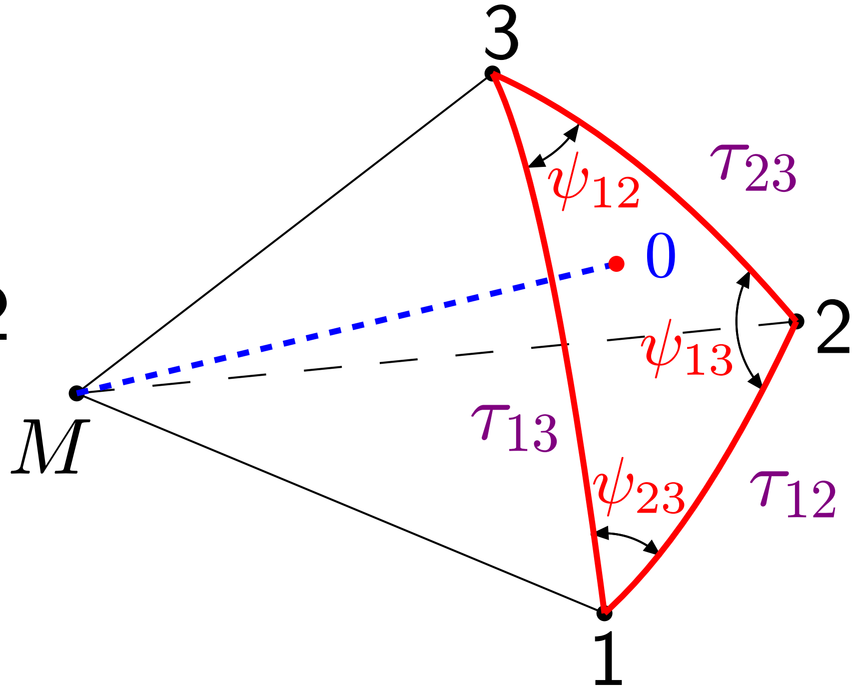
$$J^{(3)}(3;1,1,1) = -\frac{\mathrm{i}\pi^2}{2m_1 m_2 m_3} \frac{\Omega^{(3)}}{\sqrt{D^{(3)}}} = -\frac{\mathrm{i}\pi^2}{m_1 m_2 m_3 \sqrt{D^{(3)}}} \arctan\left(\frac{\sqrt{D^{(3)}}}{1 + c_{12} + c_{13} + c_{23}} \right)$$

Compare with: B. G. Nickel, J. Math. Phys. **19** (1978) 542

Three-point function: geometrical approach

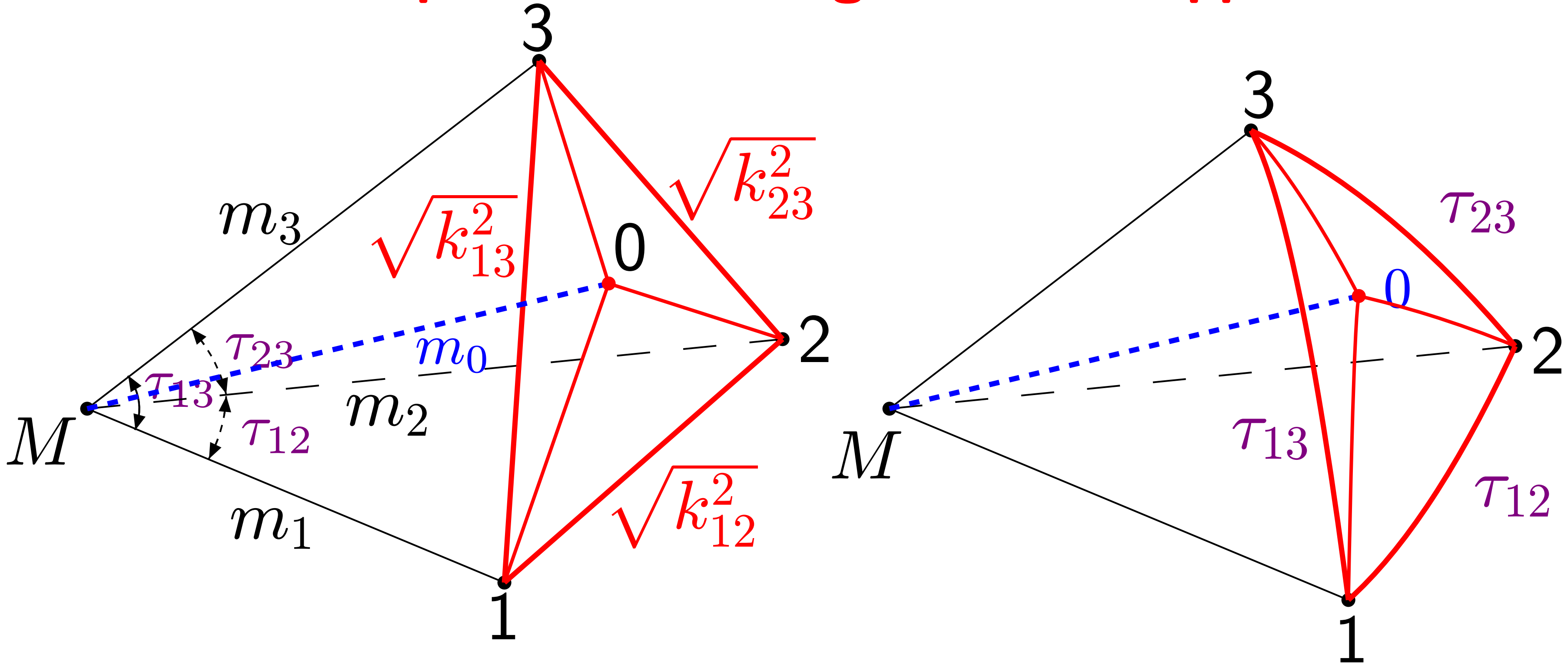


the basic tetrahedron



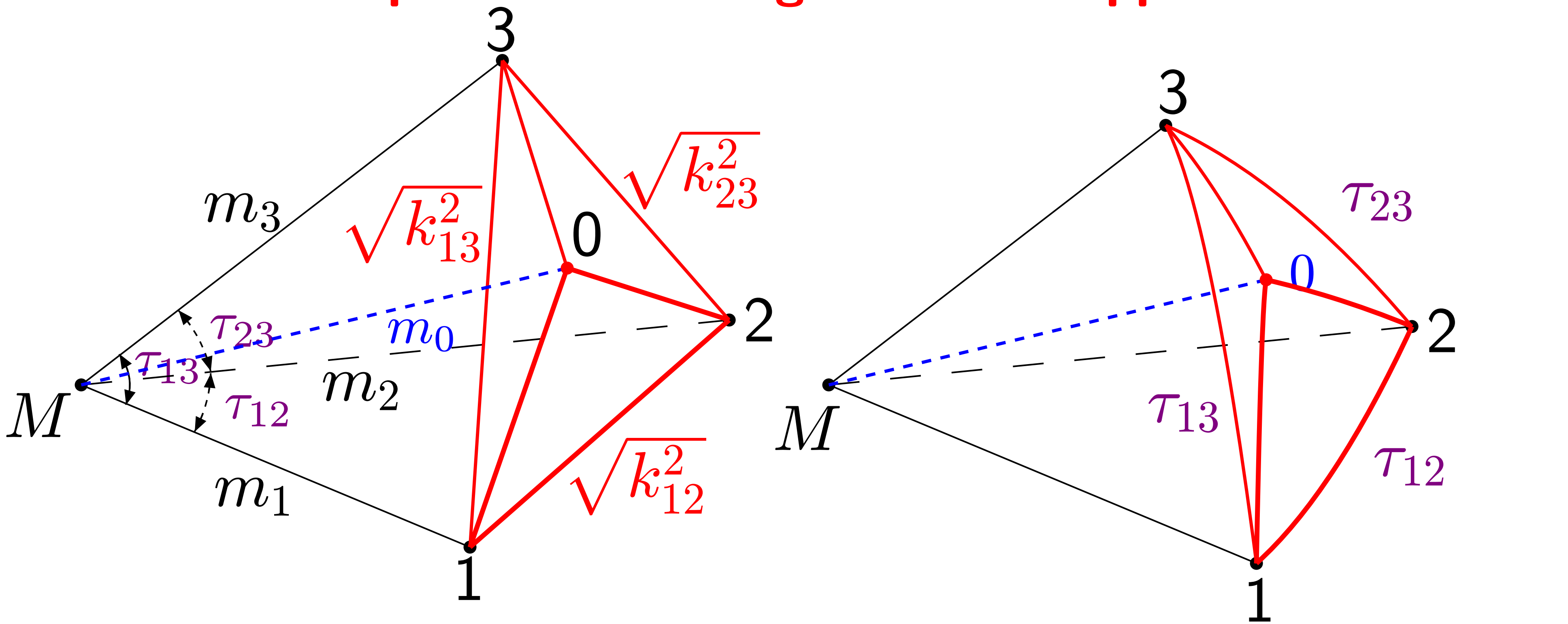
the solid angle

Three-point function: geometrical approach



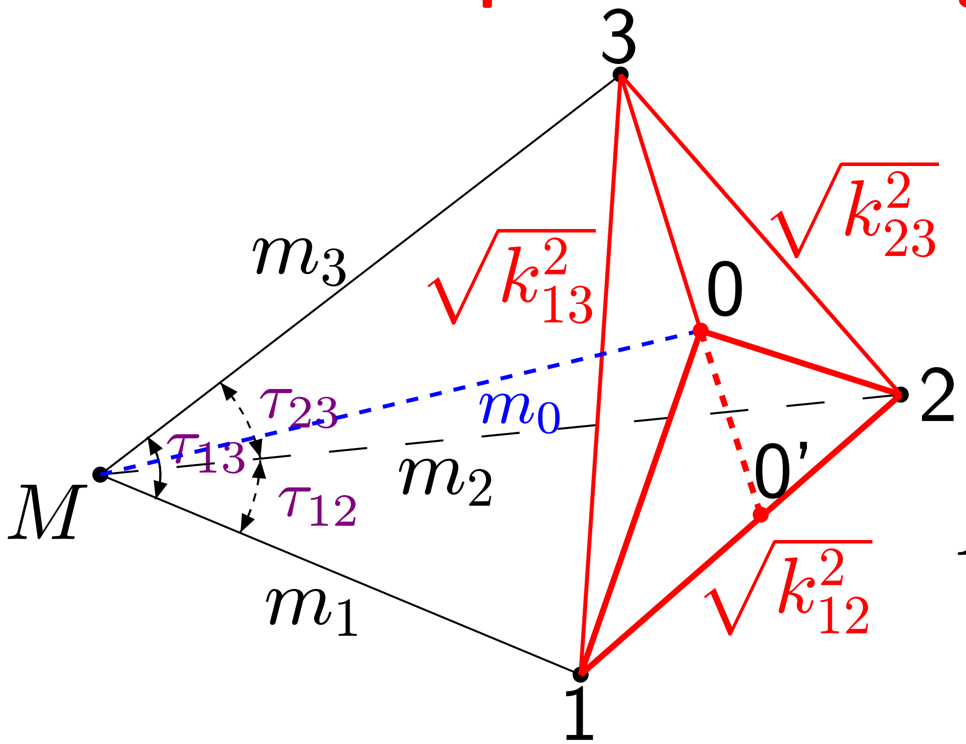
the basic tetrahedron the solid angle

Three-point function: geometrical approach

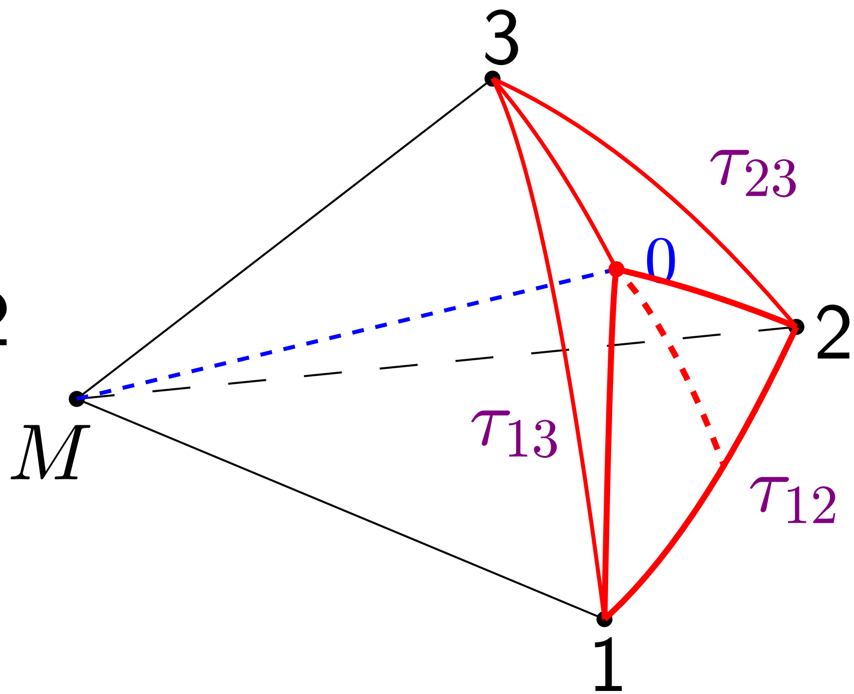


the basic tetrahedron the solid angle

Three-point function: geometrical approach



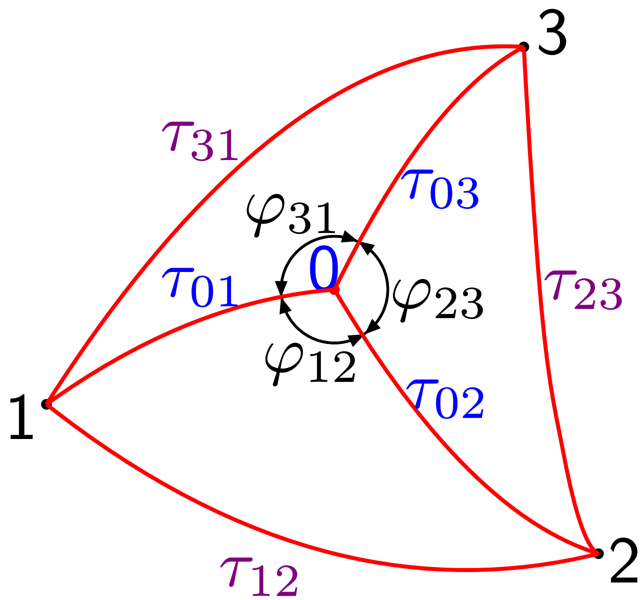
the basic tetrahedron



the solid angle

Three-point function: splitting the solid angle

Relation to the angles associated with a spherical (or hyperbolic) triangle:



$$\varphi_{12} + \varphi_{23} + \varphi_{31} = 2\pi$$

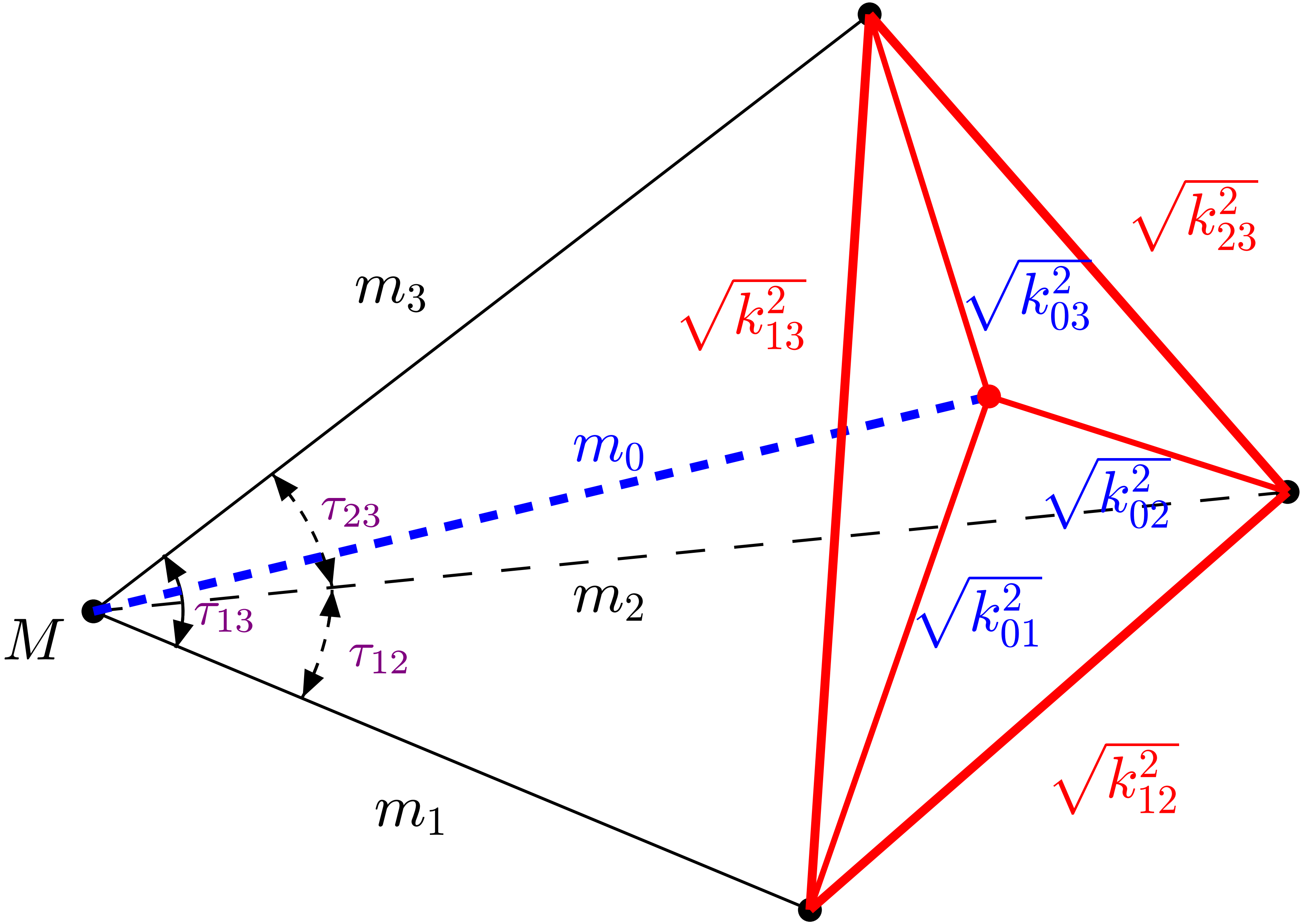
$$\cos\tau_{12} = \frac{m_1^2+m_2^2-k_{12}^2}{2m_1m_2}, \text{ etc.}$$

$$\cos\tau_{0i} = \frac{m_0}{m_i} \qquad (i=1,2,3)$$

$$m_0 = m_1 m_2 m_3 \sqrt{\frac{D^{(3)}}{\Lambda^{(3)}}}$$

$$D^{(3)} = \begin{vmatrix} 1 & c_{12} & c_{13} \\ c_{12} & 1 & c_{23} \\ c_{13} & c_{23} & 1 \end{vmatrix}, \qquad \Lambda^{(3)} = \tfrac{1}{4}\left[2k_{12}^2k_{13}^2+2k_{13}^2k_{23}^2+2k_{23}^2k_{12}^2-(k_{12}^2)^2-(k_{13}^2)^2-(k_{23}^2)^2\right]$$

Three-point function: the basic tetrahedron



Three-point function: splitting the basic tetrahedron

$$J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right)$$

$$= \frac{m_1^2 m_2^2 m_3^2}{\Lambda^{(3)}}\left\{\frac{F_1^{(3)}}{m_1^2}J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{03}^2,k_{02}^2;m_0,m_2,m_3\right)\right.$$

$$+\frac{F_2^{(3)}}{m_2^2}J^{(3)}\left(n;1,1,1\middle|k_{03}^2,k_{13}^2,k_{01}^2;m_1,m_0,m_3\right)$$

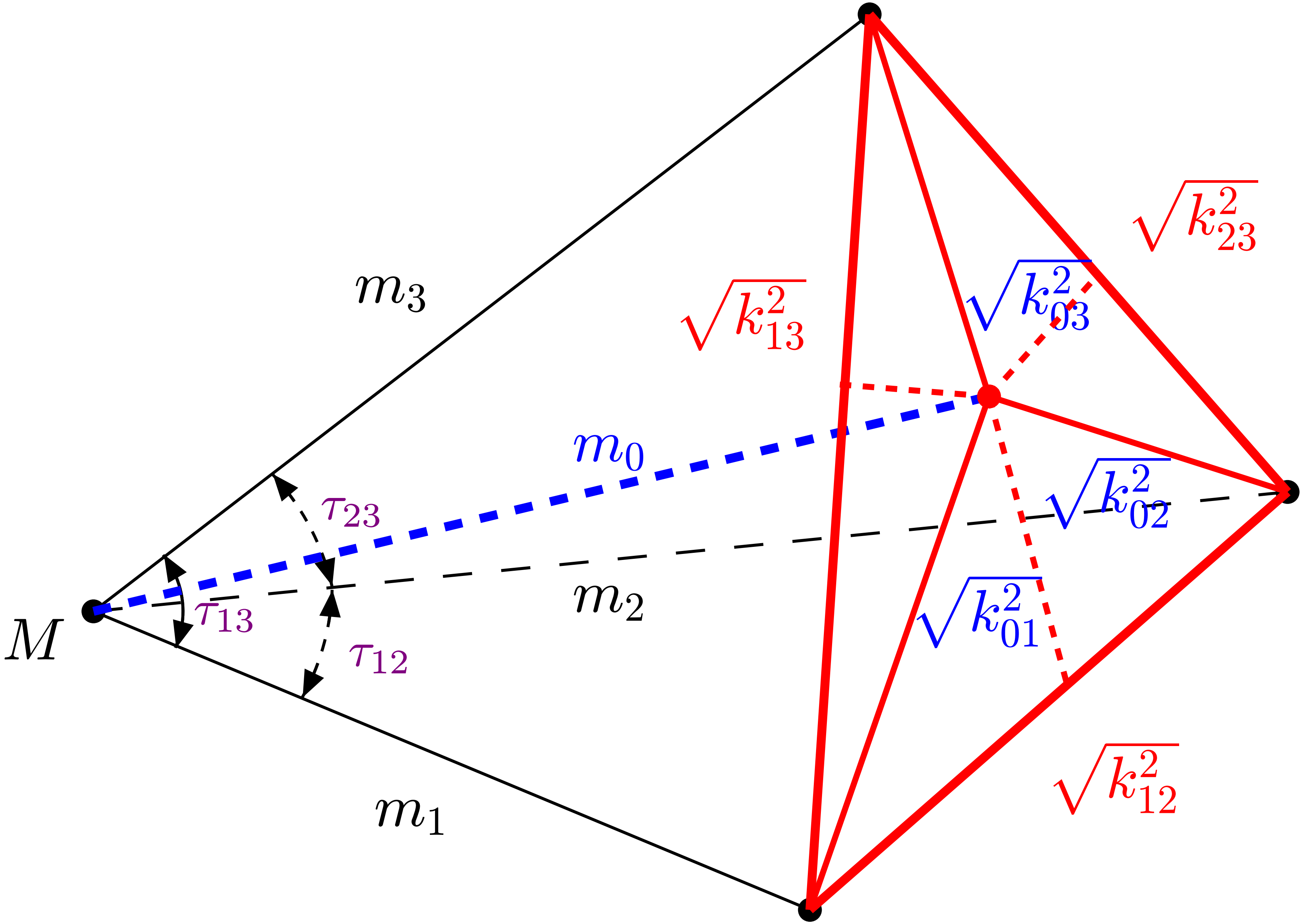
$$\left.+\frac{F_3^{(3)}}{m_3^2}J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right)\right\}$$

with $\quad k_{01}^2 = m_1^2 - m_0^2, \qquad k_{02}^2 = m_2^2 - m_0^2, \qquad k_{03}^2 = m_3^2 - m_0^2,$

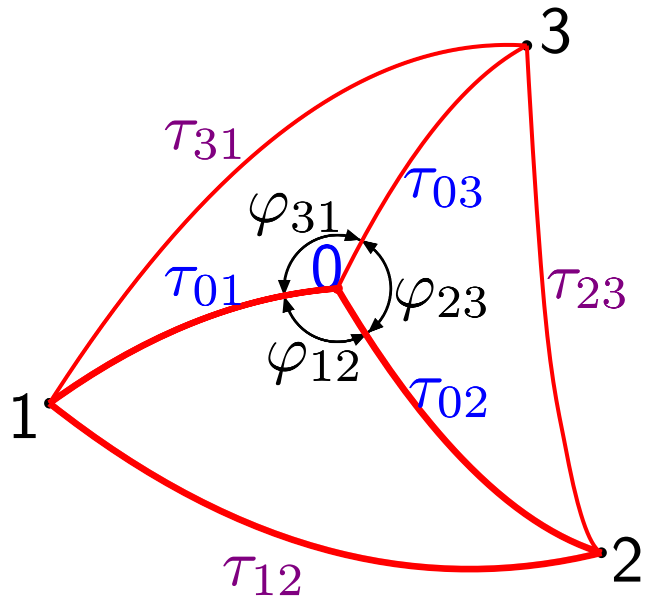
$F_3^{(3)} = \frac{1}{4m_1^2 m_2^2}\left[k_{12}^2\left(k_{13}^2+k_{23}^2-k_{12}^2+m_1^2+m_2^2-2m_3^2\right)-\left(m_1^2-m_2^2\right)\left(k_{13}^2-k_{23}^2\right)\right]$, etc.

$$\frac{F_1^{(3)}}{m_1^2}+\frac{F_2^{(3)}}{m_2^2}+\frac{F_3^{(3)}}{m_3^2}=\frac{\Lambda^{(3)}}{m_1^2 m_2^2 m_3^2}$$

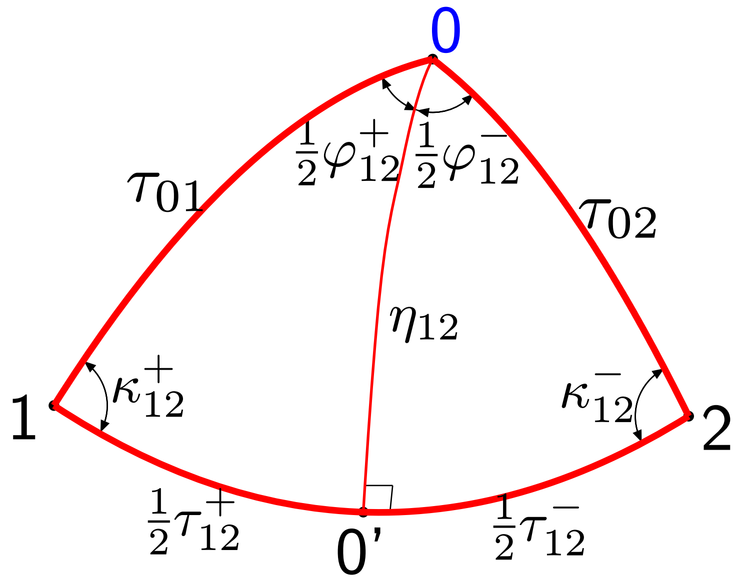
Three-point function: further splitting



Three-point function: further splitting (continued)



Choose one of the spherical triangles, 012, and split it into two parts by dropping a perpendicular η_{12}



Three-point function: further splitting (continued)

Spherical trigonometry for one of the three spherical triangles ($\frac{1}{2}(\varphi_{12}^{+}+\varphi_{12}^{-}) = \varphi_{12}$):

$$\cos\left(\tfrac{1}{2}\tau_{12}^{+}\right) = \frac{\cos\tau_{01}}{\cos\eta_{12}} = \frac{m_2\sin\tau_{12}}{\sqrt{k_{12}^2}},$$

$$\cos\left(\tfrac{1}{2}\tau_{12}^{-}\right) = \frac{\cos\tau_{02}}{\cos\eta_{12}} = \frac{m_1\sin\tau_{12}}{\sqrt{k_{12}^2}},$$

$$\sin\left(\tfrac{1}{2}\tau_{12}^{+}\right) = \sin\tau_{01}\sin\left(\tfrac{1}{2}\varphi_{12}^{+}\right) = \frac{m_1^2-m_2^2+k_{12}^2}{2m_1\sqrt{k_{12}^2}},$$

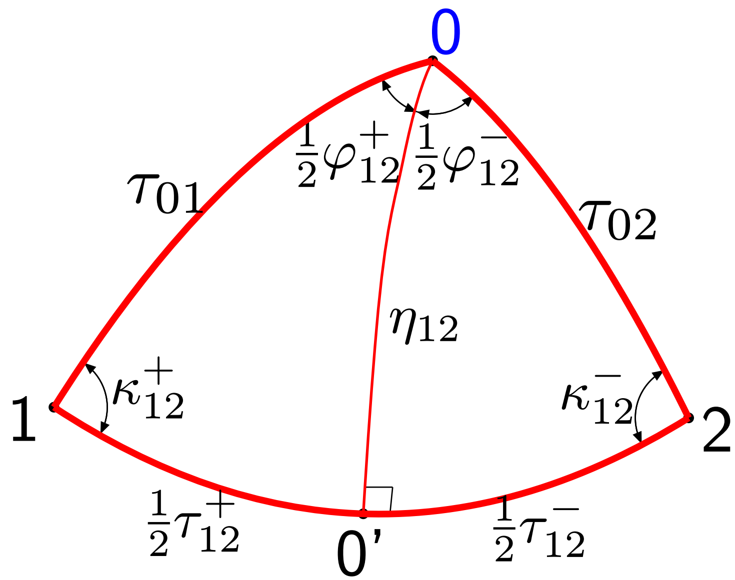
$$\sin\left(\tfrac{1}{2}\tau_{12}^{-}\right) = \sin\tau_{02}\sin\left(\tfrac{1}{2}\varphi_{12}^{-}\right) = \frac{m_2^2-m_1^2+k_{12}^2}{2m_2\sqrt{k_{12}^2}},$$

$$\cos\left(\tfrac{1}{2}\varphi_{12}^{+}\right) = \frac{\tan\eta_{12}}{\tan\tau_{01}}, \qquad \cos\left(\tfrac{1}{2}\varphi_{12}^{-}\right) = \frac{\tan\eta_{12}}{\tan\tau_{02}},$$

$$\tan\left(\tfrac{1}{2}\tau_{12}^{\pm}\right) = \sin\eta_{12}\tan\left(\tfrac{1}{2}\varphi_{12}^{\pm}\right),$$

$$\cos\kappa_{12}^{\pm} = \cos\eta_{12}\sin\left(\tfrac{1}{2}\varphi_{12}^{\pm}\right),$$

$$\cos\eta_{12} = \frac{m_0\sqrt{k_{12}^2}}{m_1 m_2\sin\tau_{12}}, \qquad \sin\eta_{12} = \sin\tau_{01}\sin\kappa_{12}^{+} = \sin\tau_{02}\sin\kappa_{12}^{-}.$$



Three-point function in an arbitrary dimension n

$$J^{(3)}(n;1,1,1) = -\frac{\mathrm{i}\pi^{n/2}}{\sqrt{\Lambda^{(3)}}}\,\Gamma\left(3-\frac{n}{2}\right)\; m_0^{n-4}\,\Omega^{(3;n)}\;,$$

$$\begin{aligned}
\Omega^{(3;n)} = \iint\limits_{\Omega^{(3)}} \frac{\sin\theta\;\mathrm{d}\theta\;\mathrm{d}\phi}{\cos^{n-3}\theta} \quad &= \quad \Omega_1^{(3;n)} + \Omega_2^{(3;n)} + \Omega_3^{(3;n)} \\
&= \quad \omega\left(\tfrac{1}{2}\varphi_{12}^{+}, \eta_{12}\right) + \omega\left(\tfrac{1}{2}\varphi_{12}^{-}, \eta_{12}\right) \\
&\qquad +\omega\left(\tfrac{1}{2}\varphi_{23}^{+}, \eta_{23}\right) + \omega\left(\tfrac{1}{2}\varphi_{23}^{-}, \eta_{23}\right) \\
&\qquad +\omega\left(\tfrac{1}{2}\varphi_{31}^{+}, \eta_{31}\right) + \omega\left(\tfrac{1}{2}\varphi_{31}^{-}, \eta_{31}\right)\,,
\end{aligned}$$

with

$$\omega\left(\tfrac{1}{2}\varphi,\eta\right) = -\frac{1}{n-4}\int\limits_0^{\varphi/2}\mathrm{d}\phi\left[1-\left(1+\frac{\tan^2\eta}{\cos^2\phi}\right)^{(n-4)/2}\right]$$

As expected, there is no singularity as $n\to 4$ (it cancels).

Three-point function in an arbitrary dimension n (continued)

The result can be presented in terms of Appell's hypergeometric function F_1,

$$\omega\left(\tfrac{1}{2}\varphi,\eta\right) = -\frac{1}{n-4}\left[\frac{\varphi}{2} - \sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\cos^{4-n}\tau_0\; F_1\left(1,1,2-\tfrac{n}{2};\tfrac{3}{2}\middle|\;\sin^2\frac{\varphi}{2},\sin^2\frac{\tau}{2}\right)\right]$$

$$= -\frac{1}{n-4}\left[\frac{\varphi}{2} - \tan\frac{\varphi}{2}\cos^{4-n}\eta\; F_1\left(\tfrac{1}{2},1,2-\tfrac{n}{2};\tfrac{3}{2}\middle|-\tan^2\frac{\varphi}{2},-\tan^2\frac{\tau}{2}\right)\right],$$

with $\cos\tau_0 = \cos\eta\;\cos\frac{\tau}{2}$,

$$F_1(a,b,b',c|x,y) = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a)_{j+l}\,(b)_j\,(b')_l}{(c)_{j+l}}\,\frac{x^j\,y^l}{j!\,l!}$$

A.I.D., hep-th/9908032, Nucl.Instr.Meth. A559 (2006) 293

Similar functions occurred in

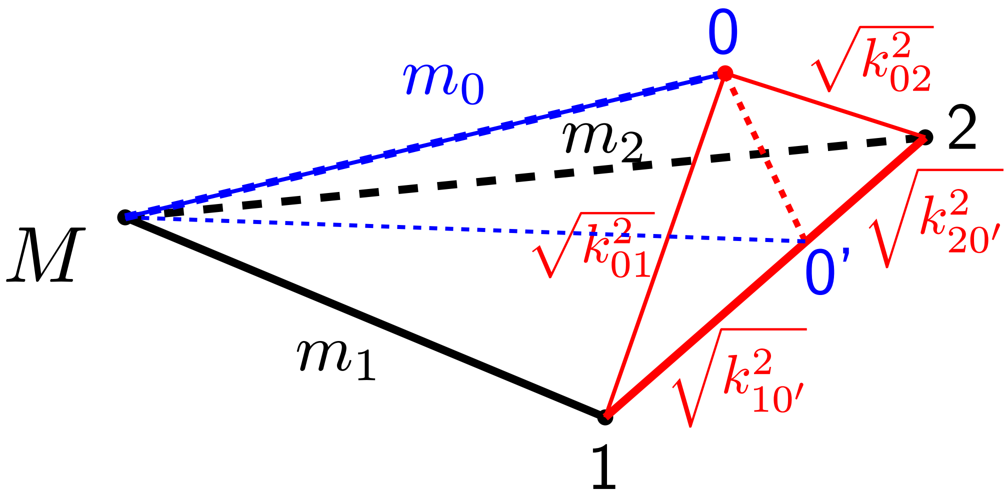
O.V. Tarasov, Nucl. Phys. B (PS) **89** (2000) 237

J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. **B672** (2003) 303

Some special cases: L.G. Cabral-Rosetti, M.A. Sanchis-Lozano, hep-ph/0206081

Reduction package: V.V. Bytev, M.Yu. Kalmykov, S.-O. Moch, Comput. Phys. Commun. **185** (2014) 3041

Three-point function: splitting of one of the three tetrahedra



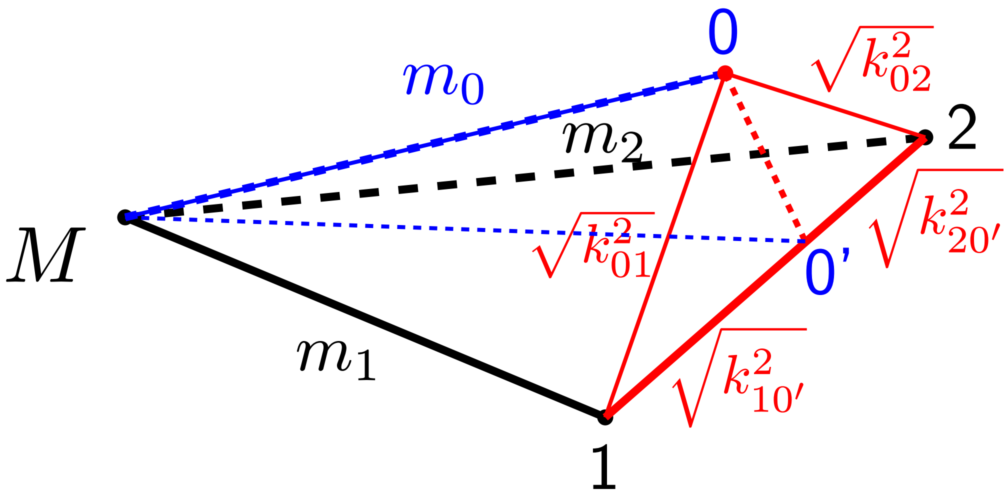
$$\sqrt{k_{12}^2} = \sqrt{k_{10'}^2} + \sqrt{k_{20'}^2}$$

$$J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right)$$
$$= \frac{1}{2k_{12}^2}\Big\{(k_{12}^2+m_1^2-m_2^2)J^{(3)}\left(n;1,1,1\middle|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right)$$
$$+(k_{12}^2-m_1^2+m_2^2)J^{(3)}\left(n;1,1,1\middle|k_{00'}^2,k_{02}^2,k_{20'}^2;m_2,m_{0'},m_0\right)\Big\}$$

with $k_{10'}^2 = \dfrac{(k_{12}^2+m_1^2-m_2^2)^2}{k_{12}^2}$, $\qquad k_{20'}^2 = \dfrac{(k_{12}^2-m_1^2+m_2^2)^2}{4k_{12}^2}$

— similarly to the reduction of the two-point function

Three-point function: reduction to integrals with two equal masses



$$\sqrt{k_{12}^2} = \sqrt{k_{10'}^2} + \sqrt{k_{20'}^2}$$

$$J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right)$$
$$= \frac{1}{2k_{12}^2}\Big\{(k_{12}^2+m_1^2-m_2^2)J^{(3)}\left(n;1,1,1\middle|k_{01}^2,k_{01}^2,4k_{10'}^2;m_1,m_1,m_0\right)$$
$$+(k_{12}^2-m_1^2+m_2^2)J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{02}^2,4k_{20'}^2;m_2,m_2,m_0\right)\Big\}$$

with $k_{10'}^2 = \dfrac{(k_{12}^2+m_1^2-m_2^2)^2}{k_{12}^2}$, $\qquad k_{20'}^2 = \dfrac{(k_{12}^2-m_1^2+m_2^2)^2}{4k_{12}^2}$

— similarly to the reduction of the two-point function

Three-point function: number of variables and the quadratic form

Number of dimensionless variables, before and after splitting:

in $J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right)$: $\quad 6-1(\text{dimension})=5$

in $J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right)$: $\quad 6-2(\text{relations})-1(\text{dimension})=3$

in $J^{(3)}\left(n;1,1,1\middle|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right)$: $\quad 6-3(\text{relations})-1(\text{dimension})=2$

Quadratic form in Feynman parametric integral:

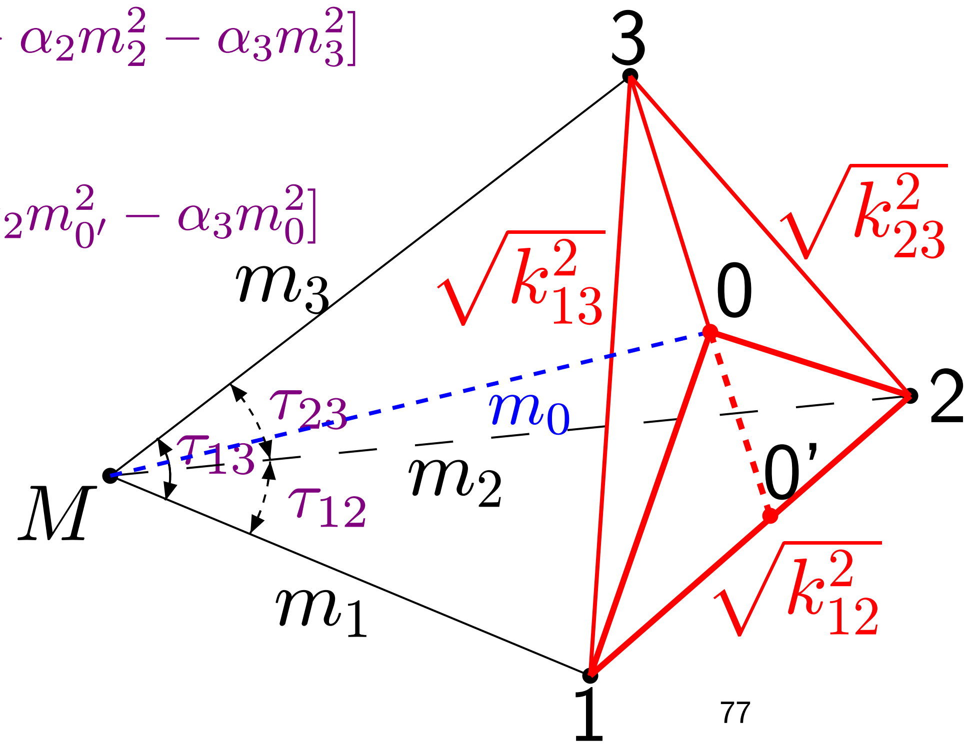
in $J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right)$:

$$[\alpha_1\alpha_2k_{12}^2+\alpha_1\alpha_3k_{13}^2+\alpha_2\alpha_3k_{23}^2-\alpha_1m_1^2-\alpha_2m_2^2-\alpha_3m_3^2]$$

in $J^{(3)}\left(n;1,1,1\middle|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right)$:

$$[\alpha_1\alpha_2k_{10'}^2+\alpha_1\alpha_3k_{01}^2+\alpha_2\alpha_3k_{00'}^2-\alpha_1m_1^2-\alpha_2m_{0'}^2-\alpha_3m_0^2]$$

$$=-[\alpha_1^2k_{10'}^2+(\alpha_1+\alpha_2)^2k_{00'}^2+m_0^2]$$



Three-point function: result in arbitrary dimension

$$J^{(3)}\left(n;1,1,1\middle|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right)$$

$$= \quad -\mathrm{i}\pi^{n/2}\Gamma(3-n/2)\int_0^1\!\!\int_0^1\!\!\int_0^1 \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3\,\delta(\alpha_1+\alpha_2+\alpha_3-1)}{[\alpha_1^2k_{10'}^2+(\alpha_1+\alpha_2)^2k_{00'}^2+m_0^2]^{3-n/2}}$$

$$= \quad -\frac{\mathrm{i}\pi^{n/2}\Gamma(2-n/2)}{2(m_0^2)^{2-n/2}k_{00'}^2}\left\{\sqrt{\frac{k_{00'}^2}{k_{10'}^2}}\arctan\sqrt{\frac{k_{10'}^2}{k_{00'}^2}}\right.$$

$$\left. -\left(\frac{m_0^2}{m_{0'}^2}\right)^{2-n/2} F_1\left(1/2,1,2-n/2;3/2\middle|-\frac{k_{10'}^2}{k_{00'}^2},-\frac{k_{10'}^2}{m_{0'}^2}\right)\right\}$$

where F_1 is Appell hypergeometric function of two variables,

$$F_1\left(a,b_1,b_2;c|x,y\right)=\sum_{j_1,j_2}\frac{(a)_{j_1+j_2}\,(b_1)_{j_1}\,(b_2)_{j_2}}{(c)_{j_1+j_2}}\,\frac{x^{j_1}\,y^{j_2}}{j_1!\,j_2!}$$

A.I.D., hep-th/9908032; Nucl.Instr.Meth. A559 (2006) 293

See also: O.V. Tarasov, Nucl. Phys. B (PS) **89** (2000) 237

J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. **B672** (2003) 303

Three-point function in $n = 4 - 2\varepsilon$ dimensions

$$J^{(3)}(n;1,1,1) = -\frac{\mathrm{i}\pi^{n/2}}{\sqrt{\Lambda^{(3)}}}\,\Gamma\left(3-\frac{n}{2}\right)\; m_0^{n-4}\,\Omega^{(3;n)}\;,$$

$$\begin{aligned}\Omega^{(3;n)} \quad &= \quad \omega\left(\tfrac{1}{2}\varphi^{+}_{12},\eta_{12}\right) + \omega\left(\tfrac{1}{2}\varphi^{-}_{12},\eta_{12}\right)\\ &\quad +\omega\left(\tfrac{1}{2}\varphi^{+}_{23},\eta_{23}\right) + \omega\left(\tfrac{1}{2}\varphi^{-}_{23},\eta_{23}\right)\\ &\quad +\omega\left(\tfrac{1}{2}\varphi^{+}_{31},\eta_{31}\right) + \omega\left(\tfrac{1}{2}\varphi^{-}_{31},\eta_{31}\right)\;,\end{aligned}$$

with

$$\begin{aligned}\omega\left(\tfrac{1}{2}\varphi,\eta\right) \quad &= \quad \frac{1}{2\varepsilon}\int\limits_0^{\varphi/2}\mathrm{d}\phi\left[1-\left(1+\frac{\tan^2\eta}{\cos^2\phi}\right)^{-\varepsilon}\right]\\ &= \quad \frac{1}{2}\sum_{j=0}^{\infty}\frac{(-\varepsilon)^j}{(j+1)!}\int\limits_0^{\varphi/2}\mathrm{d}\phi\;\ln^{j+1}\left(1+\frac{\tan^2\eta_{12}}{\cos^2\phi}\right)\end{aligned}$$

A.I.D., hep-th/9908032, Nucl.Instr.Meth. A559 (2006) 293

Special value of n: $n = 4$ $(\varepsilon \to 0)$:

$$\int\limits_0^{\varphi/2} \mathrm{d}\phi \; \ln\left(1 + \frac{\tan^2\eta}{\cos^2\phi}\right) = \tfrac{1}{2}\tau \; \ln\left(\frac{1+\sin\eta}{1-\sin\eta}\right) + \tfrac{1}{2}\mathrm{Cl}_2\left(\varphi+\tau\right) + \tfrac{1}{2}\mathrm{Cl}_2\left(\varphi-\tau\right) - \mathrm{Cl}_2\left(\varphi\right)$$

Compare with: P. Wagner, Indag. Math. 7 (1996) 527

After analytic continuation (see below), corresponds to

G. 'tHooft and M. Veltman, Nucl. Phys. B153 (1979) 365

Analytic continuation: arbitrary dimension

Consider $$\int\limits_0^{\varphi_0} \mathrm{d}\phi \left(1+\frac{\tan^2\eta}{\cos^2\phi}\right)^{-\varepsilon} .$$

Substitute $z \Rightarrow e^{2\mathrm{i}\phi}$, so that $\cos^2\phi \Rightarrow \dfrac{(1+z)^2}{4z}$,

$$1+\frac{\tan^2\eta}{\cos^2\phi} \Rightarrow \frac{(z+\rho)(z+1/\rho)}{(z+1)^2}, \quad \text{with} \quad \rho \equiv \frac{1-\sin\eta}{1+\sin\eta}$$

In this way, $$\int\limits_0^{\varphi_0} \mathrm{d}\phi \left(1+\frac{\tan^2\eta}{\cos^2\phi}\right)^{-\varepsilon} \Rightarrow \frac{\mathrm{i}}{2}\int\limits_{z_0}^{1}\frac{\mathrm{d}z}{z}\left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2}\right]^{-\varepsilon},$$

with $z_0 \leftrightarrow e^{2\mathrm{i}\varphi_0}$.

Analytic continuation: expansion in ε

Expanding in ε, we get

$$Q_j \equiv \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z} \; \ln^j \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2} \right] \; .$$

The first term, $\mathcal{O}(1)$:

$$\begin{aligned} Q_1 \;\; &\equiv \;\; \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z} \; \ln \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2} \right] \\ &= \;\; \mathrm{Li}_2\left(-z_0\rho\right) + \mathrm{Li}_2\left(-z_0/\rho\right) - 2\mathrm{Li}_2\left(-z_0\right) + \tfrac{1}{2}\ln^2\rho \end{aligned}$$

Analytic continuation: expansion in ε (continued)

$$
\begin{aligned}
Q_2 \;&\equiv\; \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z}\; \ln^2\left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2}\right] \\
&= \ln\rho \left[2\mathrm{Li}_2\left(\frac{1-\rho}{1+z_0\rho}\right) + 2\mathrm{Li}_2\left(\frac{z_0(\rho-1)}{1+z_0\rho}\right) - 2\mathrm{Li}_2\left(\frac{\rho-1}{z_0+\rho}\right) - 2\mathrm{Li}_2\left(\frac{z_0(1-\rho)}{z_0+\rho}\right)\right. \\
&\quad \left. -\mathrm{Li}_2\left(\frac{1-\rho^2}{1+z_0\rho}\right) - \mathrm{Li}_2\left(\frac{z_0(\rho^2-1)}{\rho(1+z_0\rho)}\right) + \mathrm{Li}_2\left(\frac{\rho^2-1}{\rho(z_0+\rho)}\right) + \mathrm{Li}_2\left(\frac{z_0(1-\rho^2)}{z_0+\rho}\right)\right] \\
&\quad +4\mathrm{S}_{1,2}\left(\frac{1-\rho}{1+z_0\rho}\right) - 4\mathrm{S}_{1,2}\left(\frac{z_0(\rho-1)}{1+z_0\rho}\right) + 4\mathrm{S}_{1,2}\left(\frac{\rho-1}{z_0+\rho}\right) - 4\mathrm{S}_{1,2}\left(\frac{z_0(1-\rho)}{z_0+\rho}\right) \\
&\quad -\mathrm{S}_{1,2}\left(\frac{1-\rho^2}{1+z_0\rho}\right) + \mathrm{S}_{1,2}\left(\frac{z_0(\rho^2-1)}{\rho(1+z_0\rho)}\right) - \mathrm{S}_{1,2}\left(\frac{\rho^2-1}{\rho(z_0+\rho)}\right) + \mathrm{S}_{1,2}\left(\frac{z_0(1-\rho^2)}{z_0+\rho}\right)
\end{aligned}
$$

Compare with: J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. **B672** (2003) 303

$$
Q_3 = \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z}\; \ln^3\left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2}\right], \quad \text{etc.}
$$

Some special cases considered in J.G. Körner, Z. Merebashvili, M. Rogal, Phys. Rev. D71 (2005) 054028

ε-expansion: recursive calculation

$$
\begin{aligned}
Q_0(z_0,\rho) &= -\ln z_0, \\
Q_j(z_0,\rho) &= Q_{j-1}(z_0,\rho)\ln\left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2}\right] \\
&\quad + \int\limits_{z_0}^{1} \mathrm{d}z\, Q_{j-1}(z,\rho)\left[\frac{1}{z+\rho}+\frac{1}{z+1/\rho}-\frac{2}{z+1}\right]
\end{aligned}
$$

$\Rightarrow$ higher terms can be expressed in terms of *multiple polylogarithms*

$$
\mathrm{Li}_{n_1,\ldots,n_m}(z_1,\ldots,z_m) = \sum_{0<k_1<k_2<\ldots<k_m} \frac{z_1^{k_1} z_2^{k_2}\ldots z_m^{k_m}}{k_1^{n_1} k_2^{n_2}\ldots k_m^{n_m}}
$$

A.B. Goncharov, Math. Res. Lett. 5 (1998) 497,

J. Vollinga and S. Weinzierl, Comp. Phys. Commun. **167** (2005) 177

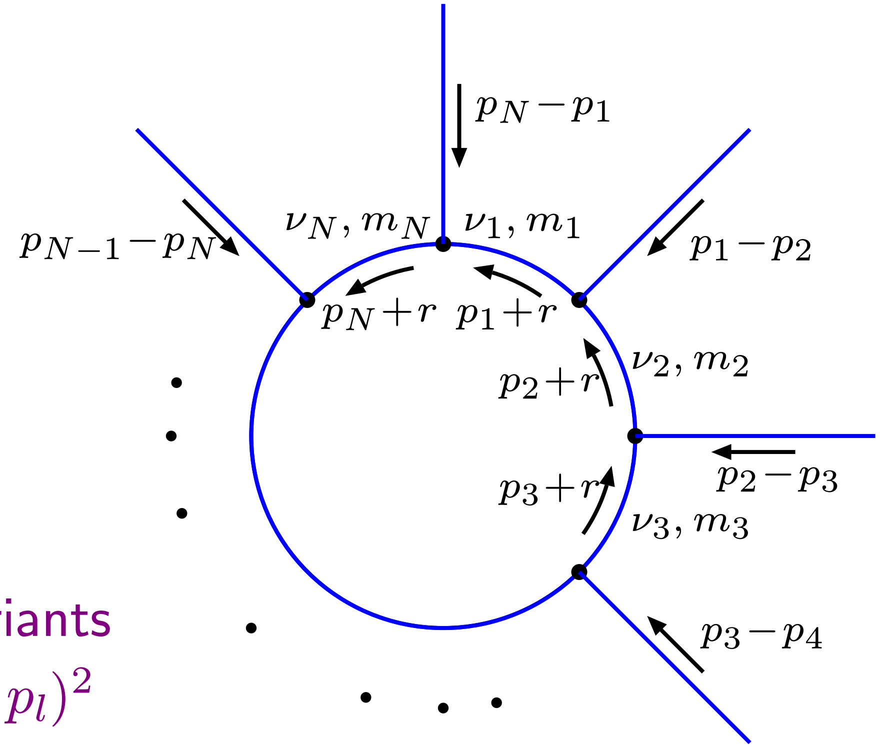
or *two-dimensional harmonic polylogarithms*,

T. Gehrmann and E. Remiddi, Nucl. Phys. **B601** (2001) 248

LECTURE #3

One-loop N-point function $J^{(N)}(n; \nu_1, \ldots, \nu_N)$

from Lecture #1



Depends on
$\frac{1}{2}N(N-1)$ invariants
$k_{jl}^2 = (p_j - p_l)^2$
and N masses m_i

$$J^{(N)}(n; \nu_1, \ldots, \nu_N) \equiv \int \frac{\mathrm{d}^n k}{[(p_1+k)^2 - m_1^2]^{\nu_1} \cdots [(p_N+k)^2 - m_N^2]^{\nu_N}}$$

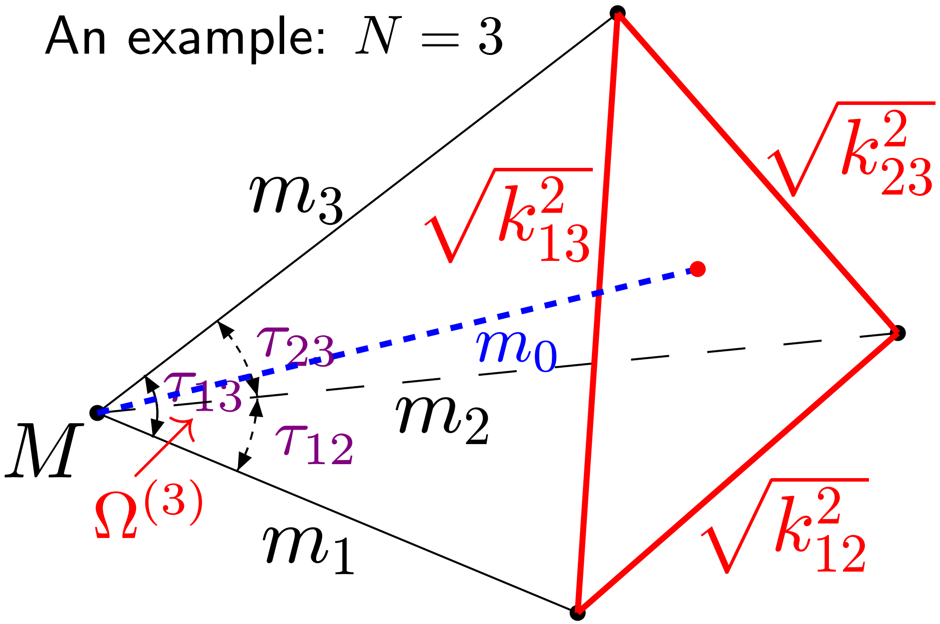
dimension powers of propagators

Geometric representation

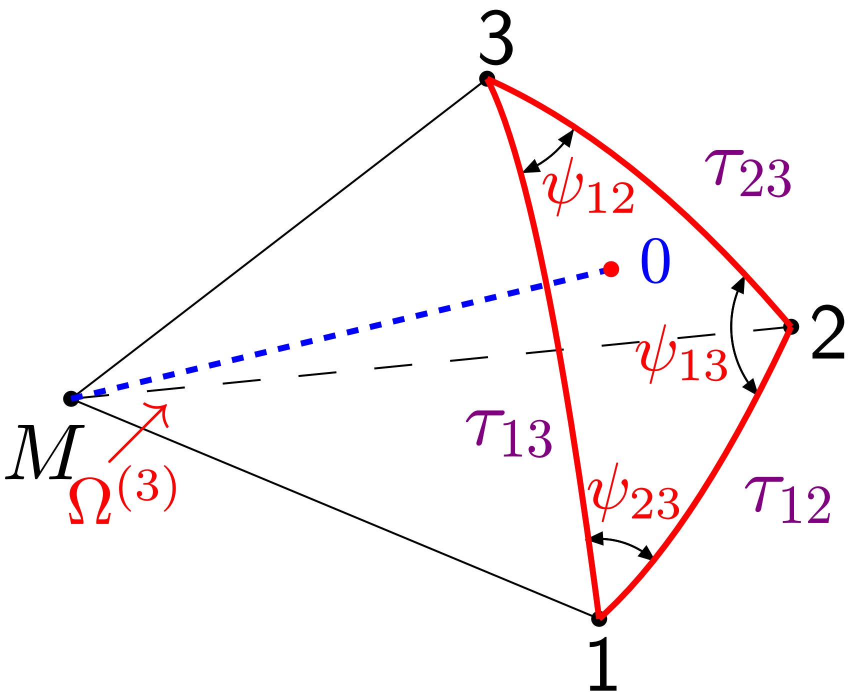
from Lecture #1

$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\tfrac{n}{2}\right)\,\frac{{m_0}^{n-N-1}}{\sqrt{\Lambda^{(N)}}}\int\limits_{\Omega^{(N)}}\ldots\int\frac{\prod \mathrm{d}\gamma_i}{\gamma_N^{\,n-N}}\,\delta\left(\sum\gamma_i^2-1\right)$$

An example: $N=3$



the basic tetrahedron



the solid angle and the integration surface

Geometric representation (continued)

from Lecture #1

$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\tfrac{n}{2}\right)\,\frac{{m_0}^{n-N-1}}{\sqrt{\Lambda^{(N)}}}\,\underset{\Omega^{(N)}}{\int\ldots\int}\frac{\prod \mathrm{d}\gamma_i}{\gamma_N^{\,n-N}}\,\delta\left(\sum\gamma_i^2-1\right)$$

- The integration goes over the hypersurface of the hypersphere cut out by the N-dimensional solid angle $\Omega^{(N)}$

- If we define the angle between the "running" unit vector and the Nth axis as θ, then $1/\gamma_N^{n-N}\Rightarrow 1/(\cos\theta)^{n-N}$, and we get

$$J^{(N)}(n;1,\ldots,1) = \mathrm{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\tfrac{n}{2}\right)\,\frac{{m_0}^{n-N}\Omega^{(N;n)}}{N!\;V^{(N)}}\,,$$

where $V^{(N)}$ is the volume of the basic simplex, m_0 – its height, and

$$\Omega^{(N;n)} \equiv \underset{\Omega^{(N)}}{\int\ldots\int}\frac{\mathrm{d}\Omega_N}{\cos^{n-N}\theta} \qquad \text{(obviously,}\quad \Omega^{(N;N)} = \Omega^{(N)}\text{)}$$

$\Rightarrow$ *everything is translated into the geometrical language!*

Feynman parameters versus geometrical approach

from Lecture #1

Feynman parametric representation (with $c_{jl} \equiv (m_j^2 + m_l^2 - k_{jl}^2)/(2m_j m_l)$):

$$J^{(N)}(n;1,\ldots,1) = \mathrm{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\tfrac{n}{2}\right)\int\limits_0^1\ldots\int\limits_0^1 \frac{\left(\prod \mathrm{d}\alpha_i\right)\cdot\delta\left(\sum\alpha_i - 1\right)}{\left[\sum\alpha_i^2 m_i^2 + 2\sum\sum\limits_{j<l}\alpha_j\alpha_l m_j m_l c_{jl}\right]^{N-n/2}}$$

- depends on the masses and k_{jl}^2 through m_i and c_{jl} in the quadratic form

Geometric representation:

$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\tfrac{n}{2}\right)\frac{{m_0}^{n-N-1}}{\sqrt{\Lambda^{(N)}}}\int\limits_{\Omega^{(N)}}\ldots\int\frac{\prod \mathrm{d}\gamma_i}{\gamma_N^{\,n-N}}\,\delta\left(\sum\gamma_i^2 - 1\right)$$

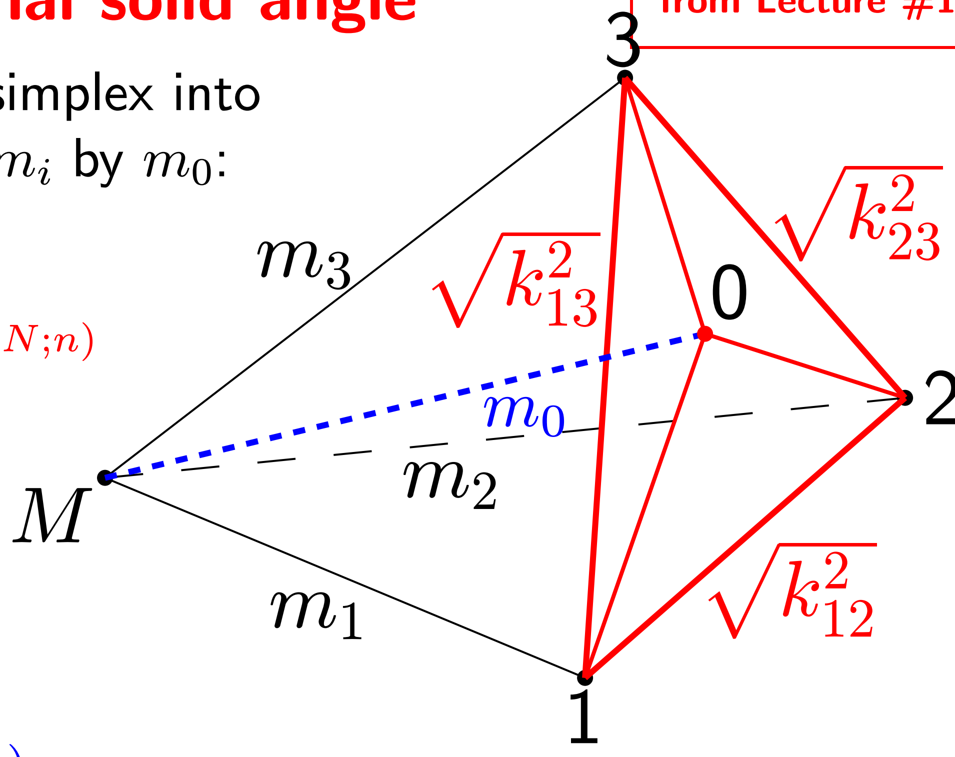
- except for the pre-factor, depends on the masses and k_{jl}^2 only through the integration limits defined by the N-dimensional solid angle $\Omega^{(N)}$
- ready for splitting!

Splitting the N-dimensional solid angle

from Lecture #1

Let us use m_0 to split the basic N-dimensional simplex into N rectangular ones, each time replacing one of m_i by m_0:

$$\Omega_i^{(N;n)} = \int\limits_{\Omega_i^{(N)}} \!\!\ldots\! \int \frac{\mathrm{d}\Omega_N}{\cos^{n-N}\theta}\,, \qquad \Omega^{(N;n)} = \sum_{i=1}^{N} \Omega_i^{(N;n)}$$



An example: $N = 3$

Each of the resulting integrals can be associated with a one-loop N-point integral

$$J^{(N)}(n;1,\ldots,1) = \sum_{i=1}^{N} \frac{V_i^{(N)}}{V^{(N)}}\, J_i^{(N)}(n;1,\ldots,1)$$

In $J_i^{(N)}$ the internal masses are $m_1, \ldots, m_{i-1}, m_0, m_{i+1}, \ldots, m_N$, and the squared momenta are k_{jl}^2 (if $j \neq i$ and $l \neq i$), $m_l^2 - m_0^2$ (if $j = i$), $m_j^2 - m_0^2$ (if $l = i$).

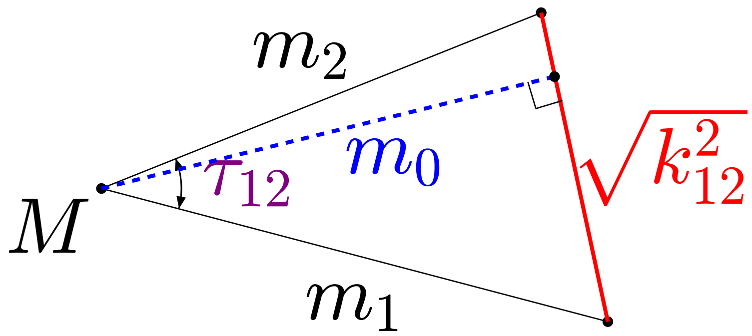
In terms of $F_i^{(N)}$,

$$J^{(N)}(n;1,\ldots,1) = \frac{1}{\Lambda^{(N)}} \left(\prod m_i^2\right) \sum_{i=1}^{N} \frac{F_i^{(N)}}{m_i^2}\, J_i^{(N)}(n;1,\ldots,1)$$

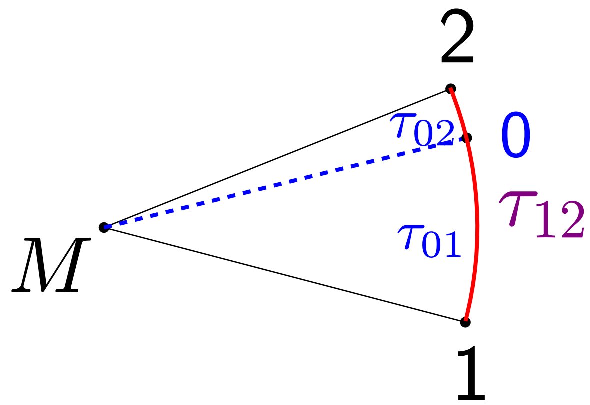
See more examples below...

Two-point function, geometrical approach

from Lecture #2



the basic triangle



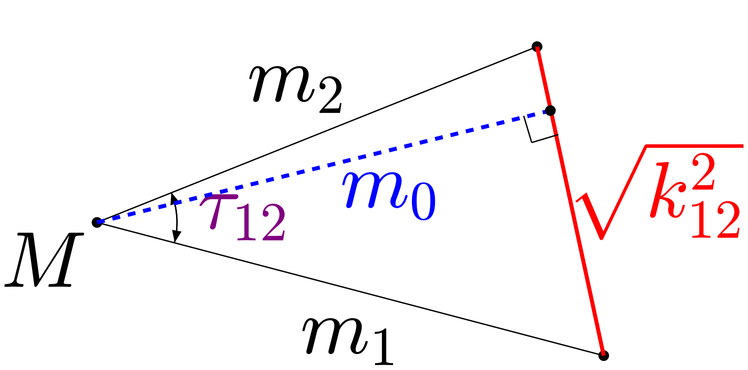
the arc τ_{12}

$$\cos\tau_{12} \equiv c_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1 m_2}, \quad D^{(2)} = 1 - c_{12}^2 = \sin^2\tau_{12}, \quad \Lambda^{(2)} = k_{12}^2,$$

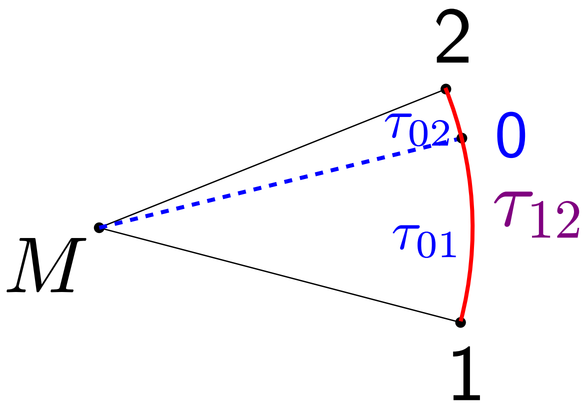
$$m_0 = m_1 m_2 \sqrt{\frac{D^{(2)}}{\Lambda^{(2)}}} = \frac{m_1 m_2 \sin\tau_{12}}{\sqrt{k_{12}^2}}, \qquad \cos\tau_{0i} = \frac{m_0}{m_i}, \quad \tau_{01} + \tau_{02} = \tau_{12}.$$

Two-point function in an arbitrary dimension n

from Lecture #2



the basic triangle



the arc τ_{12}

$$J^{(2)}(n;1,1) = \mathrm{i}\pi^{n/2}\Gamma\left(2-\frac{n}{2}\right)\frac{m_0^{n-3}}{\sqrt{k_{12}^2}}\left\{\Omega_1^{(2;n)}+\Omega_2^{(2;n)}\right\} \; ,$$

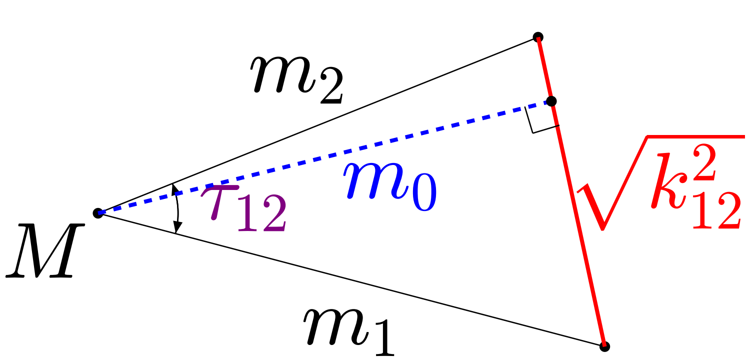
with

$$\Omega_i^{(2;n)} = \int\limits_0^{\tau_{0i}} \frac{\mathrm{d}\theta}{\cos^{n-2}\theta}$$

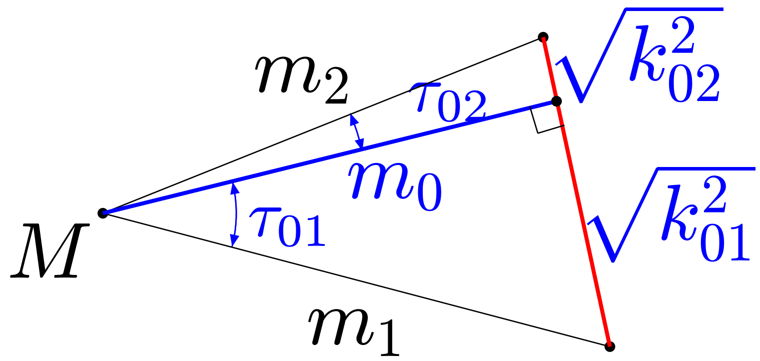
As expected, it has an "ultraviolet" singularity as $n \to 4$.

Two-point function, splitting the basic triangle

from Lecture #2



$\Rightarrow$



$$k_{01}^2 = \frac{(k_{12}^2 + m_1^2 - m_2^2)^2}{4k_{12}^2}, \qquad k_{02}^2 = \frac{(k_{12}^2 - m_1^2 + m_2^2)^2}{4k_{12}^2} \quad \Rightarrow \quad \sqrt{k_{01}^2} + \sqrt{k_{02}^2} = \sqrt{k_{12}^2}$$

$$F_1^{(2)} = \frac{k_{12}^2 - m_1^2 + m_2^2}{2m_2^2}, \qquad F_2^{(2)} = \frac{k_{12}^2 + m_1^2 - m_2^2}{2m_1^2} \quad \Rightarrow \quad \frac{F_1^{(2)}}{m_1^2} + \frac{F_2^{(2)}}{m_2^2} = \frac{k_{12}^2}{m_1^2 m_2^2} = \frac{\Lambda^{(2)}}{m_1^2 m_2^2}$$

$$J^{(2)}\left(n;1,1|k_{12}^2;m_1,m_2\right) = \frac{1}{2k_{12}^2}\Big\{(k_{12}^2 + m_1^2 - m_2^2)J^{(2)}\left(n;1,1|k_{01}^2;m_1,m_0\right)$$
$$+(k_{12}^2 - m_1^2 + m_2^2)J^{(2)}\left(n;1,1|k_{02}^2;m_2,m_0\right)\Big\}$$

This is an example of a functional relation between integrals with different momenta and masses, similar to those described in

O.V. Tarasov, Phys.Lett. **B670** (2008) 67

Two-point function: number of variables and the quadratic form

from Lecture #2

Number of dimensionless variables, before and after splitting:

in $J^{(2)}\left(n;1,1\middle|k_{12}^2;m_1,m_2\right)$: $\quad 3-1(\text{dimension})=2$

in $J^{(2)}\left(n;1,1\middle|k_{01}^2;m_1,m_0\right)$: $\quad 3-1(k_{01}^2=m_1^2-m_0^2)-1(\text{dimension})=1$

Quadratic form in Feynman parametric integral:

in $J^{(2)}\left(n;1,1\middle|k_{12}^2;m_1,m_2\right)$: $\quad [\alpha_1\alpha_2k_{12}^2-\alpha_1m_1^2-\alpha_2m_2^2]$

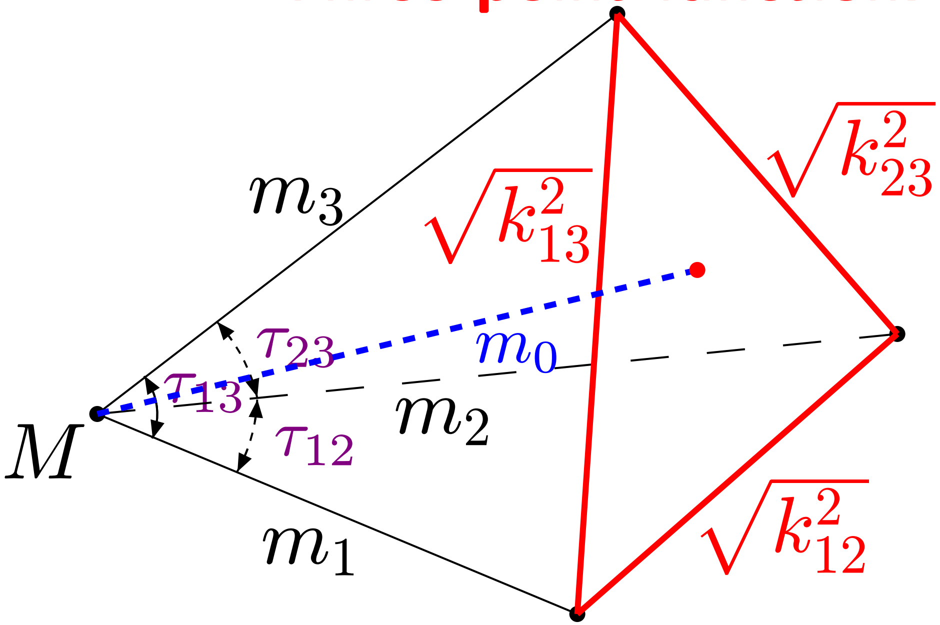
in $J^{(2)}\left(n;1,1\middle|k_{01}^2;m_1,m_0\right)$: $\quad [\alpha_1\alpha_2k_{01}^2-\alpha_1m_1^2-\alpha_2m_0^2]=-[\alpha_1^2k_{01}^2+m_0^2]$

Result in arbitrary dimension:

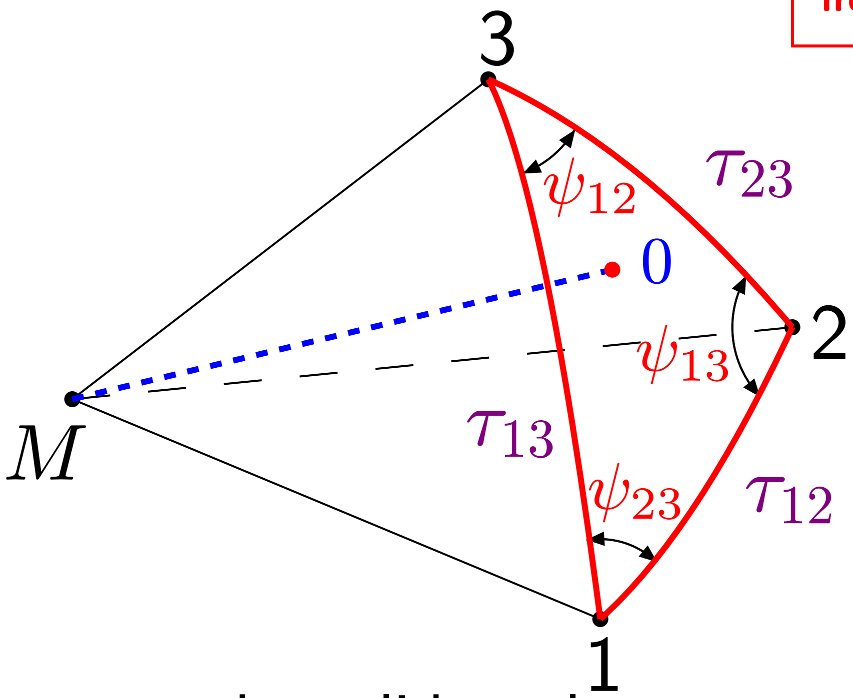
$$
\begin{aligned}
J^{(2)}\left(n;1,1\middle|k_{01}^2;m_1,m_0\right) &= \mathrm{i}\pi^{n/2}\Gamma(2-n/2)\int\limits_0^1\int\limits_0^1\frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\delta(\alpha_1+\alpha_2-1)}{[\alpha_1^2k_{01}^2+m_0^2]^{2-n/2}} \\
&= \mathrm{i}\pi^{n/2}\frac{\Gamma(2-n/2)}{(m_0^2)^{2-n/2}}\;{}_2F_1\left(\begin{matrix}1/2,\;2-n/2\\3/2\end{matrix}\middle|-\frac{k_{01}^2}{m_0^2}\right)
\end{aligned}
$$

Three-point function: geometrical approach

from Lecture #2



the basic tetrahedron



the solid angle

Tetrahedron volume : $V^{(3)} = \frac{1}{6}\, m_1 m_2 m_3 \sqrt{D^{(3)}}$, where $D^{(3)} = \begin{vmatrix} 1 & c_{12} & c_{13} \\ c_{12} & 1 & c_{23} \\ c_{13} & c_{23} & 1 \end{vmatrix}$

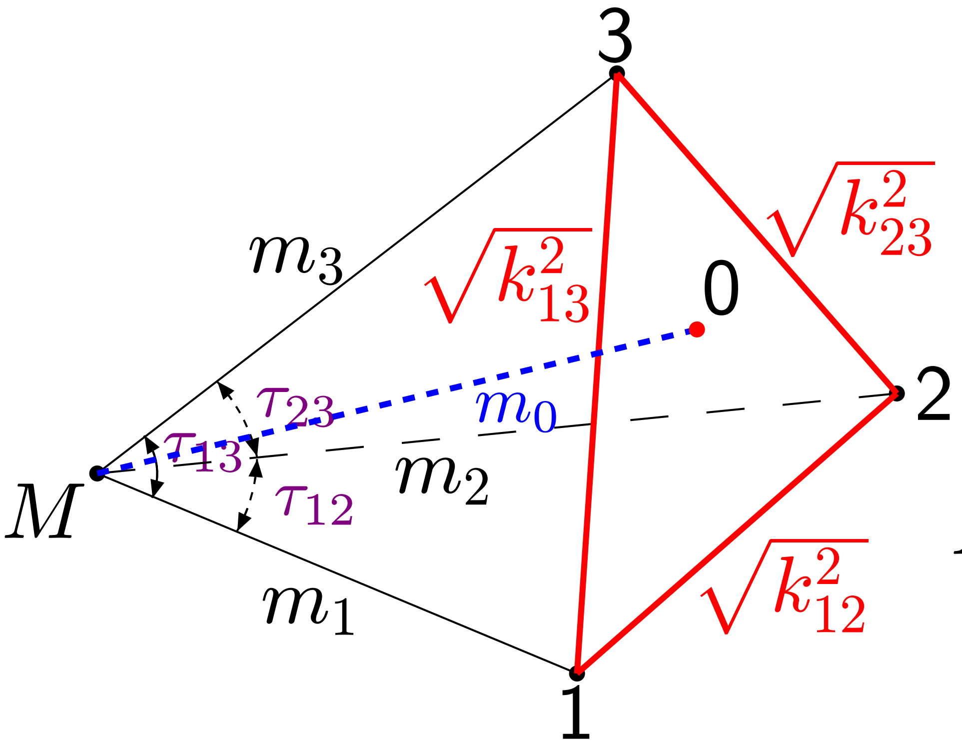
Red triangle area: $\overline{V}_0^{(2)} = \frac{1}{2}\sqrt{\Lambda^{(3)}}$,

with $\Lambda^{(3)} = \frac{1}{4}\left[2k_{12}^2k_{13}^2+2k_{13}^2k_{23}^2+2k_{23}^2k_{12}^2-(k_{12}^2)^2-(k_{13}^2)^2-(k_{23}^2)^2\right] = -\frac{1}{4}\lambda(k_{12}^2, k_{13}^2, k_{23}^2)$,

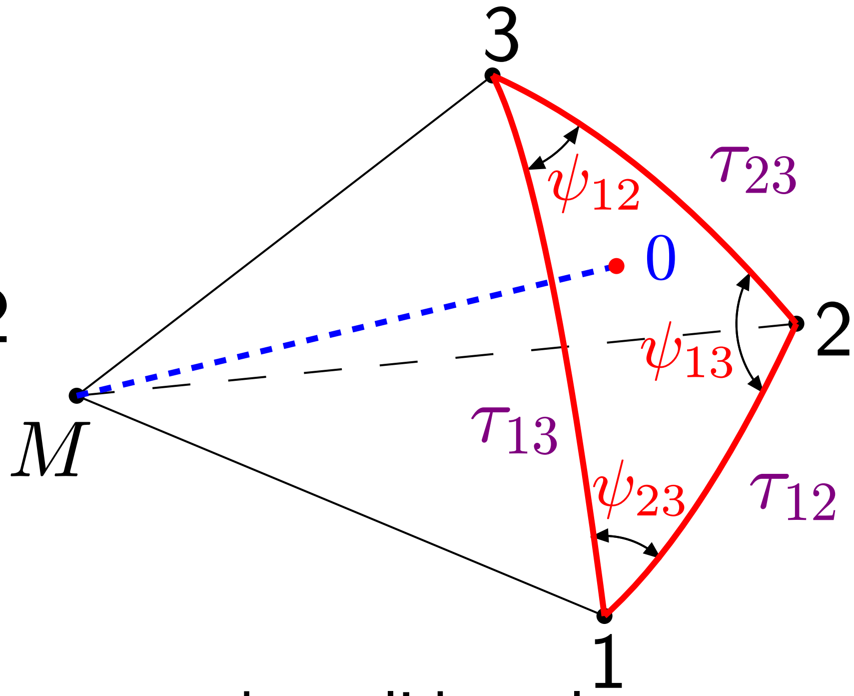
where $\lambda(x,y,z)$ is the Källen function.

Three-point function: geometrical approach

from Lecture #2



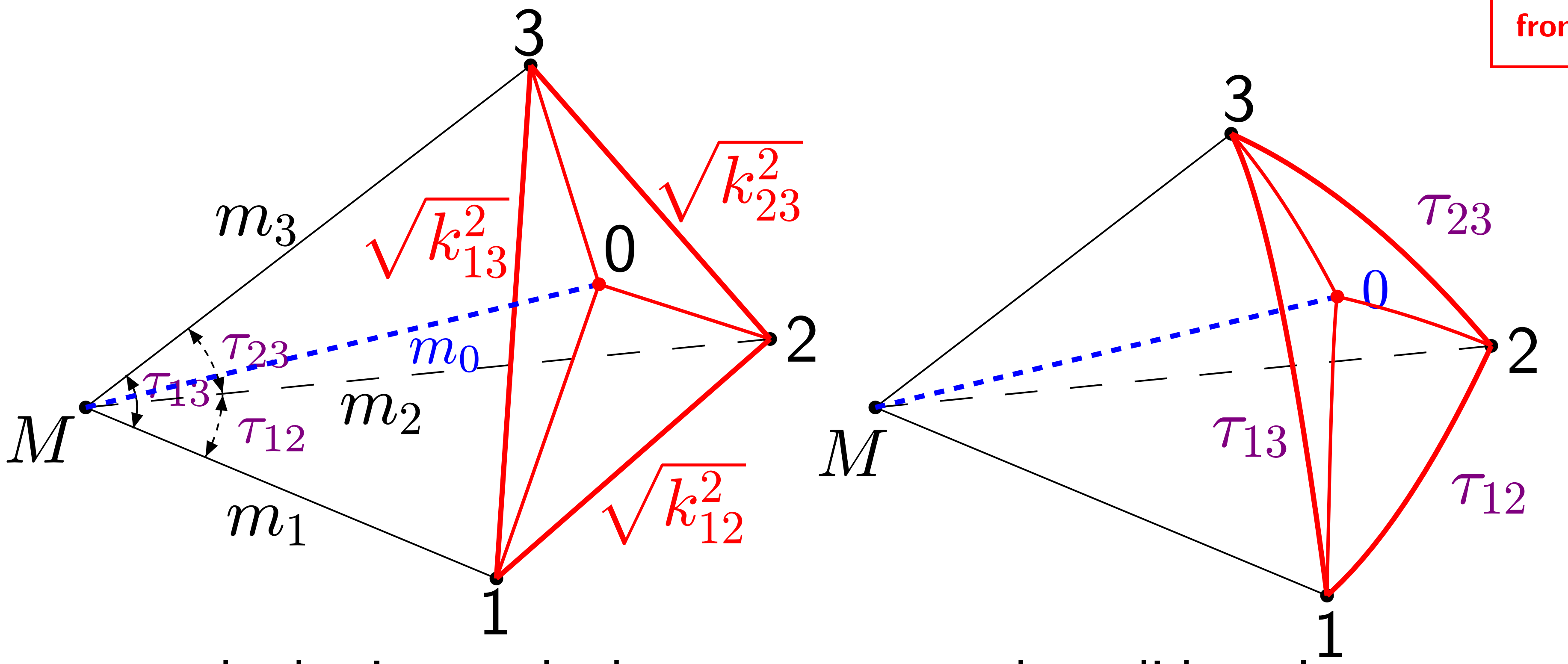
the basic tetrahedron



the solid angle

Three-point function: geometrical approach

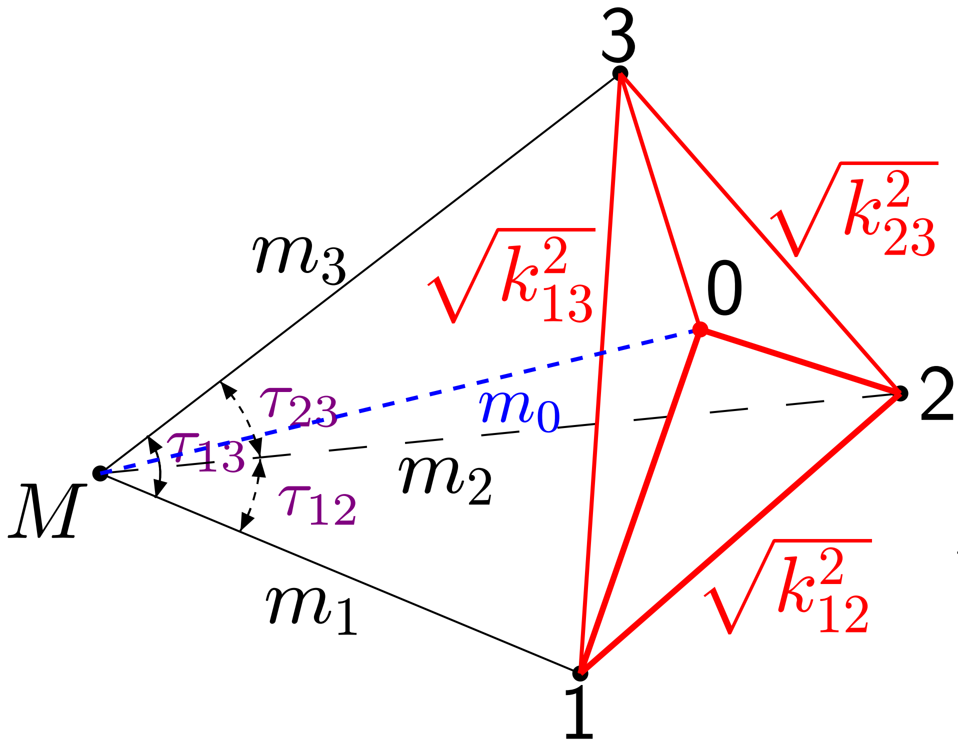
from Lecture #2



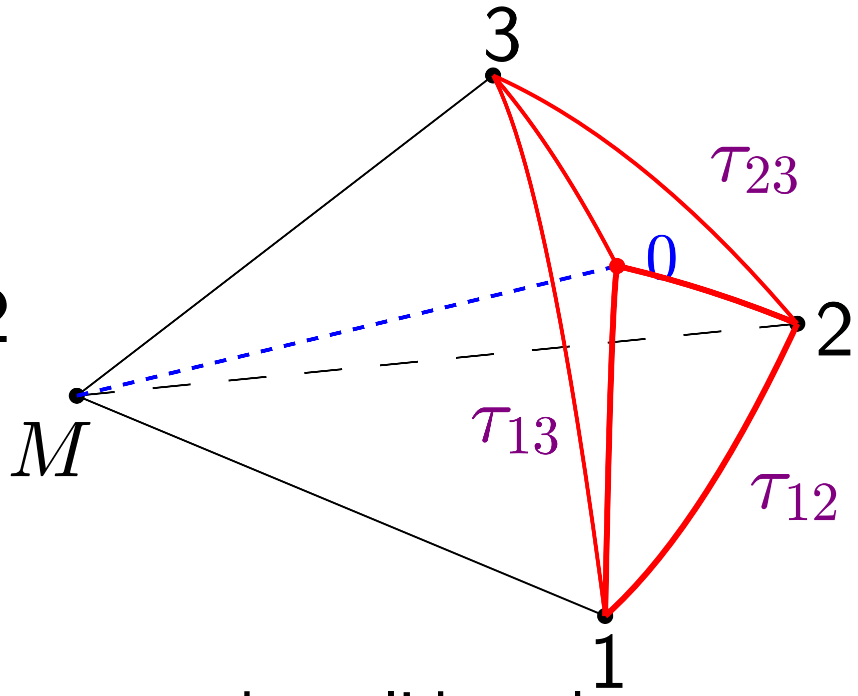
the basic tetrahedron the solid angle

Three-point function: geometrical approach

from Lecture #2



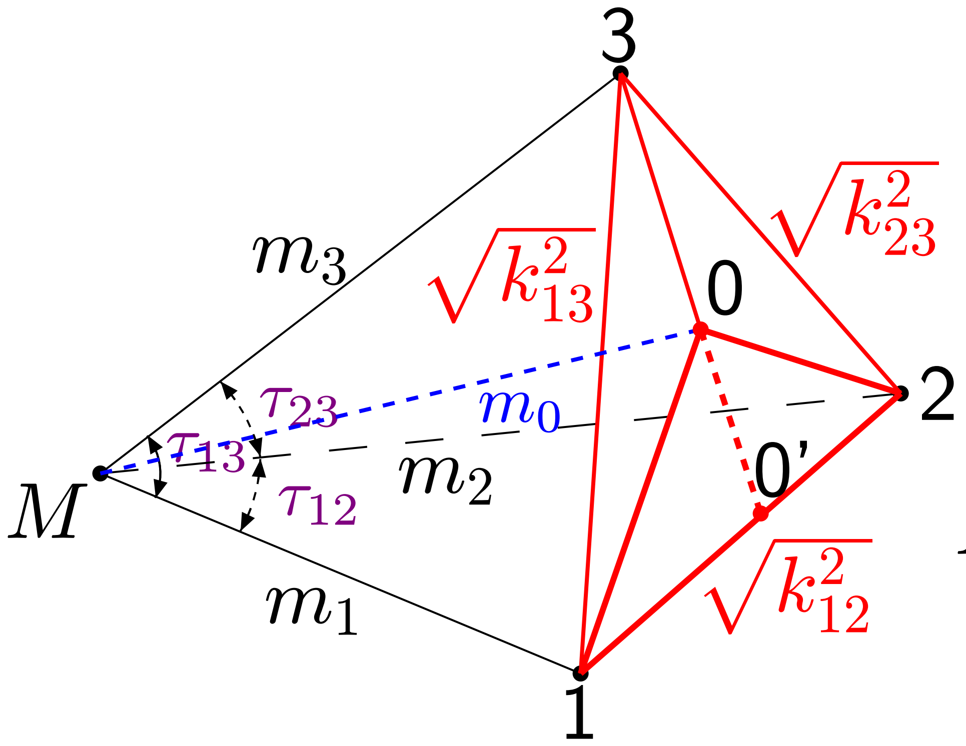
the basic tetrahedron



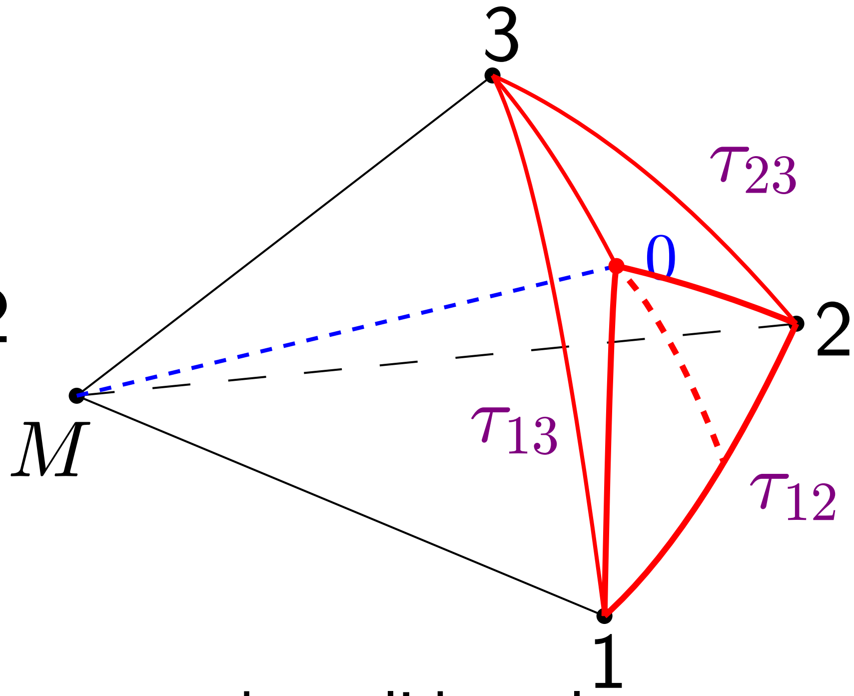
the solid angle

Three-point function: geometrical approach

from Lecture #2



the basic tetrahedron



the solid angle

Three-point function: number of variables and the quadratic form

Number of dimensionless variables, before and after splitting:

from Lecture #2

in $J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right)$: $6-1(\text{dimension})=5$

in $J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right)$: $6-2(\text{relations})-1(\text{dimension})=3$

in $J^{(3)}\left(n;1,1,1\middle|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right)$: $6-3(\text{relations})-1(\text{dimension})=2$

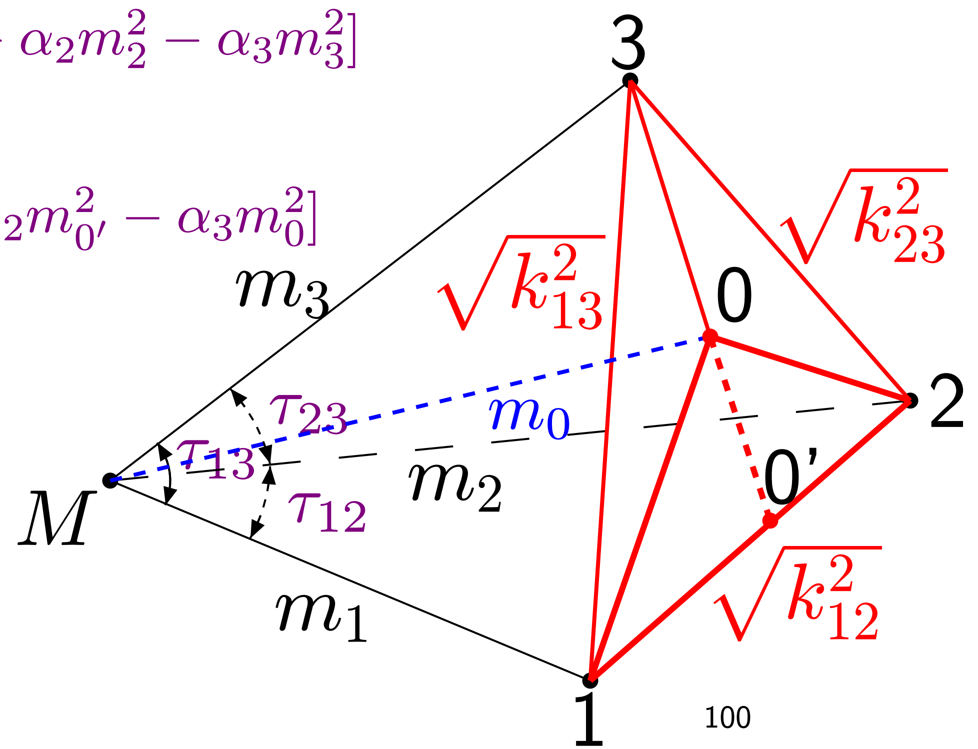
Quadratic form in Feynman parametric integral:

in $J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right)$:

$$[\alpha_1\alpha_2k_{12}^2+\alpha_1\alpha_3k_{13}^2+\alpha_2\alpha_3k_{23}^2-\alpha_1m_1^2-\alpha_2m_2^2-\alpha_3m_3^2]$$

in $J^{(3)}\left(n;1,1,1\middle|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right)$:

$$[\alpha_1\alpha_2k_{10'}^2+\alpha_1\alpha_3k_{01}^2+\alpha_2\alpha_3k_{00'}^2-\alpha_1m_1^2-\alpha_2m_{0'}^2-\alpha_3m_0^2]$$
$$=-[\alpha_1^2k_{10'}^2+(\alpha_1+\alpha_2)^2k_{00'}^2+m_0^2]$$



Three-point function: result in arbitrary dimension

from Lecture #2

$$J^{(3)}\left(n;1,1,1\middle|k_{00'}^2,k_{01}^2,k_{10'}^2;m_1,m_{0'},m_0\right)$$

$$= \quad -\mathrm{i}\pi^{n/2}\Gamma(3-n/2)\int_0^1\int_0^1\int_0^1 \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3\,\delta(\alpha_1+\alpha_2+\alpha_3-1)}{[\alpha_1^2 k_{10'}^2+(\alpha_1+\alpha_2)^2 k_{00'}^2+m_0^2]^{3-n/2}}$$

$$= \quad -\frac{\mathrm{i}\pi^{n/2}\Gamma(2-n/2)}{2(m_0^2)^{2-n/2}k_{00'}^2}\left\{\sqrt{\frac{k_{00'}^2}{k_{10'}^2}}\arctan\sqrt{\frac{k_{10'}^2}{k_{00'}^2}}\right.$$

$$\left. -\left(\frac{m_0^2}{m_{0'}^2}\right)^{2-n/2} F_1\left(1/2,1,2-n/2;3/2\middle|-\frac{k_{10'}^2}{k_{00'}^2},-\frac{k_{10'}^2}{m_{0'}^2}\right)\right\}$$

where F_1 is Appell hypergeometric function of two variables,

$$F_1\left(a,b_1,b_2;c\middle|x,y\right)=\sum_{j_1,j_2}\frac{(a)_{j_1+j_2}\,(b_1)_{j_1}\,(b_2)_{j_2}}{(c)_{j_1+j_2}}\,\frac{x^{j_1}\,y^{j_2}}{j_1!\,j_2!}$$

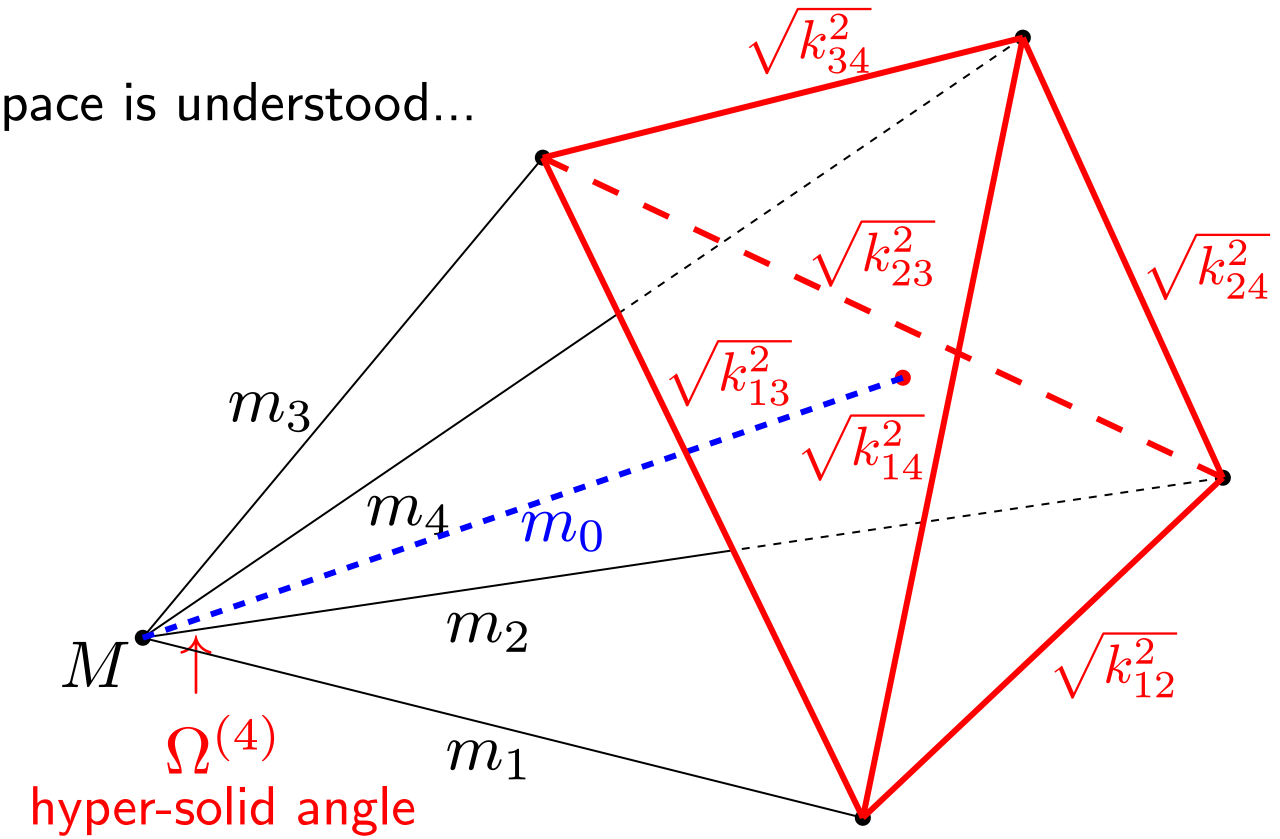
A.I.D., hep-th/9908032; Nucl.Instr.Meth. A559 (2006) 293

See also: O.V. Tarasov, Nucl. Phys. B (PS) **89** (2000) 237

J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. **B672** (2003) 303

Four-point function: the basic simplex for $N = 4$

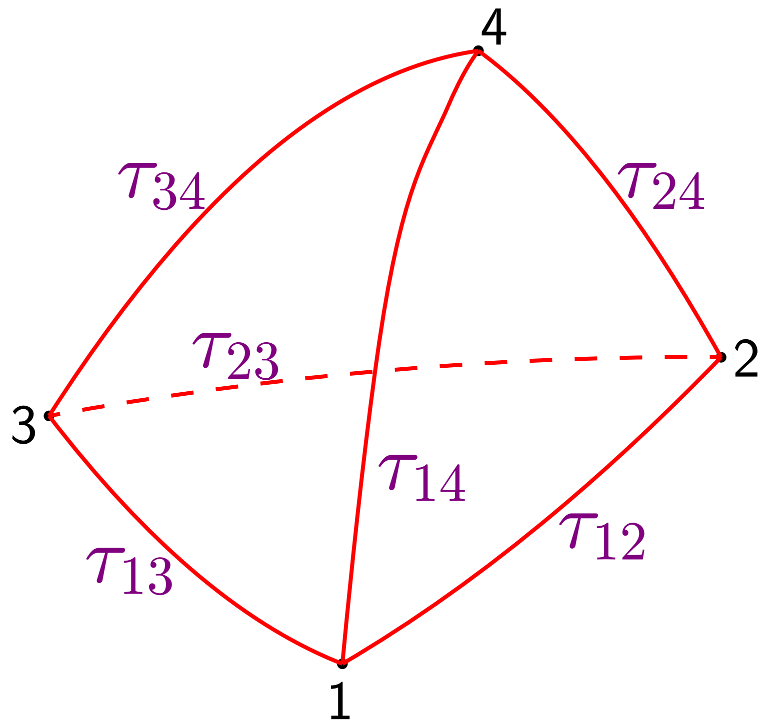
4-dim. space is understood...



$$D^{(4)} = \det \| c_{jl} \| , \qquad \Lambda^{(4)} = \det \| (k_{jN} \cdot k_{lN}) \| ,$$

$$V^{(4)} = \frac{(\Pi m_i)}{4!} \sqrt{D^{(4)}}, \qquad \overline{V}_0^{(3)} = \frac{1}{3!} \sqrt{\Lambda^{(4)}}, \qquad m_0 = (\Pi m_i) \sqrt{\frac{D^{(4)}}{\Lambda^{(4)}}}$$

Geometrical approach: four-point function

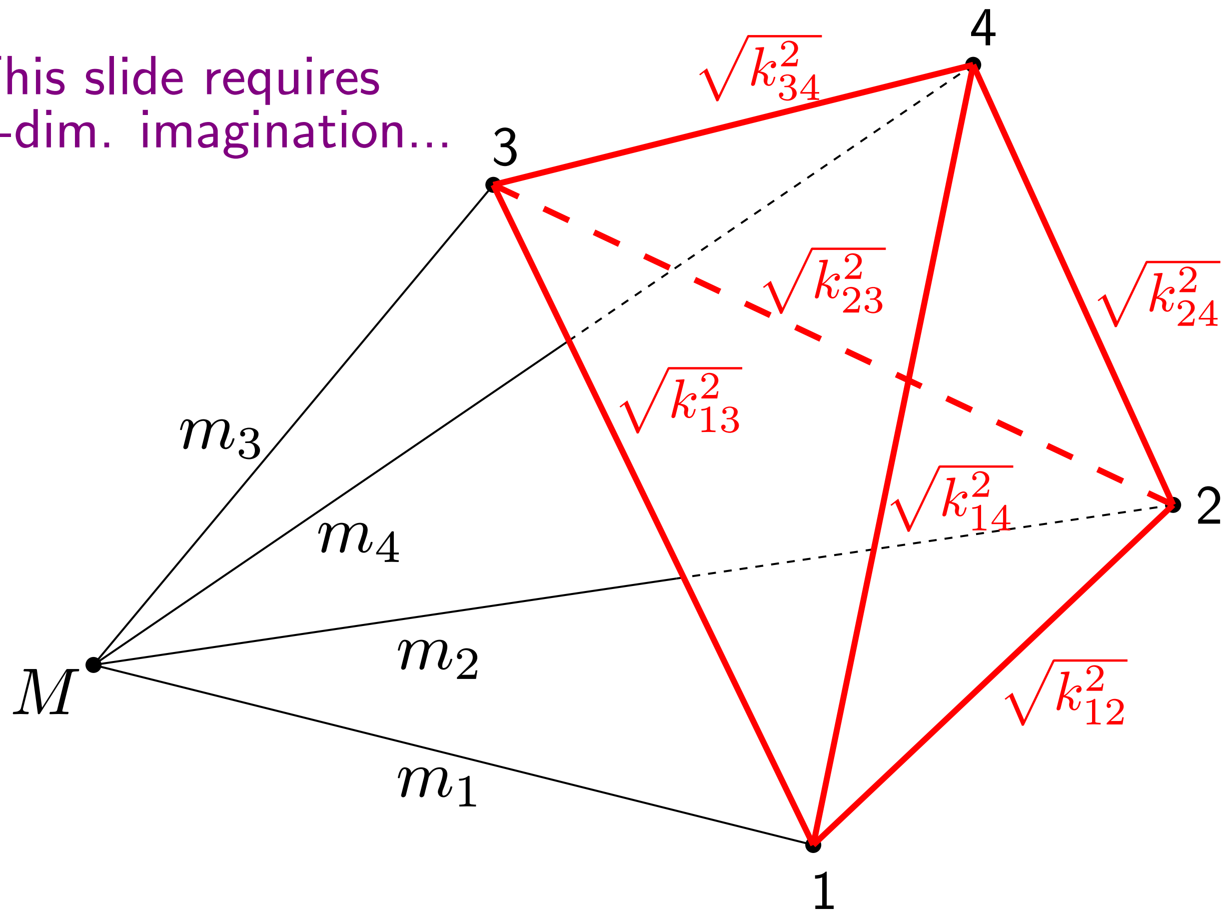


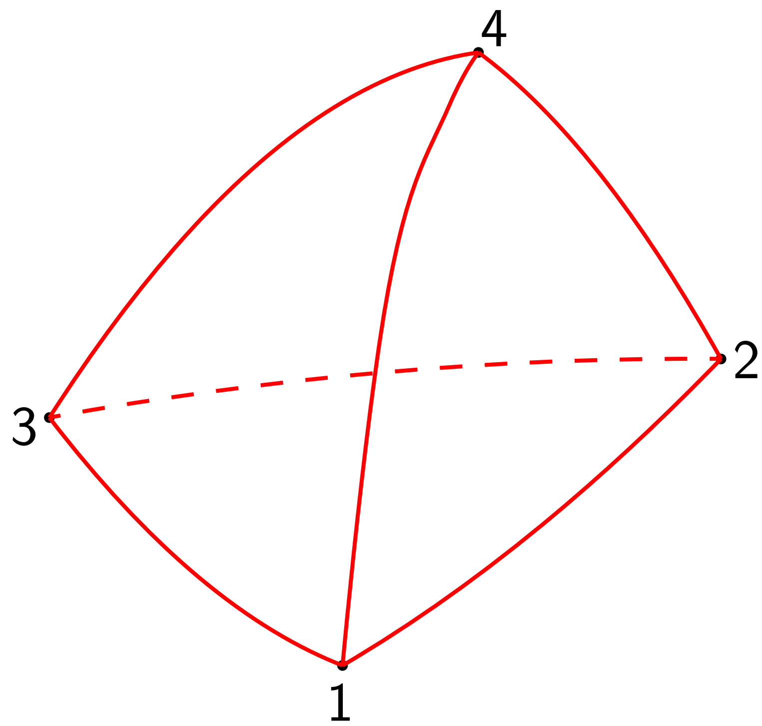
The non-Euclidean (spherical or hyperbolic) tetrahedon $\leftrightarrow \Omega^{(4)}$

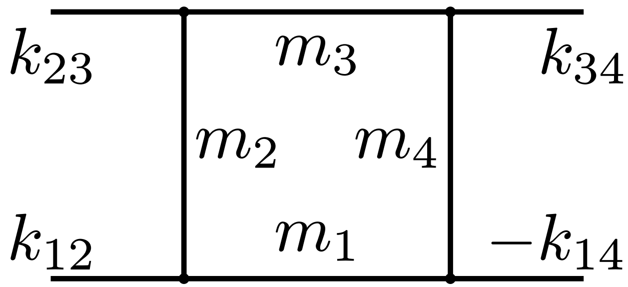
$$J^{(4)}(n;1,1,1,1) = \mathrm{i}\pi^{n/2}\,\Gamma\left(4-\frac{n}{2}\right)\frac{m_0^{n-4}\ \Omega^{(4;n)}}{m_1 m_2 m_3 m_4 \sqrt{D^{(4)}}}, \quad \text{with} \quad \Omega^{(4;n)} = \iiint\limits_{\Omega^{(4)}} \frac{\mathrm{d}\Omega_4}{\cos^{n-4}\theta}$$

Four-point function: basic simplex and non-Euclidean tetrahedron

This slide requires
4-dim. imagination...



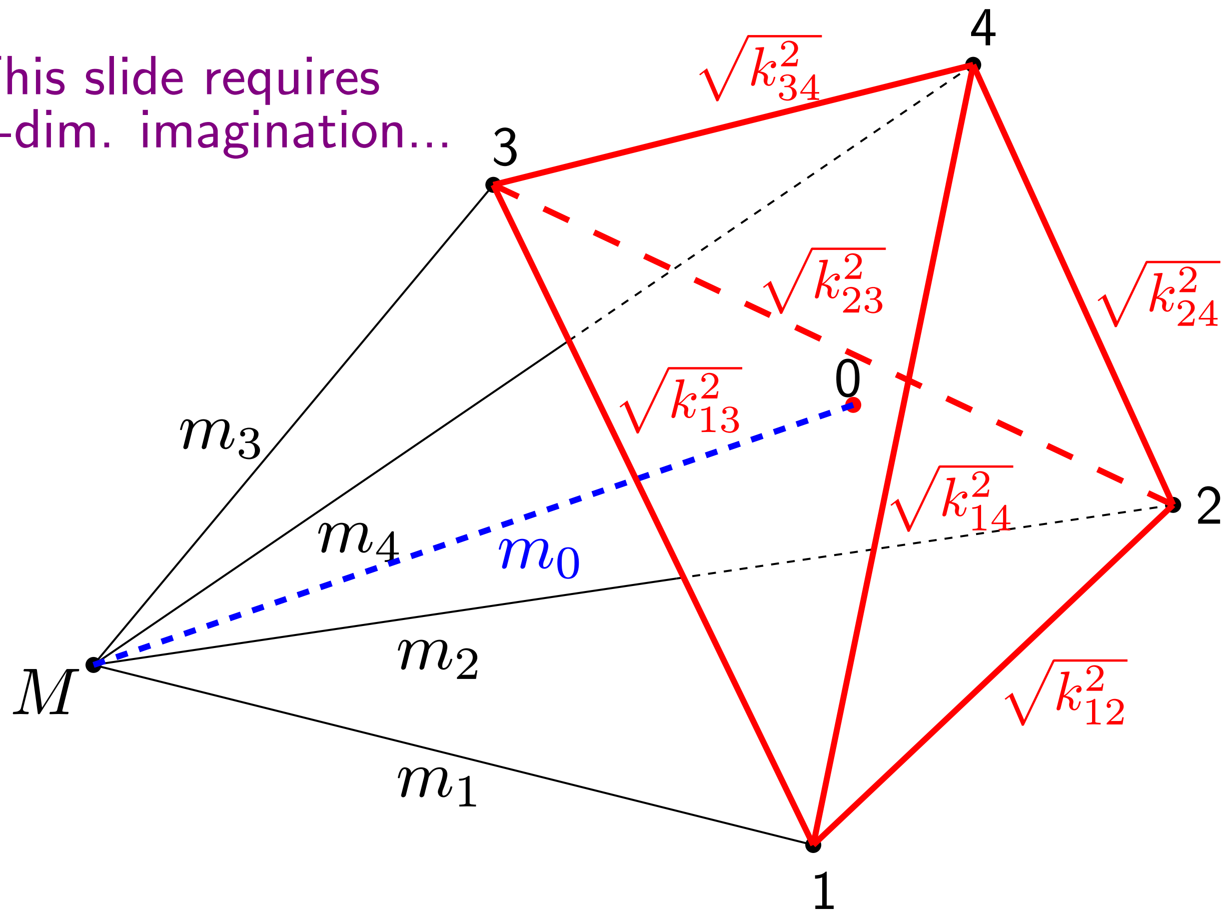


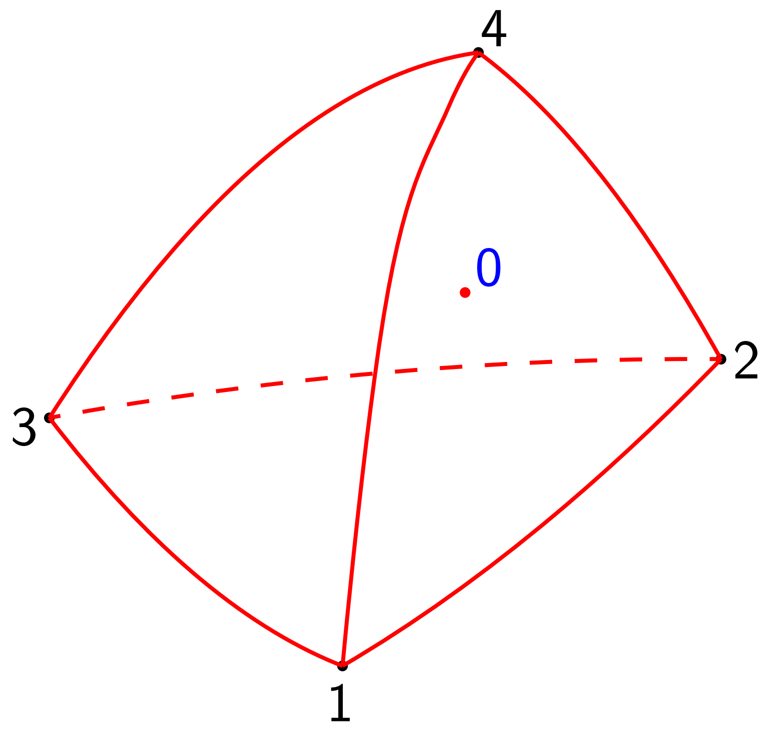


$k_{12}^2, k_{23}^2, k_{34}^2, k_{14}^2$ – external momenta squared
k_{13}^2, k_{24}^2 – Mandelstam variables s and t

Four-point function: basic simplex and non-Euclidean tetrahedron

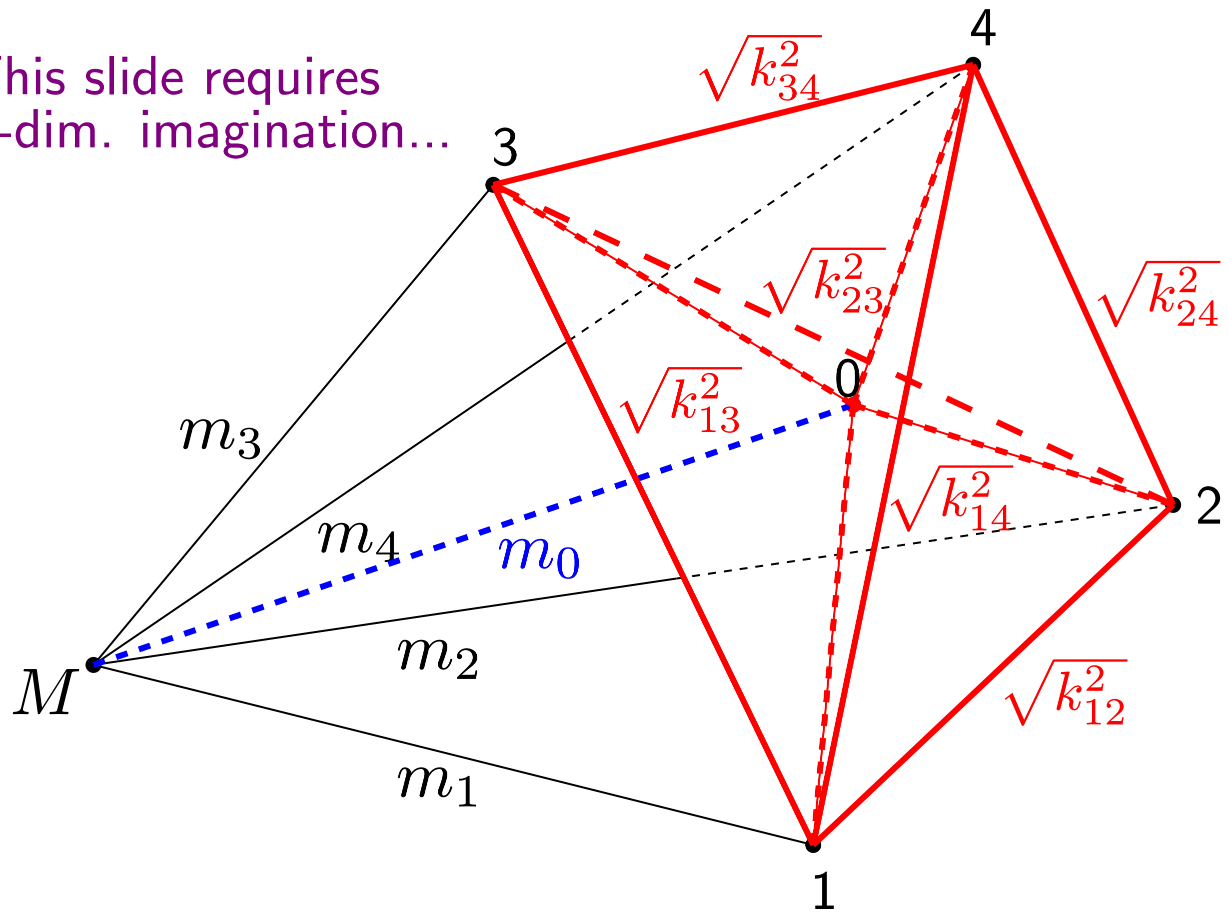
This slide requires
4-dim. imagination...

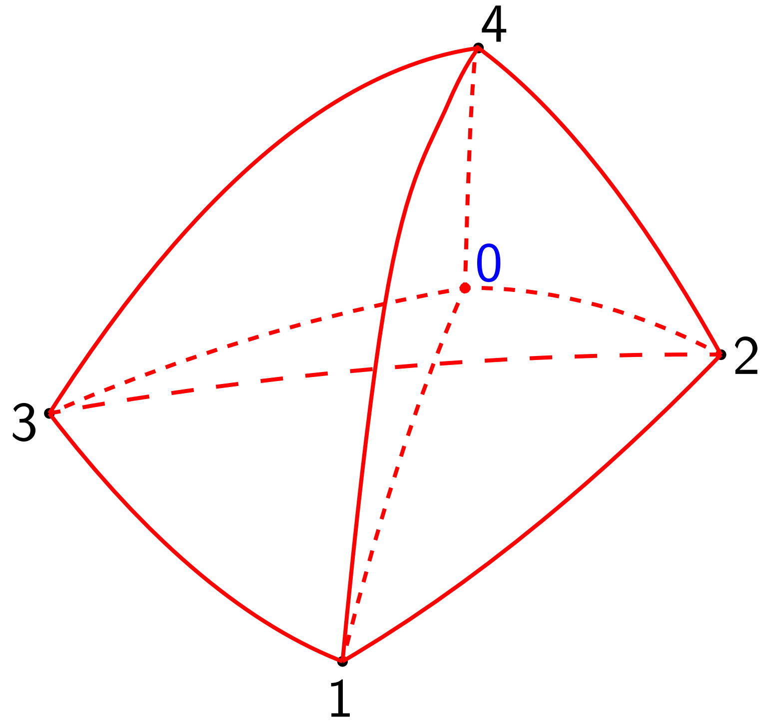




Four-point function: basic simplex and non-Euclidean tetrahedron

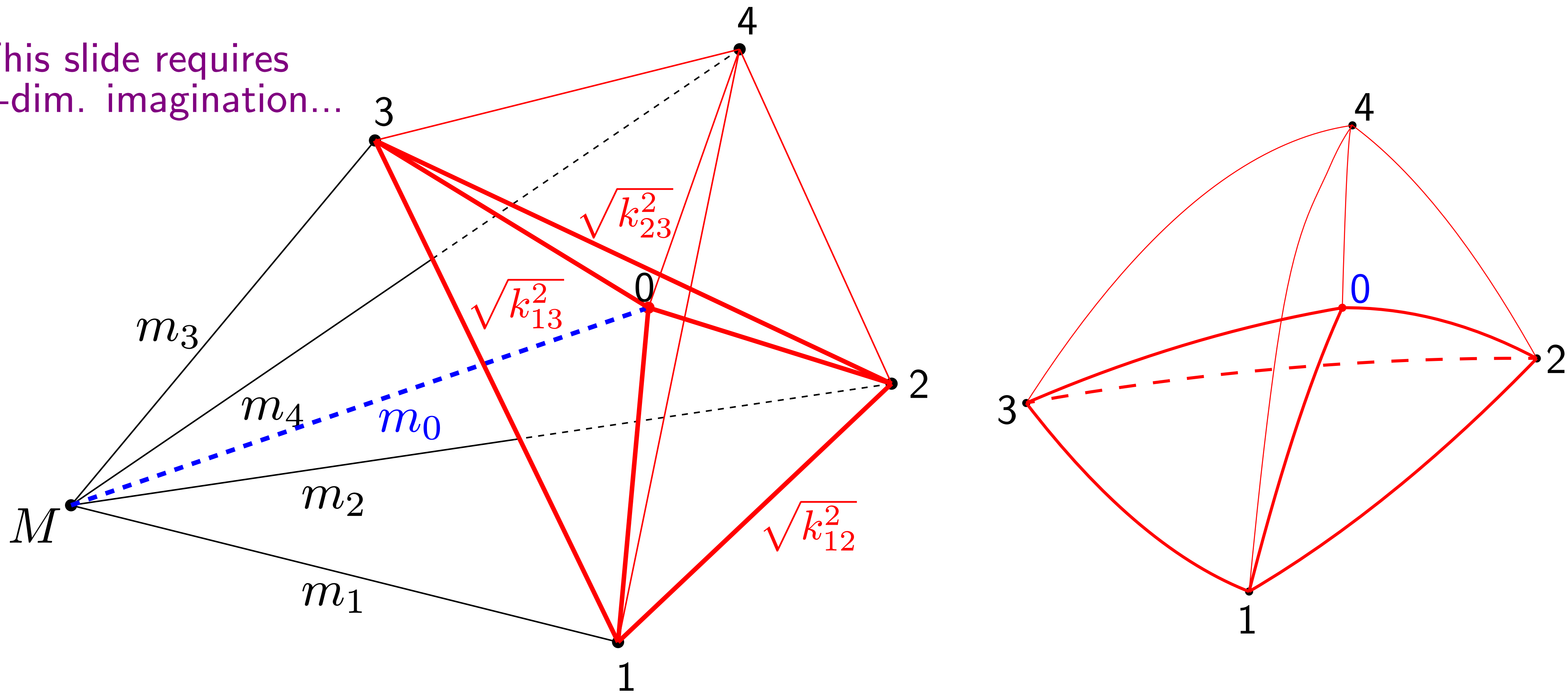
This slide requires
4-dim. imagination...





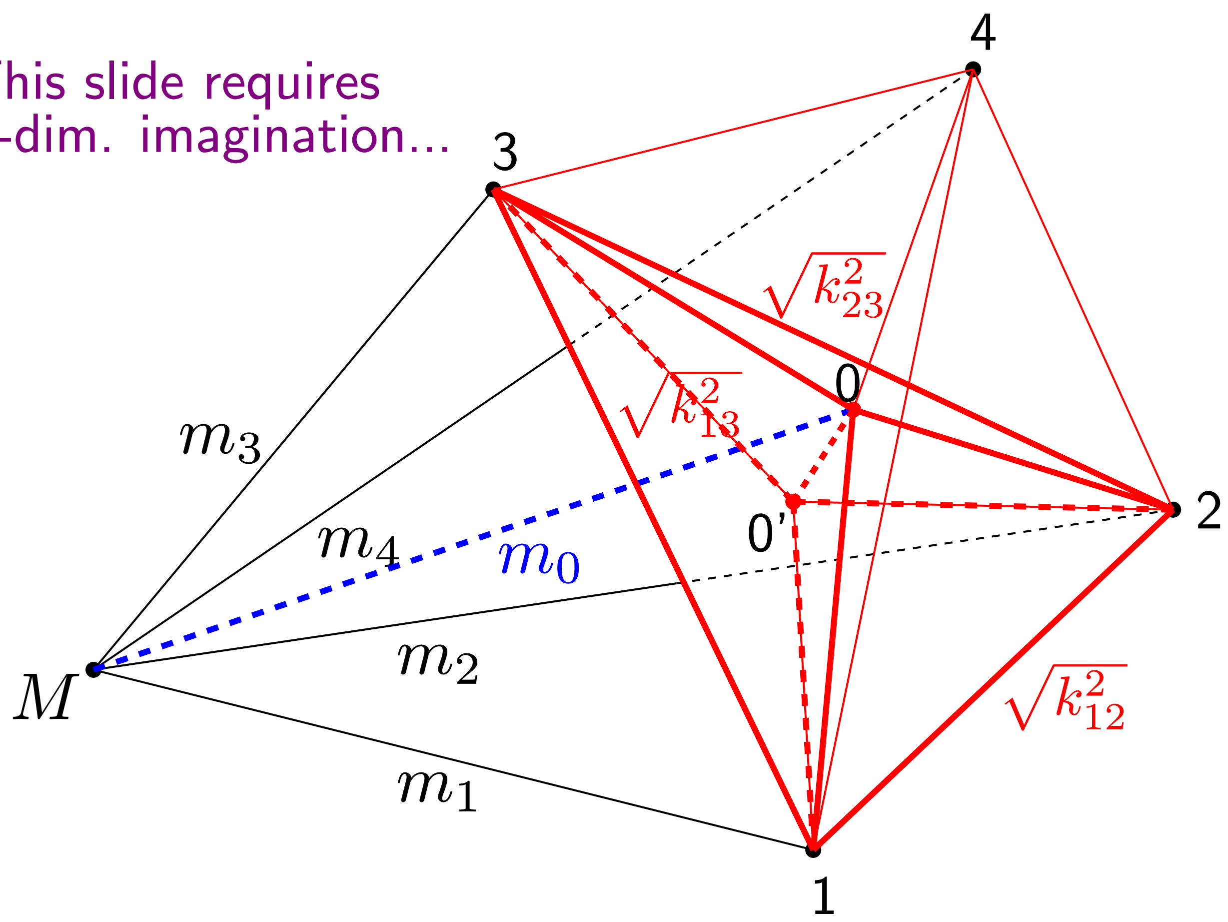
Four-point function: basic simplex and non-Euclidean tetrahedron

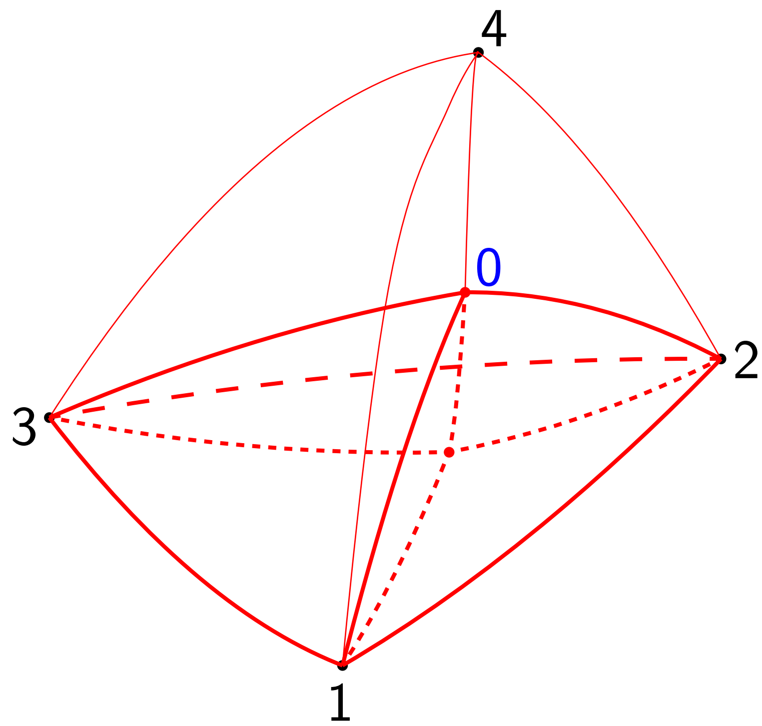
This slide requires
4-dim. imagination...



Four-point function: basic simplex and non-Euclidean tetrahedron

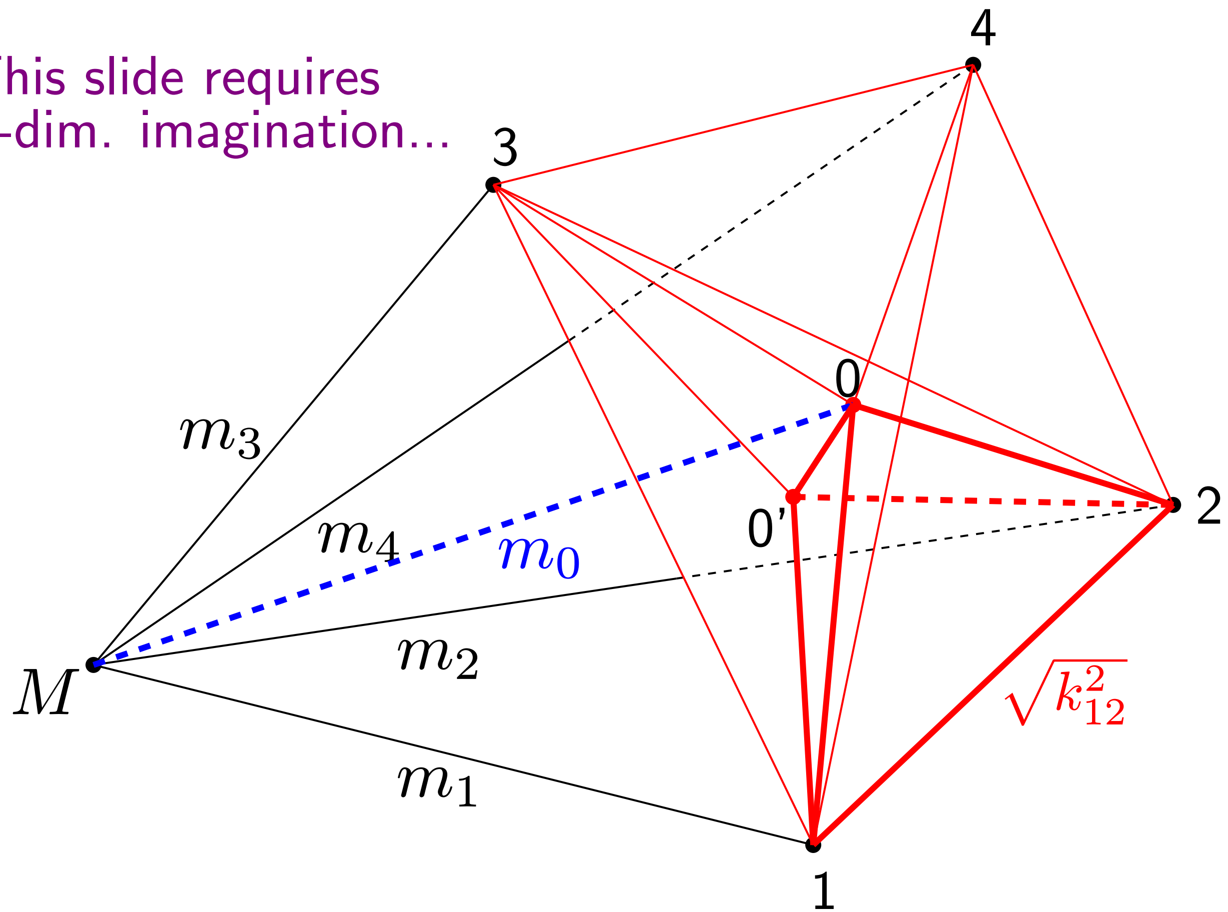
This slide requires
4-dim. imagination...

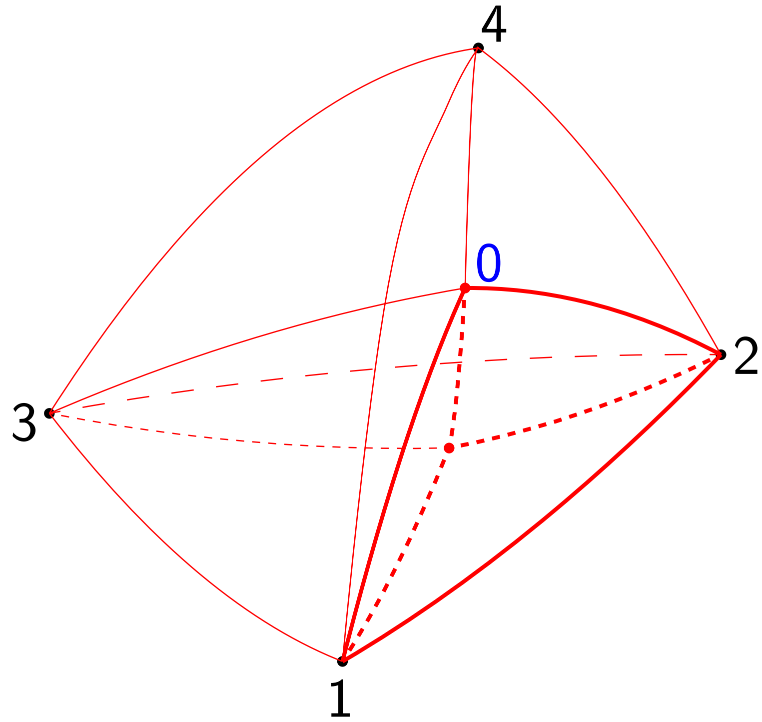




Four-point function: basic simplex and non-Euclidean tetrahedron

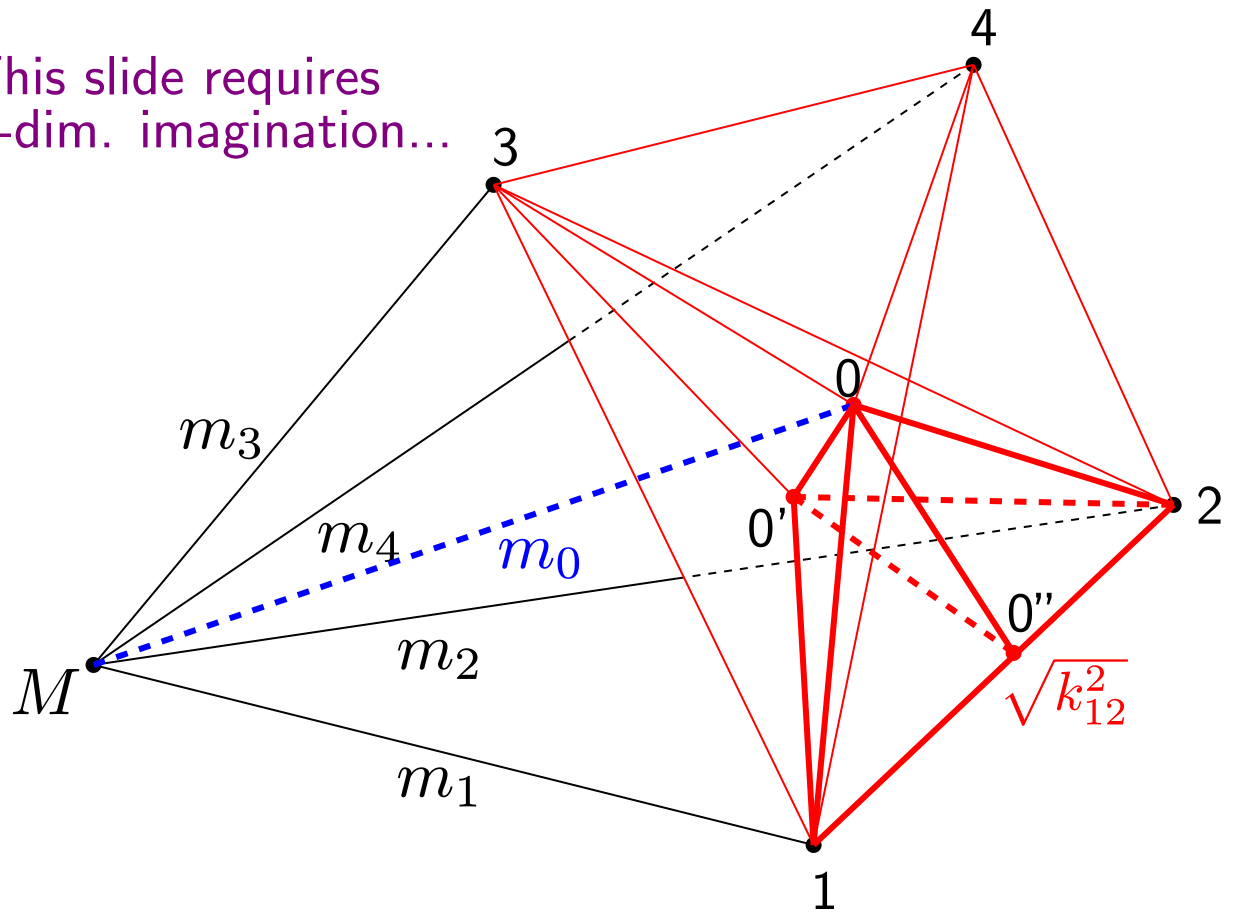
This slide requires
4-dim. imagination...

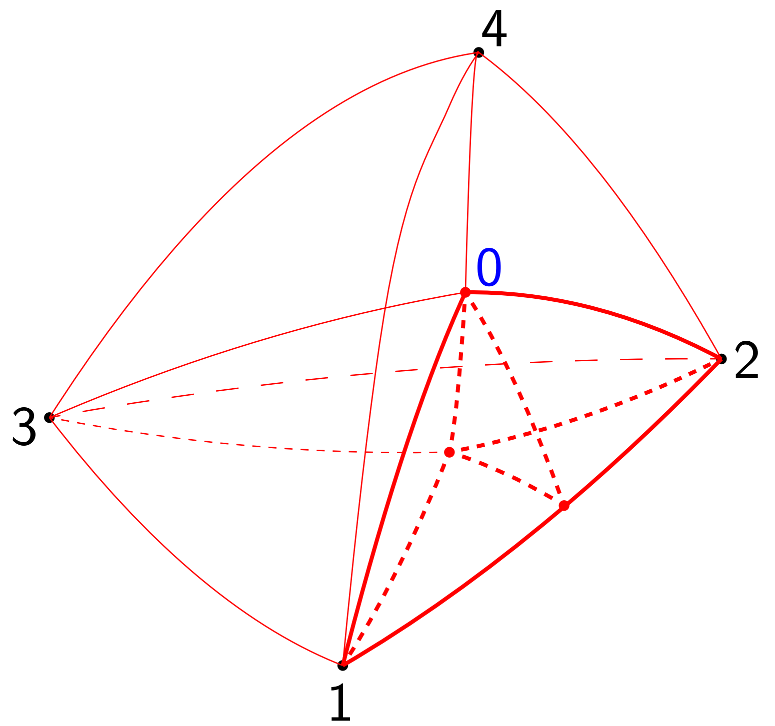




Four-point function: basic simplex and non-Euclidean tetrahedron

This slide requires
4-dim. imagination...





Four-point function: number of dimensionless variables

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{12}^2,k_{23}^2,k_{34}^2,k_{14}^2,k_{13}^2,k_{24}^2\};\{m_1,m_2,m_3,m_4\}\right)$:

$10-1(\text{dimension})=9$

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{12}^2,k_{23}^2,k_{03}^2,k_{01}^2,k_{13}^2,k_{02}^2\};\{m_1,m_2,m_3,m_0\}\right)$

(after splitting the tetrahedron 1234 into four tetrahedra):

$10-3(\text{relations})-1(\text{dimension})=6$

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{12}^2,k_{20'}^2,k_{00'}^2,k_{01}^2,k_{10'}^2,k_{02}^2\};\{m_1,m_2,m_{0'},m_0\}\right)$

(after splitting the tetrahedron 0123 into three tetrahedra):

$10-5(\text{relations})-1(\text{dimension})=4$

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{10''}^2,k_{0'0''}^2,k_{00'}^2,k_{01}^2,k_{10'}^2,k_{00''}^2\};\{m_1,m_{0''},m_{0'},m_0\}\right)$

(after splitting each of the resulting tetrahedra into two):

$10-6(\text{relations})-1(\text{dimension})=3$

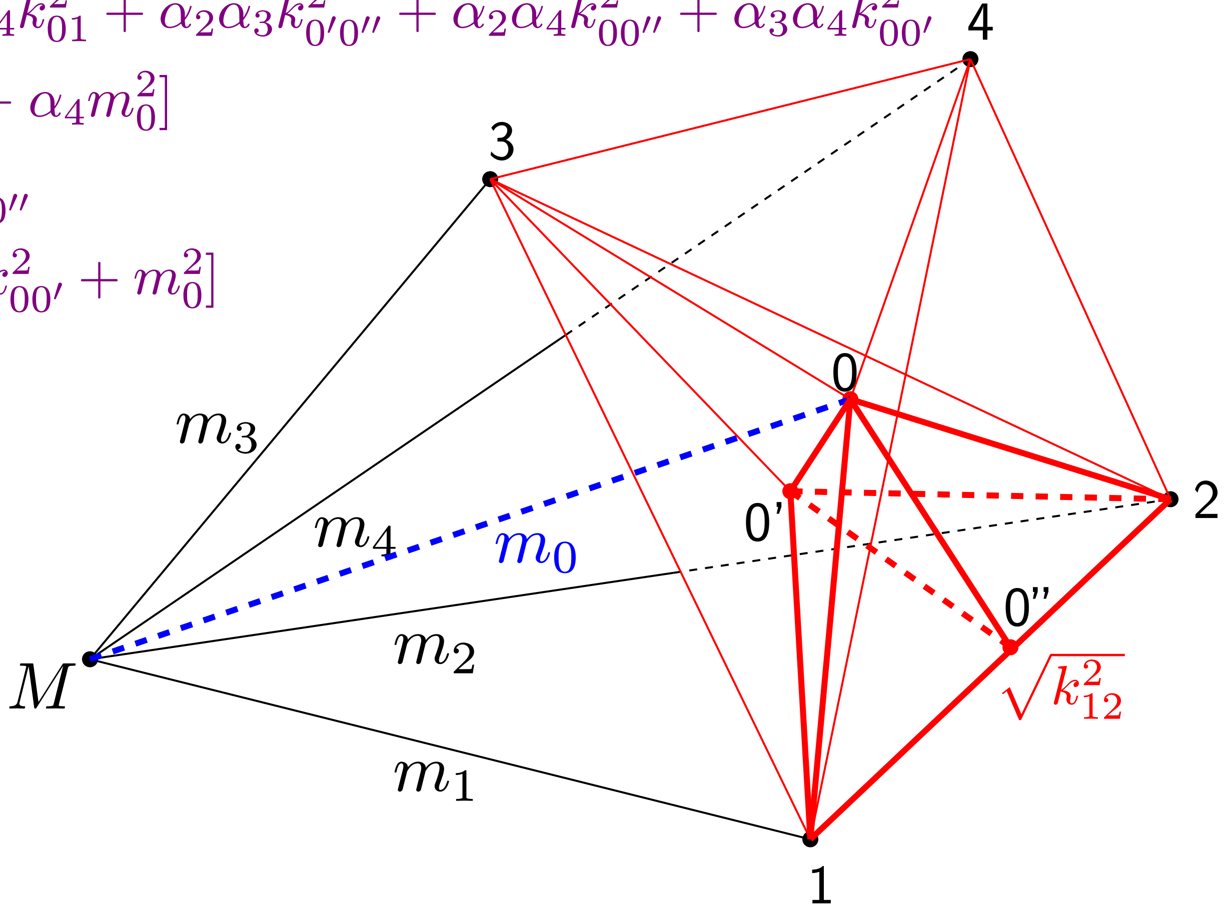
Four-point function: quadratic form in Feynman parametric integral

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{12}^2,k_{23}^2,k_{34}^2,k_{14}^2,k_{13}^2,k_{24}^2\};\{m_1,m_2,m_3,m_4\}\right)$:

$$[\alpha_1\alpha_2k_{12}^2+\alpha_1\alpha_3k_{13}^2+\alpha_1\alpha_4k_{14}^2+\alpha_2\alpha_3k_{23}^2+\alpha_2\alpha_4k_{24}^2+\alpha_3\alpha_4k_{34}^2$$
$$-\alpha_1m_1^2-\alpha_2m_2^2-\alpha_3m_3^2-\alpha_4m_4^2]$$

in $J^{(4)}\left(n;1,1,1,1\middle|\{k_{10''}^2,k_{0'0''}^2,k_{00'}^2,k_{01}^2,k_{10'}^2,k_{00''}^2\};\{m_1,m_{0''},m_{0'},m_0\}\right)$:

$$[\alpha_1\alpha_2k_{10''}^2+\alpha_1\alpha_3k_{10'}^2+\alpha_1\alpha_4k_{01}^2+\alpha_2\alpha_3k_{0'0''}^2+\alpha_2\alpha_4k_{00''}^2+\alpha_3\alpha_4k_{00'}^2$$
$$-\alpha_1m_1^2-\alpha_2m_{0''}^2-\alpha_3m_{0'}^2-\alpha_4m_0^2]$$
$$=-[\alpha_1^2k_{10''}^2+(\alpha_1+\alpha_2)^2k_{0'0''}^2$$
$$+(\alpha_1+\alpha_2+\alpha_3)^2k_{00'}^2+m_0^2]$$



Four-point function: result in arbitrary dimension

$$J^{(4)}\left(n;1,1,1,1\middle|\{k^2_{10''},k^2_{0'0''},k^2_{00'},k^2_{01},k^2_{10'},k^2_{00''}\};\{m_1,m_{0''},m_{0'},m_0\}\right)$$

$$= \mathrm{i}\pi^{n/2}\Gamma(4-n/2)\int\limits_0^1\int\limits_0^1\int\limits_0^1\int\limits_0^1 \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3\,\mathrm{d}\alpha_4\,\delta(\alpha_1+\alpha_2+\alpha_3+\alpha_4-1)}{[\alpha_1^2k^2_{10''}+(\alpha_1+\alpha_2)^2k^2_{0'0''}+(\alpha_1+\alpha_2+\alpha_3)^2k^2_{00'}+m_0^2]^{4-n/2}}$$

$$= \frac{\mathrm{i}\pi^{n/2}\Gamma(3-n/2)}{2k^2_{0'0''}(m_0^2)^{3-n/2}}\left\{\sqrt{\frac{k^2_{0'0''}}{k^2_{10''}}}\arctan\sqrt{\frac{k^2_{10''}}{k^2_{0'0''}}}\;{}_2F_1\left(\begin{array}{c}1/2,3-n/2\\3/2\end{array}\middle|-\frac{k^2_{00'}}{m_0^2}\right)\right.$$

$$\left.-\left(\frac{m_0^2}{m_{0'}^2}\right)^{2-n/2}F_N\left(1,1,3-n/2,1/2,(n-3)/2,1/2;3/2,3/2,3/2\middle|-\frac{k^2_{10''}}{k^2_{0'0''}},-\frac{k^2_{00'}}{m_0^2},-\frac{k^2_{10'}}{m_{0'}^2}\right)\right\}$$

where F_N is one of the Lauricella-Saran functions,

$$F_N(a_1,a_2,a_3,b_1,b_2,b_1;c_1,c_2,c_2|x,y,z)=\sum_{j_1,j_2,j_3}\frac{(a_1)_{j_1}(a_2)_{j_2}(a_3)_{j_3}(b_1)_{j_1+j_3}(b_2)_{j_2}}{(c_1)_{j_1}(c_2)_{j_2+j_3}}\frac{x^{j_1}y^{j_2}z^{j_3}}{j_1!j_2!j_3!}$$

A.I.D., arXiv:1711.07351

See also: J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. **B672** (2003) 303 (F_S can be transformed into F_N)

J. Blümlein, K. H. Phan, T. Riemann, Nucl. Part. Phys. Proc. **270-272** (2016) 227

Reduction package (F_S): V.V. Bytev, M.Yu. Kalmykov, S.-O. Moch, Comput. Phys. Commun. **185** (2014) 3041

Reduced number of variables and simplified quadratic forms

	total # of dimensionless variables	# of splitting pieces	reduced # of variables
$N = 2$	$3 - 1 = 2$	2	1
$N = 3$	$6 - 1 = 5$	6	2
$N = 4$	$10 - 1 = 9$	24	3
arbitrary N	$\frac{1}{2}(N-1)(N+2)$	$N!$	$N-1$

$J^{(2)}\left(n; 1, 1 \middle| k_{01}^2; m_1, m_0\right)$:

$$\Rightarrow -[\alpha_1^2 k_{01}^2 + m_0^2]$$

$J^{(3)}\left(n; 1, 1, 1 \middle| k_{00'}^2, k_{01}^2, k_{10'}^2; m_1, m_{0'}, m_0\right)$:

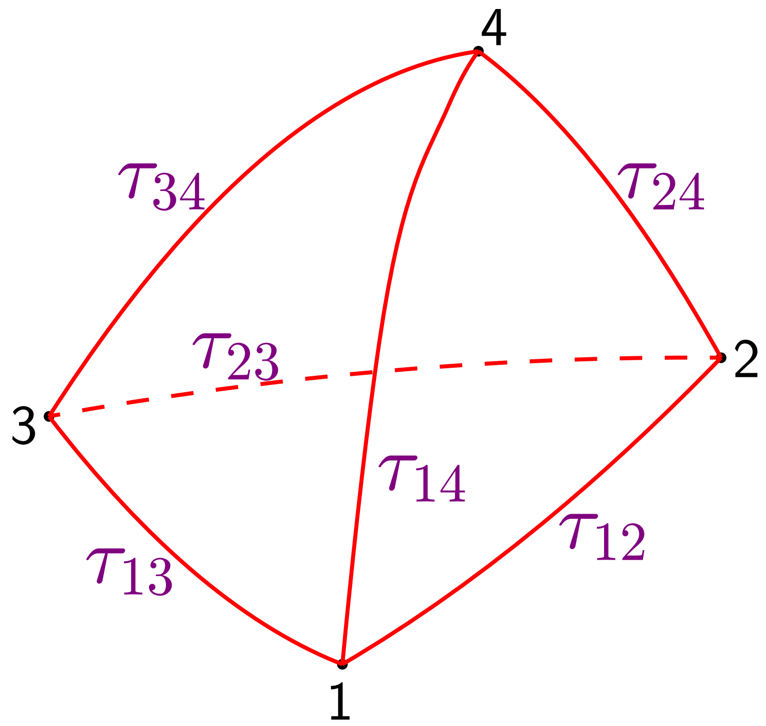
$$\Rightarrow -[\alpha_1^2 k_{10'}^2 + (\alpha_1 + \alpha_2)^2 k_{00'}^2 + m_0^2]$$

$J^{(4)}\left(n; 1, 1, 1, 1 \middle| \{k_{10''}^2, k_{0'0''}^2, k_{00'}^2, k_{01}^2, k_{10'}^2, k_{00''}^2\}; \{m_1, m_{0''}, m_{0'}, m_0\}\right)$:

$$\Rightarrow -[\alpha_1^2 k_{10''}^2 + (\alpha_1 + \alpha_2)^2 k_{0'0''}^2 + (\alpha_1 + \alpha_2 + \alpha_3)^2 k_{00'}^2 + m_0^2]$$

$\Rightarrow$ for $N > 4$ we should also expect squares of sums of partial sums of α's

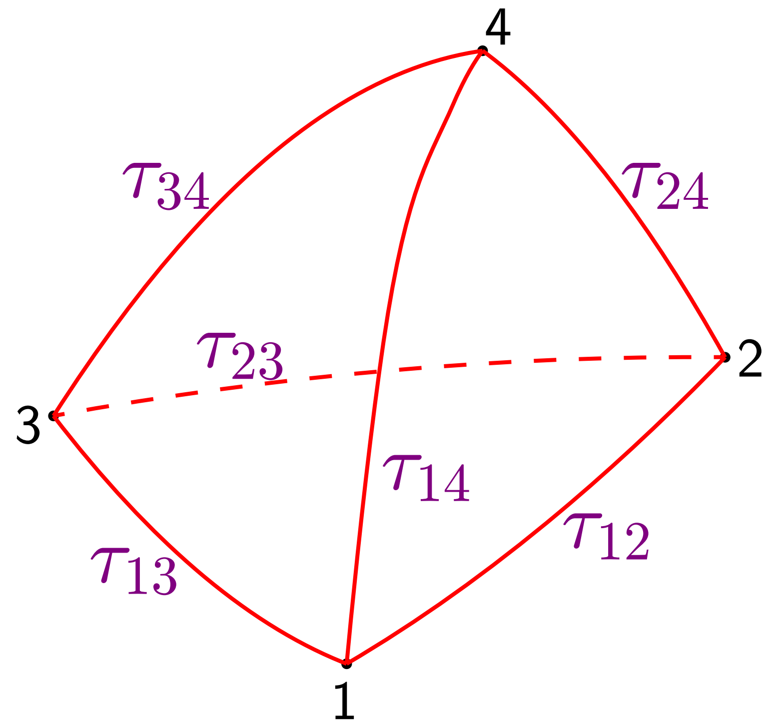
Four-point function in four dimension



The non-Euclidean (spherical or hyperbolic) tetrahedon $\leftrightarrow \Omega^{(4)}$

$$J^{(4)}(4;1,1,1,1) = \mathrm{i}\pi^2 \, \frac{\Omega^{(4)}}{m_1 m_2 m_3 m_4 \sqrt{D^{(4)}}}$$

Four-point function in four dimension



To calculate the volume integral $\Omega^{(4)}$, one may split this tetrahedron into tetrahedra with some rectangular angles – birectangular tetrahedra ("orthoschemes"), etc.

N.I. Lobachevsky (1836), L. Schläfli (1858), H.S. Coxeter (1935), R. Kellerhals, E.B. Vinberg, e.a.

For the volume of three-dimensional tetrahedron in the space of constant curvature, nice closed formulae exist, in terms of dilogarithmic functions,

Yu. Cho and H. Kim (1999), J. Murakami and M. Yano (2001), A. Ushijima (2002)

We can also use the result in terms of F_N (at $n = 4$) and its integral representation.

Four-point function in four dimension: the massless limit

When all masses $m_i \to 0$, the quantities c_{jl} become infinite and should be considered as hyperbolic cosines. For the Gram determinant $D^{(4)}$ we get

$$\left(m_1^2 m_2^2 m_3^2 m_4^2 D^{(4)} \right)\Big|_{m_i \to 0} \;\Rightarrow\; \tfrac{1}{16}\Big[(k_{12}^2 k_{34}^2)^2 + (k_{13}^2 k_{24}^2)^2 + (k_{14}^2 k_{23}^2)^2$$
$$-2k_{12}^2 k_{34}^2 k_{13}^2 k_{24}^2 - 2k_{12}^2 k_{34}^2 k_{14}^2 k_{23}^2 - 2k_{13}^2 k_{24}^2 k_{14}^2 k_{23}^2 \Big]$$
$$= \tfrac{1}{16} \lambda \left(k_{12}^2 k_{34}^2, k_{13}^2 k_{24}^2, k_{14}^2 k_{23}^2 \right)$$

The elements of the dual matrix $\|\widetilde{c}_{jl}\|$ still can be interpreted as cosines of the dihedral angles

$$\widetilde{c}_{12} = \widetilde{c}_{34} = -\cos\psi_{34} = -\cos\psi_{12} = \frac{k_{13}^2 k_{24}^2 + k_{14}^2 k_{23}^2 - k_{12}^2 k_{34}^2}{\sqrt{k_{13}^2 k_{24}^2 k_{14}^2 k_{23}^2}} ,$$

$$\widetilde{c}_{13} = \widetilde{c}_{24} = -\cos\psi_{24} = -\cos\psi_{13} = \frac{k_{14}^2 k_{23}^2 + k_{12}^2 k_{34}^2 - k_{13}^2 k_{24}^2}{\sqrt{k_{14}^2 k_{23}^2 k_{12}^2 k_{34}^2}} ,$$

$$\widetilde{c}_{14} = \widetilde{c}_{23} = -\cos\psi_{23} = -\cos\psi_{14} = \frac{k_{12}^2 k_{34}^2 + k_{13}^2 k_{24}^2 - k_{14}^2 k_{23}^2}{\sqrt{k_{12}^2 k_{34}^2 k_{13}^2 k_{24}^2}} .$$

Four-point function in four dimension: the massless limit (continued)

- Therefore, in this situation we get nothing but an *ideal* hyperbolic tetrahedron, i.e., the tetrahedron whose vertices are all at infinity.

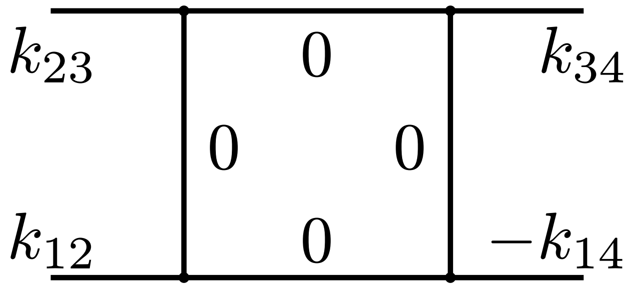
- The pairs of opposite dihedral angles of the ideal tetrahedron are equal, $\psi_{12} = \psi_{34}$, $\psi_{13} = \psi_{24}$, $\psi_{14} = \psi_{23}$, whereas its volume is given by

$$\Omega^{(4)} = \frac{1}{2\mathrm{i}} \Big[\mathrm{Cl}_2\left(2\psi_{12}\right) + \mathrm{Cl}_2\left(2\psi_{13}\right) + \mathrm{Cl}_2\left(2\psi_{23}\right) \Big] \, , \qquad \psi_{12} + \psi_{13} + \psi_{23} = \pi \, .$$

- In this case, all equations depend only on the products $k_{12}^2 k_{34}^2$, $k_{13}^2 k_{24}^2$, $k_{14}^2 k_{23}^2$, and the result for $\Omega^{(4)}$ is of the same form as the massless three-point function in four dimensions
  $\leftrightarrow$ "glueing" of the arguments

  N.I. Ussyukina and A.I.D., Phys. Lett. B298 (1993) 363

$k_{12}^2, k_{23}^2, k_{34}^2, k_{14}^2$ – external momenta squared
k_{13}^2, k_{24}^2 – Mandelstam variables s and t



Reduction of integrals with $N > n$

Use the height m_0 to split the N-dimensional simplex in the case when $n = N - 1$,

$$J^{(N)}(N-1;1,\ldots,1) = \frac{1}{\Lambda^{(N)}}\left(\prod m_i^2\right)\sum_{i=1}^{N}\frac{F_i^{(N)}}{m_i^2}J_i^{(N)}(N-1;1,\ldots,1)\;,$$

where $J_i^{(N)}(N-1;1,\ldots,1)$ is obtained via substituting the ith mass side by m_0.

Using the representation with $\delta\left((\alpha^T\|c\|\alpha)-1\right)$ we can show that

$$J_i^{(N)}(N-1;1,\ldots,1) = -\frac{1}{2m_0^2}\,J^{(N-1)}(N-1;1,\ldots,1)\Big|_{\text{without } i}\;.$$

Therefore,

$$J^{(N)}(N-1;1,\ldots,1) = -\frac{1}{2D^{(N)}}\sum_{i=1}^{N}\frac{F_i^{(N)}}{m_i^2}J^{(N-1)}(N-1;1,\ldots,1)\Big|_{\text{without } i}\;.$$

in agreement with known results.

B.G. Nickel, J. Math. Phys. **19** (1978) 542,

B. Petersson, J. Math. Phys. **6** (1965) 1955, and other papers

Summary

- A geometrical way to calculate dimensionally-regulated Feynman diagrams is reviewed.

- All variables (k_{jl}^2 and m_i) acquire direct geometrical meaning; the integration region corresponds to an N-dimensional solid angle; thresholds (and pseudothresholds) can be associated with situations when some hypervolumes vanish; the dependence on the momenta and masses is moved from the integrand (as in Feynman parametric representation) into the integration limits

- In the one-loop N-point case, results can be related to certain volume integrals in non-Euclidean geometry. For example, the result for the four-point function can be associated with the content of a spherical or hyperbolic tetrahedron in three-dimensional spherical or hyperbolic space (Lobachevsky, Schläfli, ...)

- Analytical continuation of the results to other regions of kinematical variables (momenta and masses of the particles) is discussed. In a number of cases, analytic results can be presented in terms of the (generalized) polylogarithms and associated functions. In more complicated cases, multiple polylogarithms may appear.

Summary (continued)

- Geometrical splitting provides straightforward way to reduce general integrals to those with lesser number of independent variables and predict the set and the number of these variables in the resulting integrals; it also allows to derive functional relations between integrals with different momenta and masses.

- Resulting integrals (after splitting) can be calculated either within geometrical approach (by integrating over non-Euclidean simplices), or by going back to the Feynman parametric representation, which becomes greatly simplified due to right-triangle connections between the invariants.

- Explicit results for general N-point integrals in arbitrary dimension can be presented in terms of hypergeometric functions of $(N-1)$ variables, in particular:
  - for the 2-point diagram we get the hypergeometric function ${}_2F_1$;
  - for the 3-point diagram we get Appell hypergeometric function F_1;
  - for the 4-point diagram we get the Lauricella-Saran function F_N (which can be transformed into F_S).