Introduction to hydrodynamic description of Heavy-Ion Collisions

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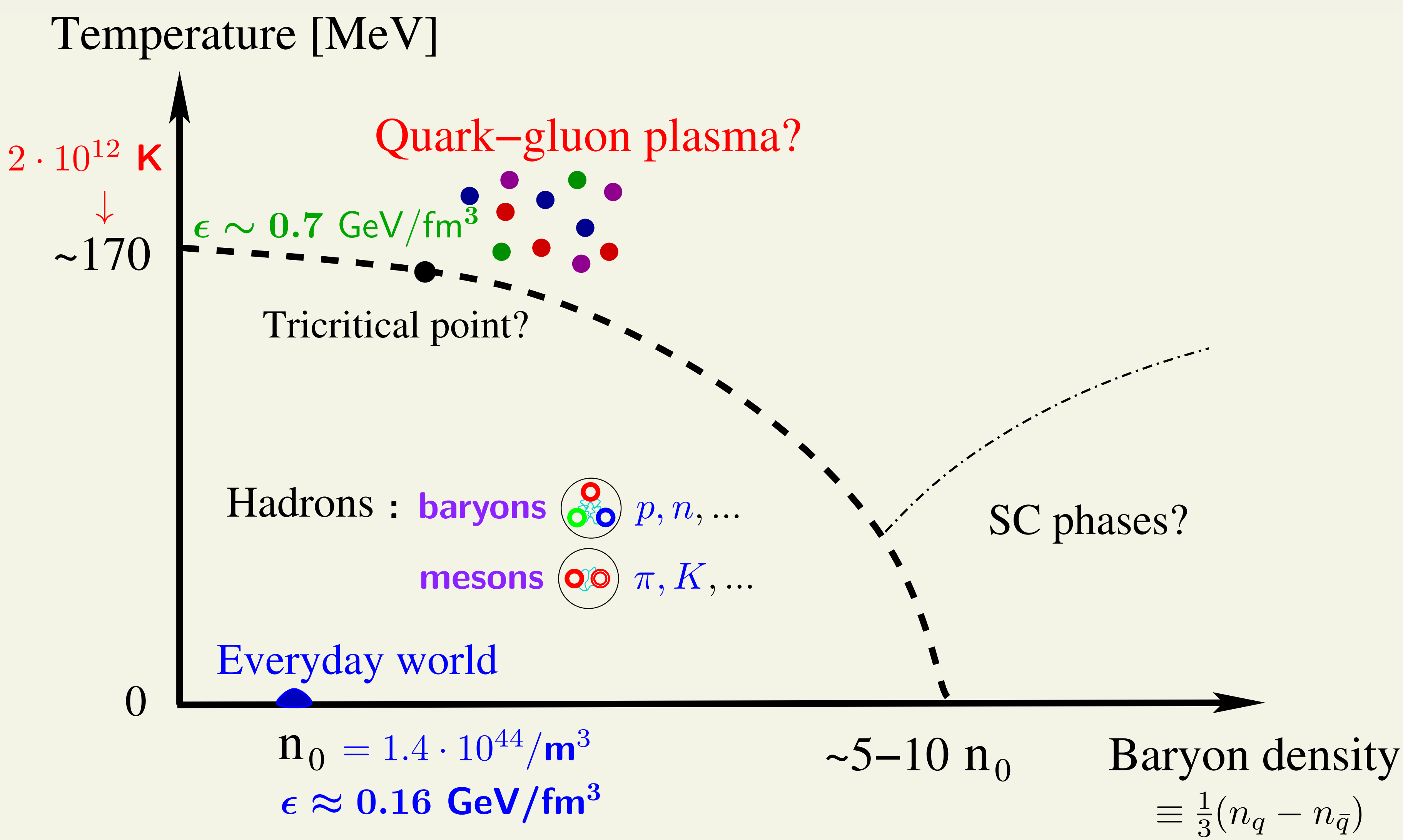
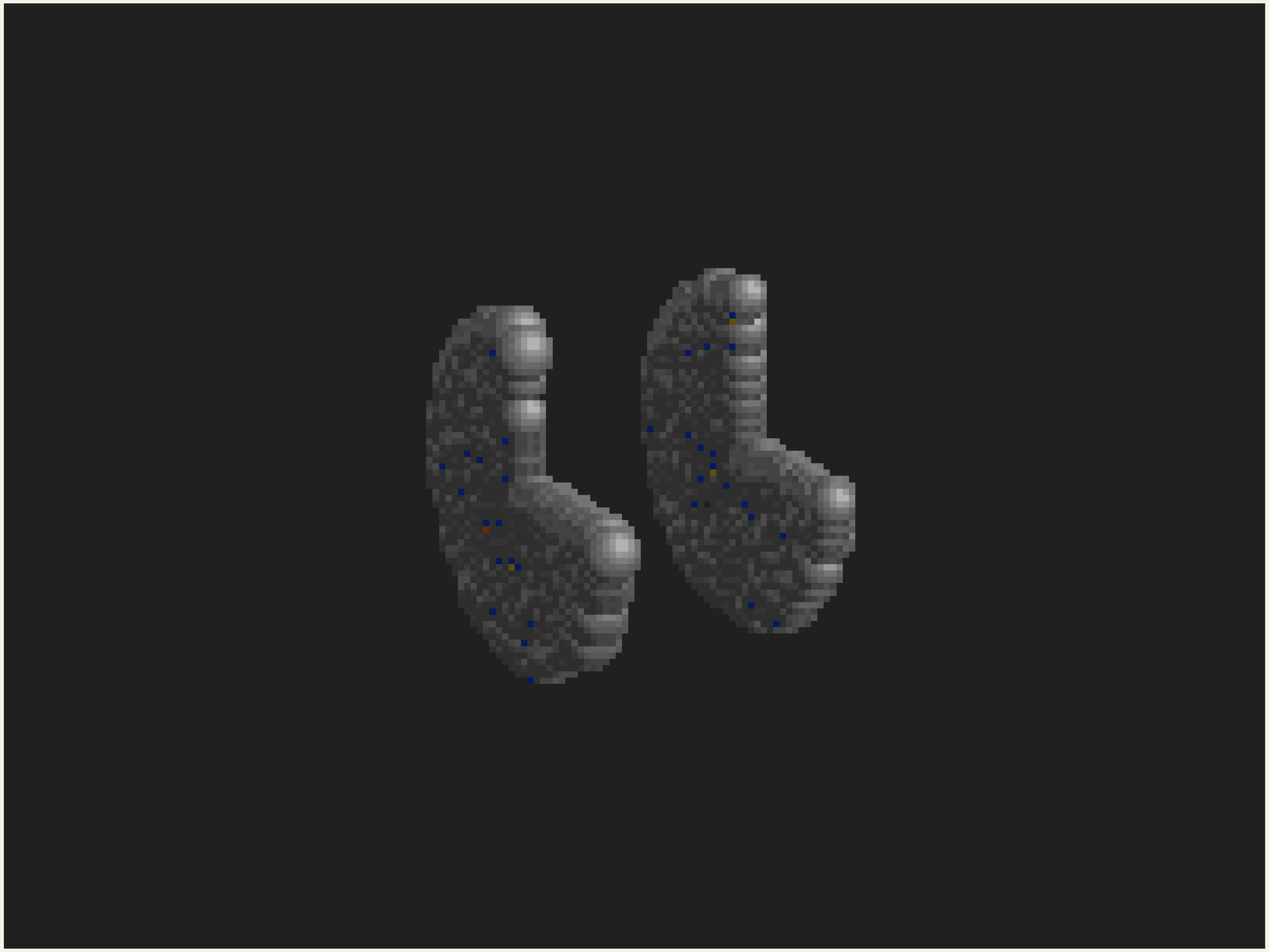
Nuclear phase diagram



Heavy-ion collision



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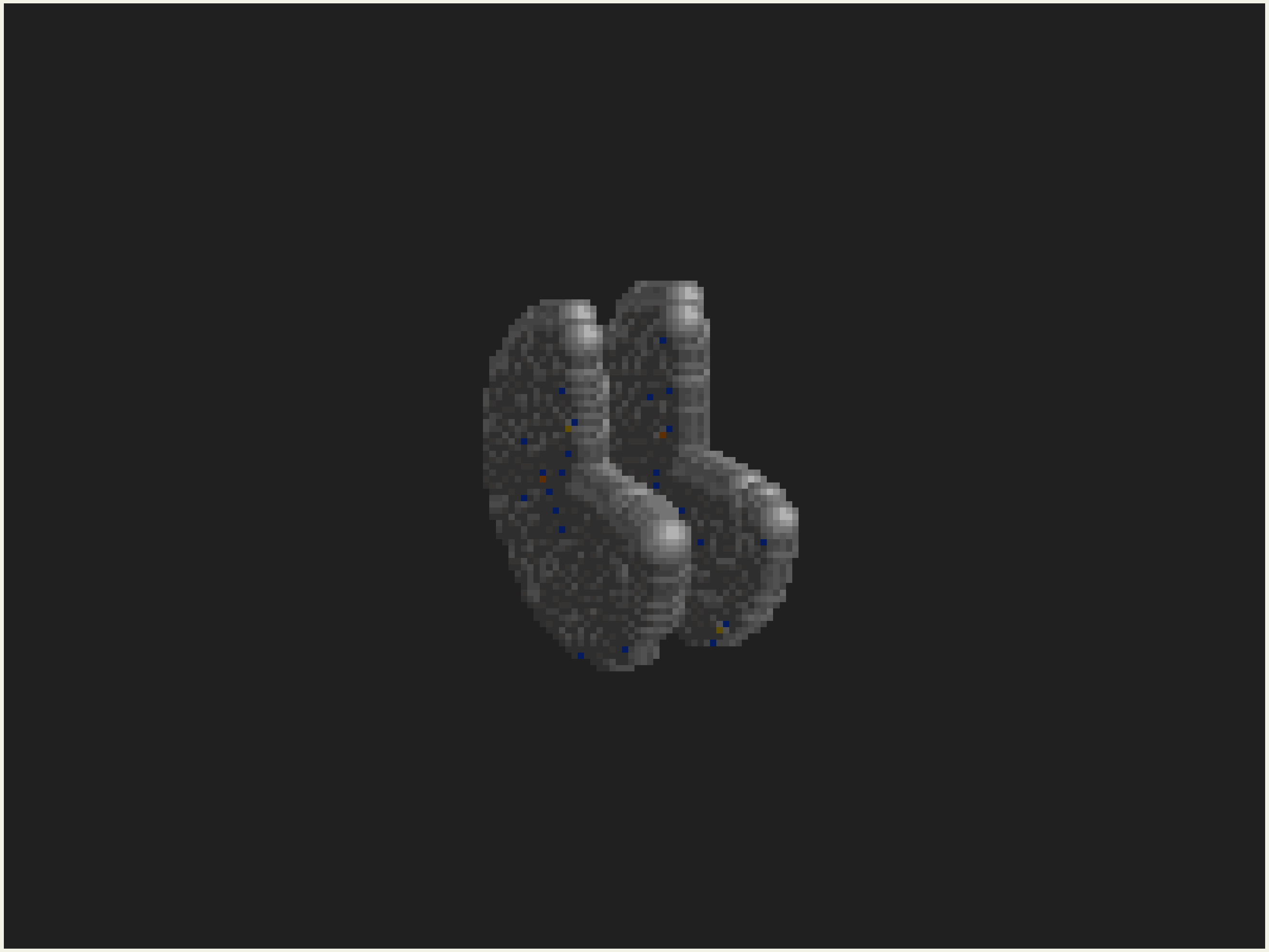
Heavy-ion collision



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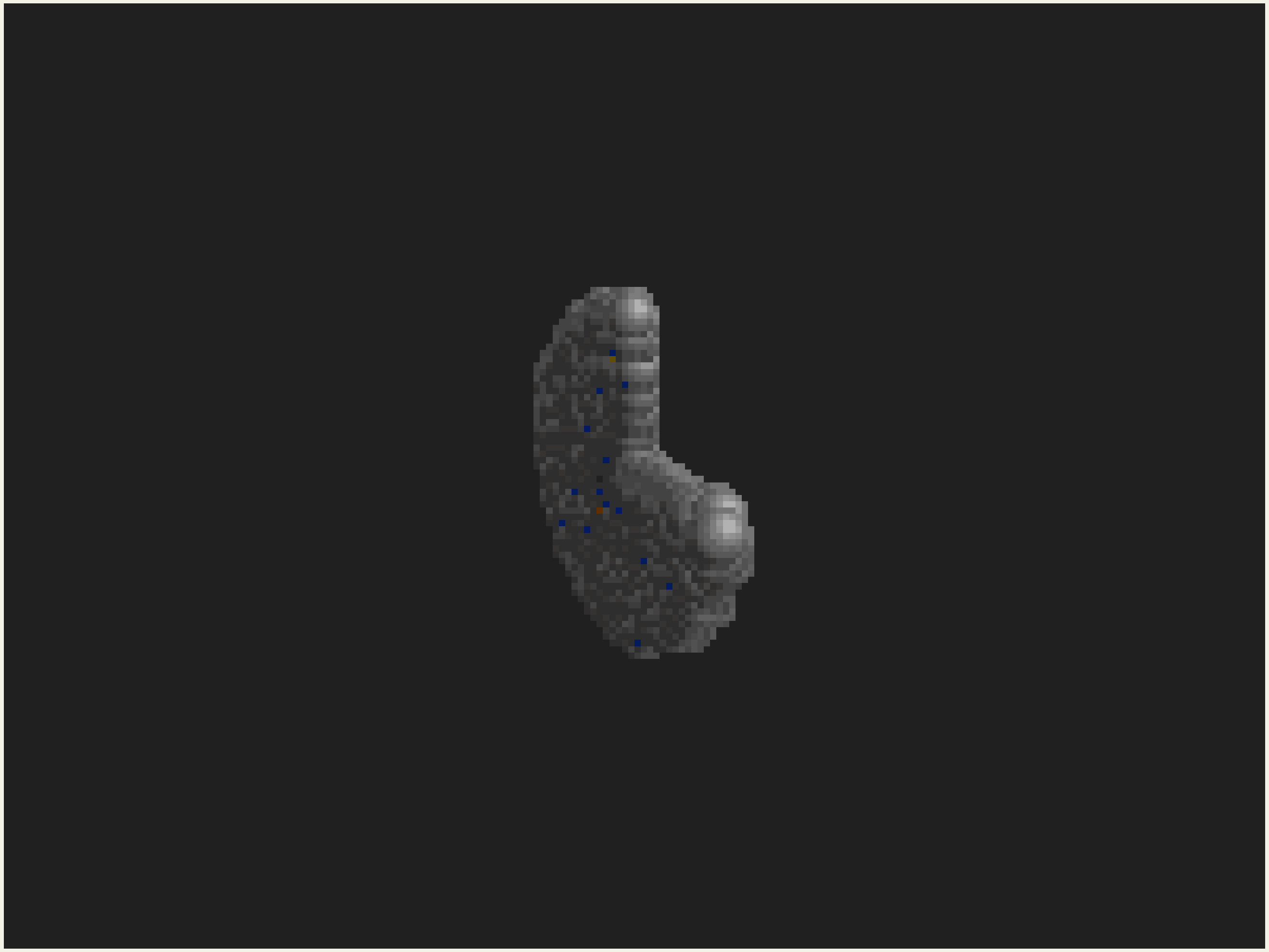
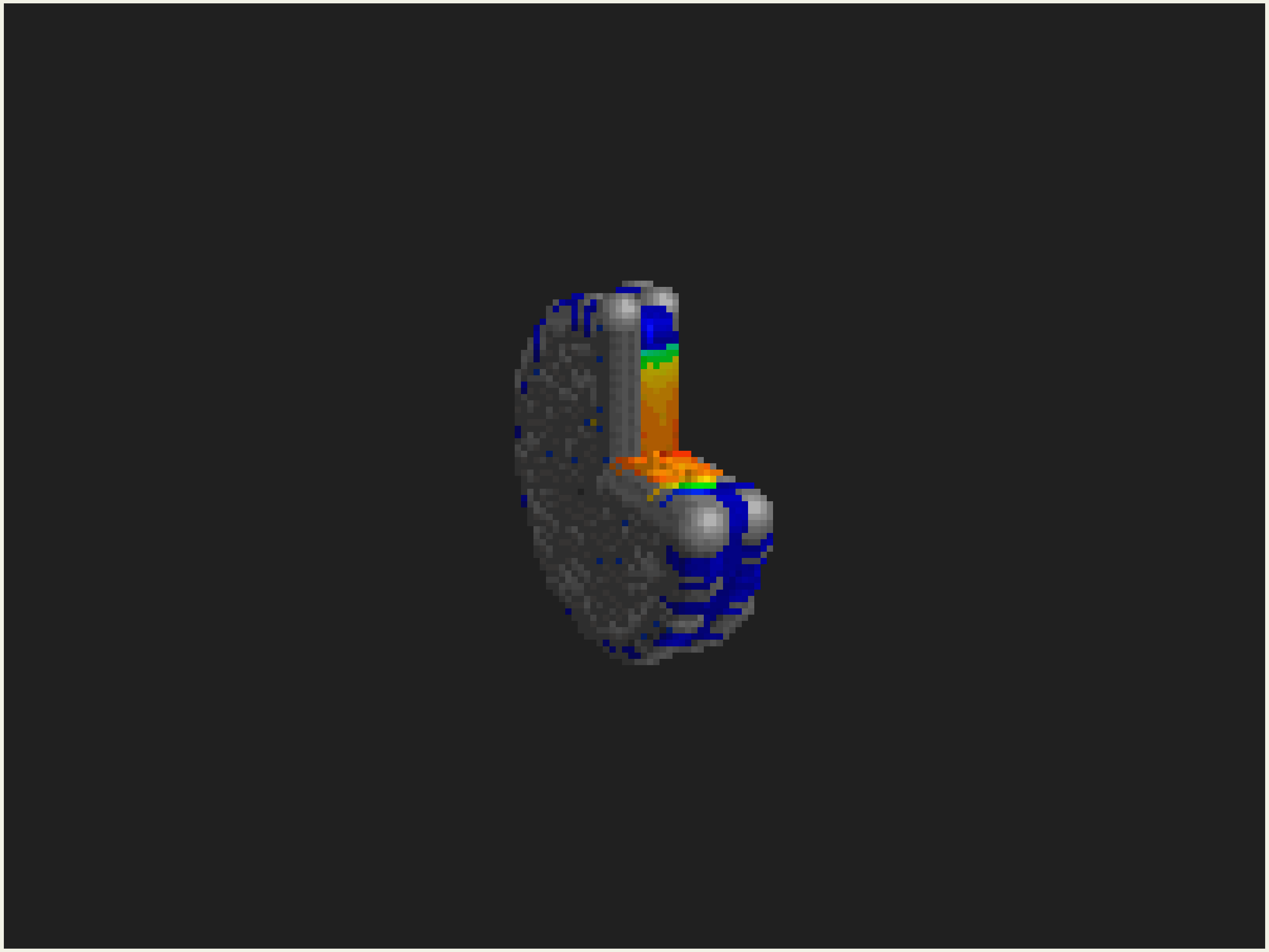


Heavy-ion collision



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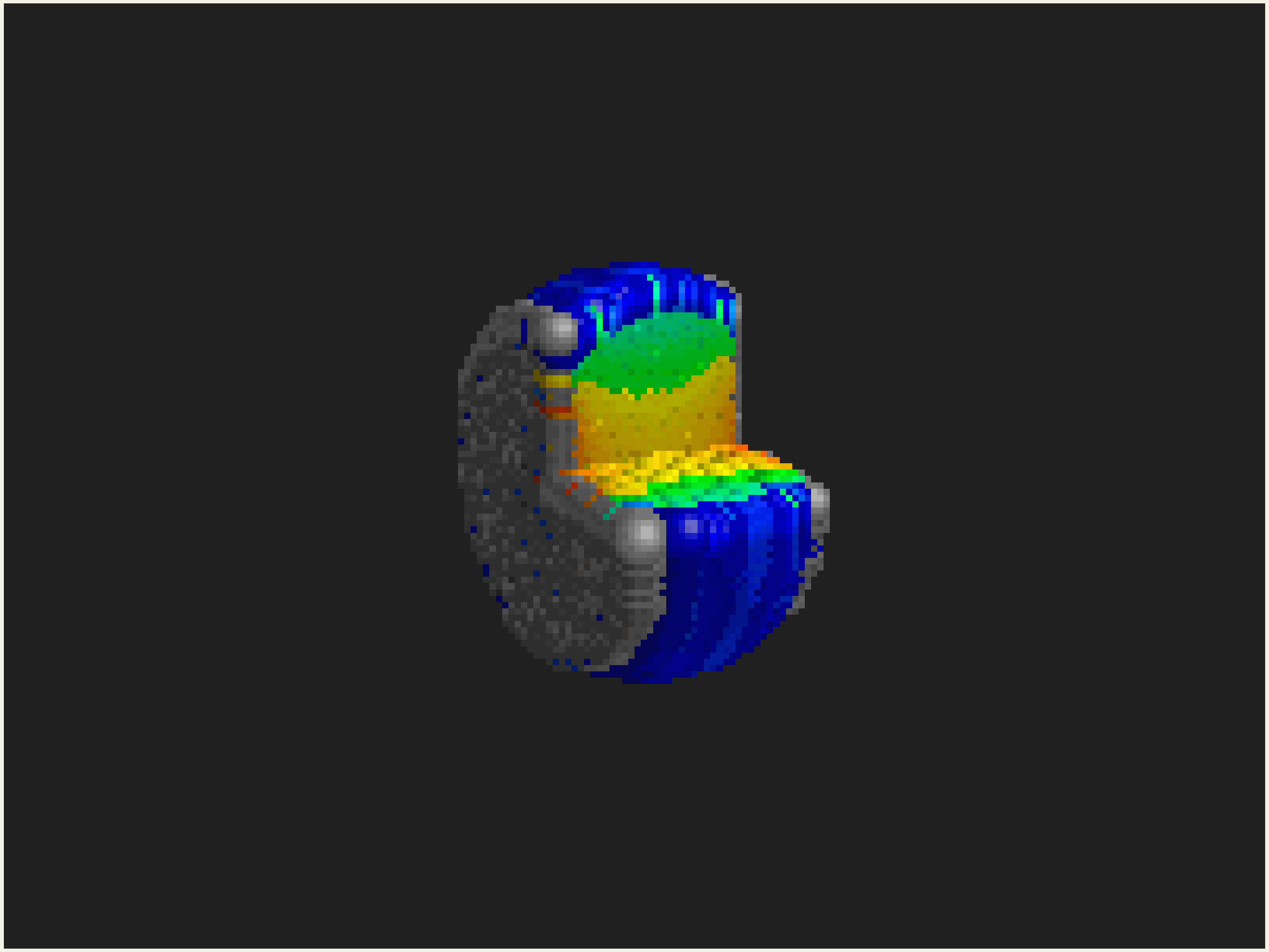
Heavy-ion collision



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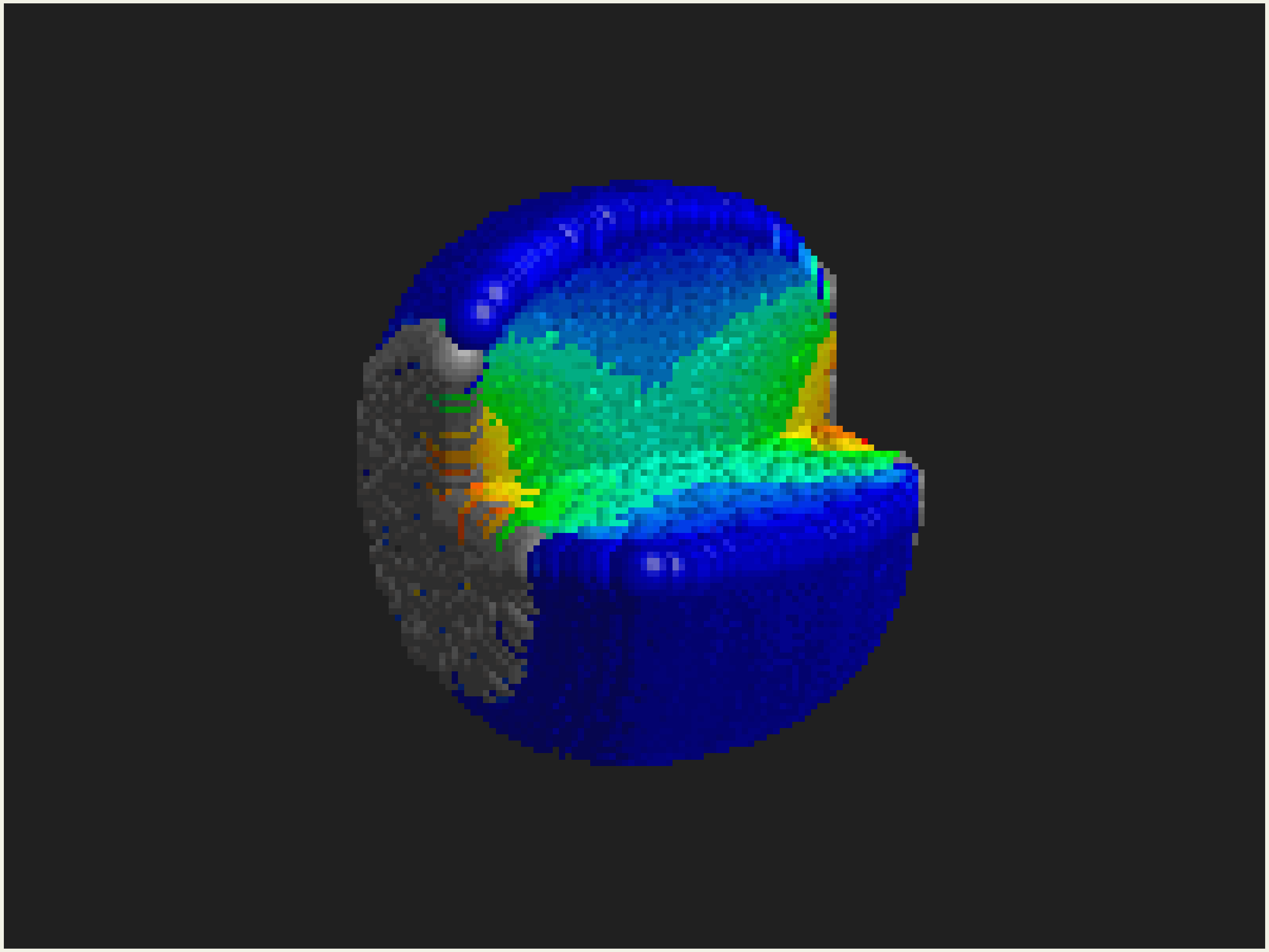
Heavy-ion collision



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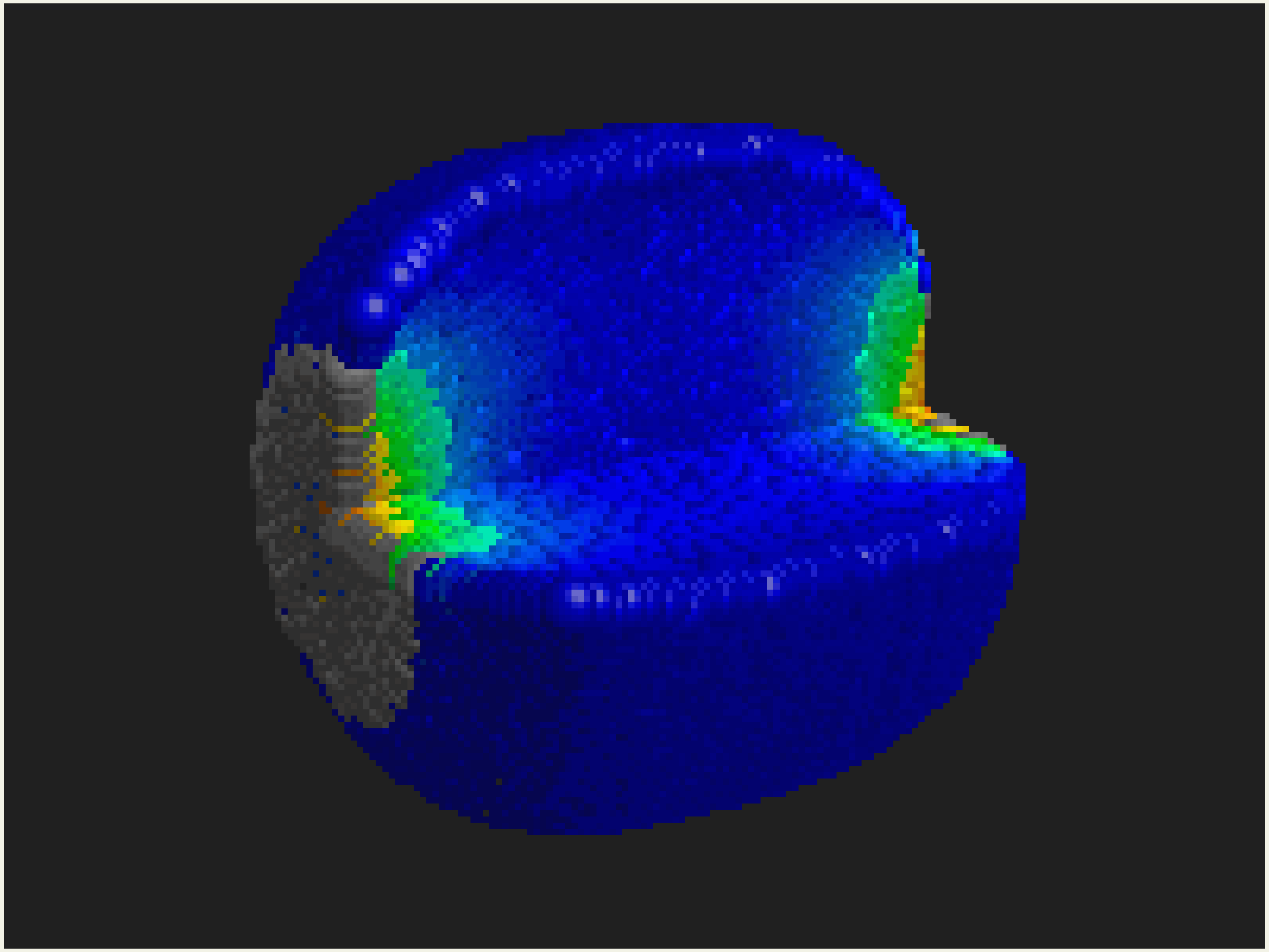
Heavy-ion collision



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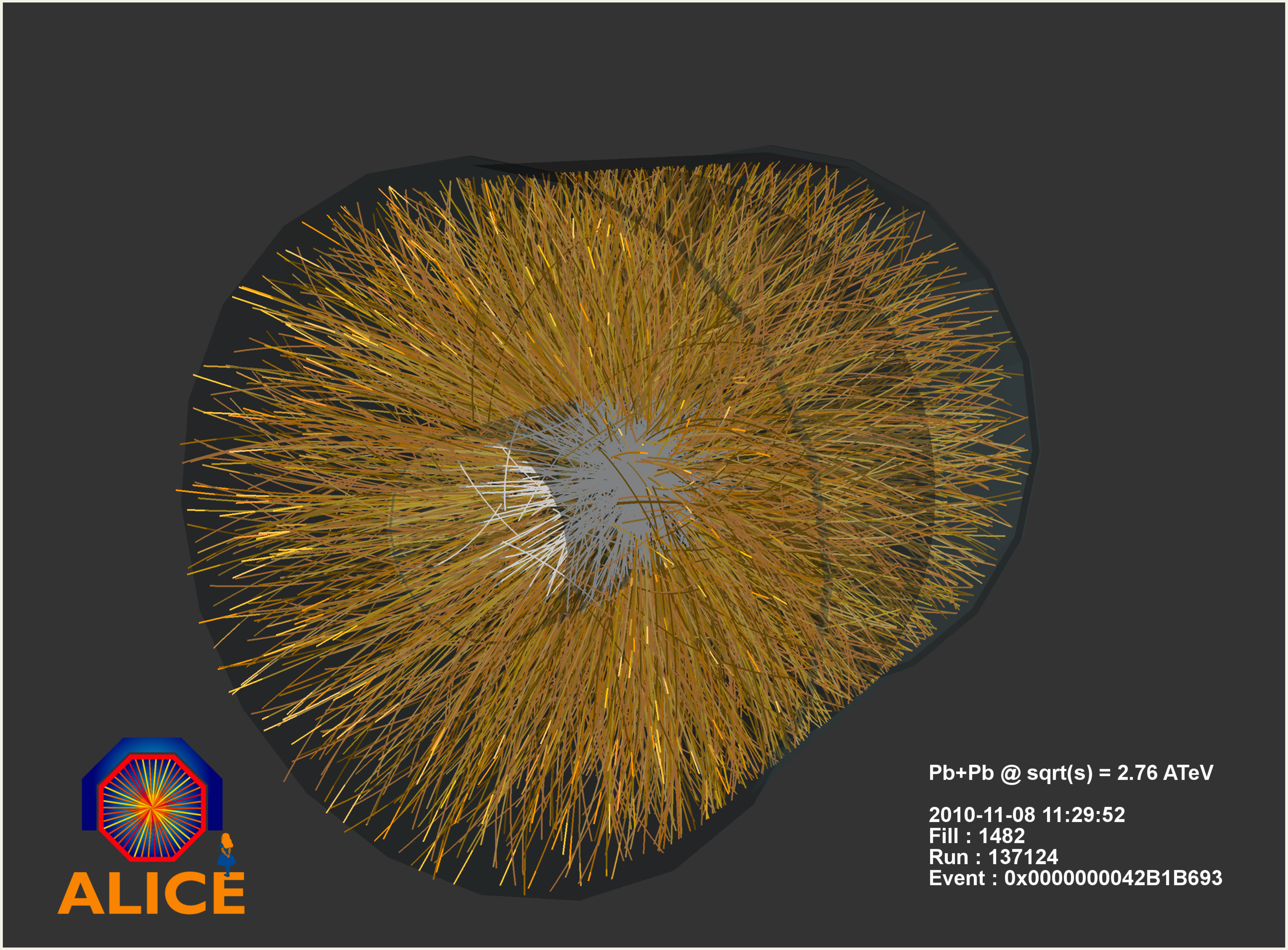
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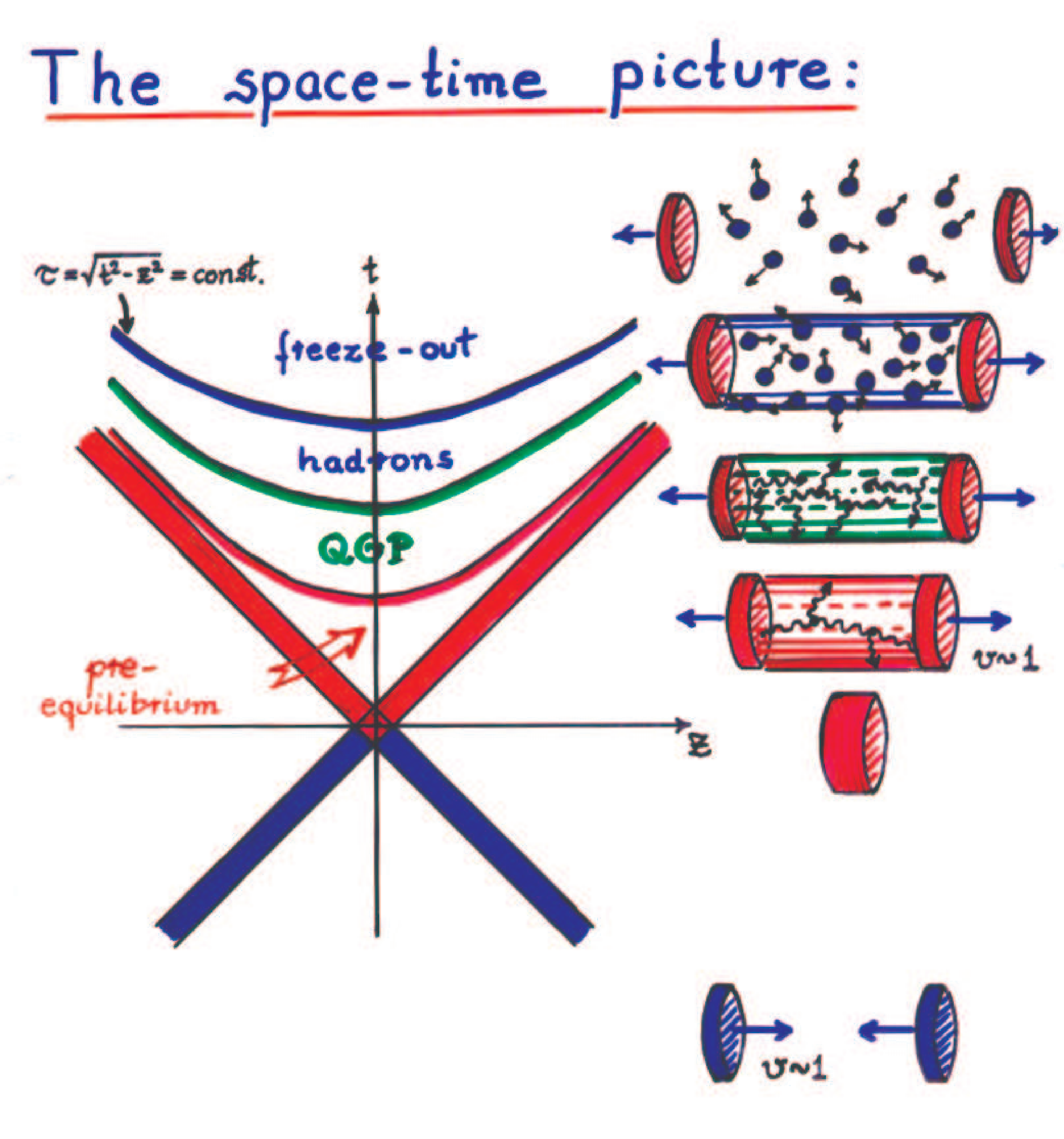

Transient matter

- **lifetime**
  $t \sim 10\,\mathrm{fm/c}$
  $\sim 10^{-23}$ **seconds**
- **small size**
  $r \sim 10\,\mathrm{fm}$
  $\sim 10^{-14}\,\mathrm{m}$
- **rapid expansion**

Multiplicity @ LHC
~ 15000

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Conservation laws

Conservation of energy and momentum:

$$\partial_\mu T^{\mu\nu}(x) = 0$$

Conservation of charge:

$$\partial_\mu N^\mu(x) = 0$$

Local conservation of particle number and energy-momentum.

$$\Longleftrightarrow$$ **Hydrodynamics!**

This can be generalized to multicomponent systems and systems with several conserved charges:

$$\partial_\mu N_i^\mu = 0,$$

$i =$ **baryon number, strangeness, charge. . .**

Consider only baryon number conservation, $i = B$.

$\Rightarrow$ **5 equations contain 14 unknowns!**

$\Rightarrow$ **The system of equations does not close.**

$\Rightarrow$ **Provide 9 additional equations or Eliminate 9 unknowns.**

So what are the components of $T^{\mu\nu}$ and N^{μ}?

- N^{μ} **and** $T^{\mu\nu}$ **can be decomposed with respect to arbitrary, normalized, time-like 4-vector** u^{μ}**,**

$$u_{\mu}u^{\mu} = 1$$

- **Define a projection operator**

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu}u^{\nu}, \quad \Delta^{\mu\nu}u_{\nu} = 0,$$

which projects on the 3-space orthogonal to u^{μ}**.**

- **Then**

$$\boxed{N^\mu = nu^\mu + \nu^\mu}$$

where

$n = N^\mu u_\mu$ is (baryon) charge density in the frame where $u = (1, \mathbf{0})$, local rest frame, LRF

$\nu^\mu = \Delta^{\mu\nu} N_\nu$ is charge flow in LRF,

and

$$\boxed{T^{\mu\nu} = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu}}$$

$\epsilon \equiv u_\mu T^{\mu\nu} u_\nu$ energy density in LRF

$P \equiv -\frac{1}{3}\Delta^{\mu\nu} T_{\mu\nu}$ isotropic pressure in LRF

$q^\mu \equiv \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta$ energy flow in LRF

$\pi^{\mu\nu} \equiv [\frac{1}{2}(\Delta^\mu{}_\alpha \Delta^\nu{}_\beta + \Delta^\nu{}_\beta \Delta^\mu{}_\alpha) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}] T^{\alpha\beta}$

(trace-free) stress tensor in LRF

- **The 14 unknowns in 5 equations:**

$$\left.\begin{array}{cc} N^\mu & 4 \\ T^{\mu\nu} & 10 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{cc} n,\ \epsilon,\ P & 3 \\ q^\mu & 3 \\ \nu^\mu & 3 \\ \pi^{\mu\nu} & 5 \end{array}\right.$$

- **So far u^μ is arbitrary. It attains a physical meaning by relating it to N^μ or $T^{\mu\nu}$:**

1. **Eckart frame:**
$$u_E^\mu \equiv \frac{N^\mu}{\sqrt{N_\nu N^\nu}}$$
u^μ is 4-velocity of charge flow, $\nu^\mu = 0$.
The 14 unknowns are n, ϵ, P, q^μ, $\pi^{\mu\nu}, u^\mu$.
2. **Landau frame:**
$$u_L^\mu \equiv \frac{T^{\mu\nu}u_\nu}{\sqrt{u_\alpha T^{\alpha\beta}T_{\beta\gamma}u^\gamma}}$$
u^μ is 4-velocity of energy flow, $q^\mu = 0$.
The 14 unknowns are n, ϵ, P, ν^μ, $\pi^{\mu\nu}, u^\mu$.

- **In general, the hydrodynamical equations are not closed and cannot be solved uniquely.**

Ideal hydrodynamics

Suppose particles are in **local thermodynamical equilibrium**, i.e., single particle phase space distribution function is given by:

$$f_i(x,k) = \frac{g}{(2\pi)^3}\left[\exp\left(\frac{k_\mu u^\mu(x) - \mu(x)}{T(x)}\right) \pm 1\right]^{-1}$$

where

$T(x)$ and $\mu(x)$: local temperature and chemical potential
$u(x)^\mu$: local 4-velocity of fluid flow.

Then kinetic theory definitions give

$$N^\mu(x) \equiv \sum_i q_i \int \frac{\mathrm{d}^3\mathbf{k}}{E}\, k^\mu f_i(x,k) = n(T,\mu)\, u^\mu$$

$$T^{\mu\nu}(x) \equiv \sum_i \int \frac{\mathrm{d}^3\mathbf{k}}{E}\, k^\mu k^\nu f_i(x,k)$$

$$= (\epsilon(T,\mu) + P(T,\mu))\, u^\mu u^\nu - P(T,\mu)\, g^{\mu\nu}$$

where

$$n(T,\mu) = \sum_i q_i \int \mathrm{d}^3\mathbf{k}\, f_i(x,E)$$ **is local charge density,**

$$\epsilon(T,\mu) = \sum_i \int \mathrm{d}^3\mathbf{k}\, E f_i(x,E)$$ **is local energy density and**

$$P(T,\mu) = \sum_i \int \mathrm{d}^3\mathbf{k}\, \frac{\mathbf{k}^2}{3E} f_i(x,E)$$ **is local pressure.**

Note! $f(x,E)$ **is distribution in local rest frame:** $u^\mu = (1,\mathbf{0})$**.**

$\rightarrow$ **Local thermodynamical equilibrium implies there is no viscosity:**

$$\boxed{\nu^\mu = q^\mu = \pi^{\mu\nu} = 0.}$$

Ideal fluid approximation:

$$N^\mu = nu^\mu$$
$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - Pg^{\mu\mu}$$

- **Local equilibrium** $\Rightarrow$ **no viscosity**: $\nu^\mu = q^\mu = \pi^{\mu\nu} = 0$.
- **Now N^μ and $T^{\mu\nu}$ contain 6 unknowns, ϵ, P, n and u^μ, but there are still only 5 equations!**
- **In thermodynamical equilibrium ϵ, P and n are not independent! They are specified by two variables, T and μ.**
- **The equation of state (EoS), $P(T,\mu)$ eliminates one unknown!**
- **Any equation of state of the form**

  $$P = P(\epsilon, n)$$

  closes the system of hydrodynamic equations and makes it uniquely solvable (given initial conditions).

Remark: $P = P(\epsilon, n)$ **is not a complete equation of state**
in a thermodynamical sense.

A complete equation of state allows to compute all thermodynamic variables.

For example, $s = s(\epsilon, n)$**:** $\mathrm{d}s = 1/T\mathrm{d}\epsilon - \mu/T\mathrm{d}n$ **(1st law of thermod.)**

$$\frac{1}{T} = \frac{\partial s}{\partial \epsilon}|_n, \qquad \frac{\mu}{T} = -\frac{\partial s}{\partial n}|_\epsilon, \qquad P = Ts + \mu n - \epsilon$$

$P = P(\epsilon, n)$ **does not work!**

$$\frac{\partial P}{\partial \epsilon}|_n =? \qquad \frac{\partial P}{\partial n}|_\epsilon =?$$

However, $P = P(T, \mu)$ **does work!**

$$\mathrm{d}P = s\mathrm{d}T + n\mathrm{d}\mu \quad \Rightarrow \quad s = \frac{\partial P}{\partial T}|_\mu, \qquad n = \frac{\partial P}{\partial \mu}|_T$$

Entropy in ideal fluid

is conserved!

$$\partial_\mu S^\mu = 0$$

where $S^\mu = su^\mu$**.**

Equations of motion

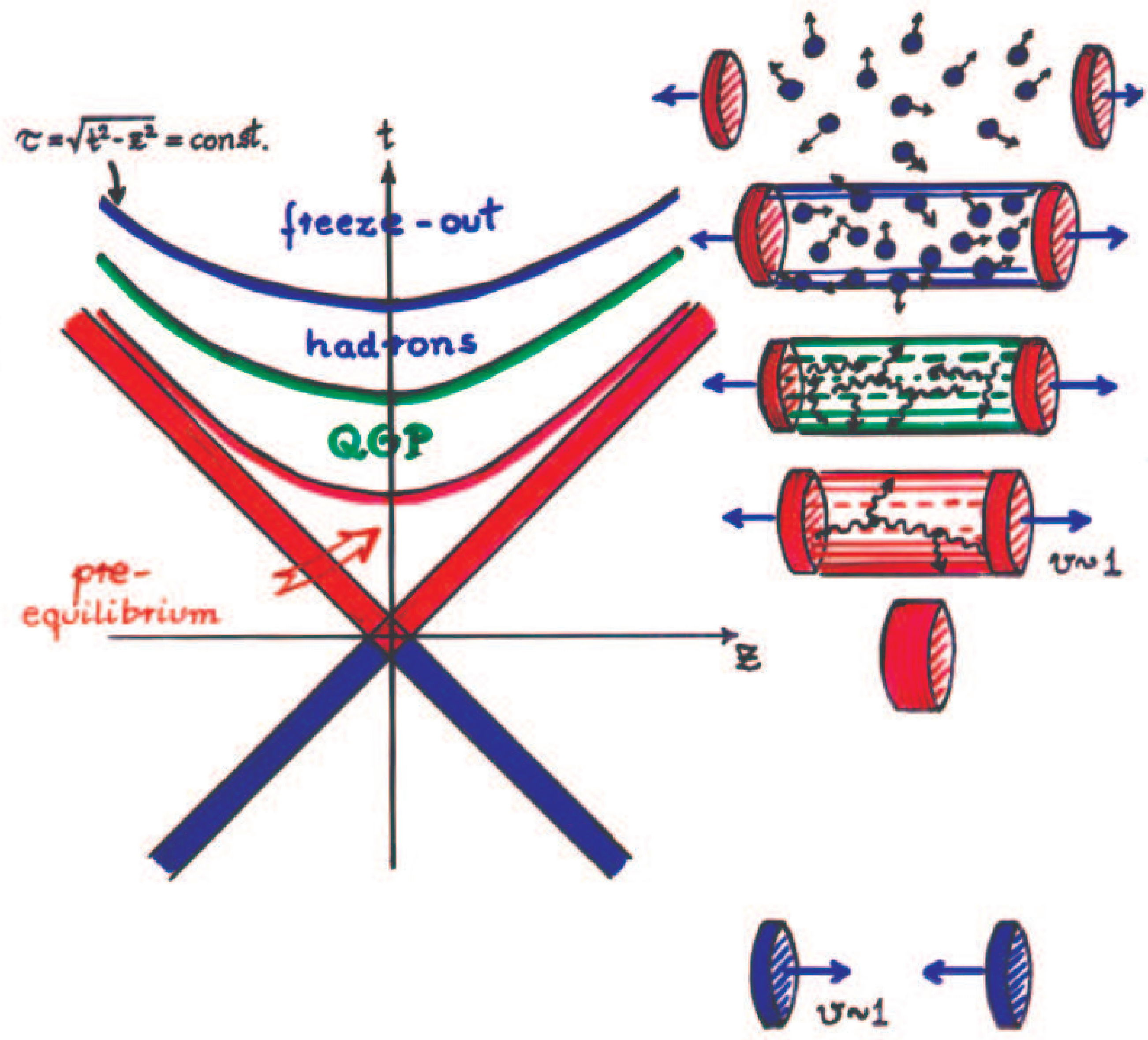
Conservation laws lead to the equations of motion for relativistic fluid:

$$
\begin{aligned}
Dn &= -n\partial_\mu u^\mu \\
D\epsilon &= -(\epsilon + P)\partial_\mu u^\nu \\
(\epsilon + P)Du^\mu &= \nabla^\mu P,
\end{aligned}
$$

where

$$D = u^\mu \partial_\mu \qquad \text{and} \qquad \nabla^\mu = \Delta^{\mu\nu}\partial_\nu.$$

The space-time picture:



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Usefulness of hydro?

- **Initial state:** **unknown**
- **Equation of state:** **unknown**
- **Transport coefficients:** **unknown**
- **Freeze-out:** **unknown**

$\Bigg\} \Rightarrow$ **Predictive power?**

– ***"Hydro doesn't know where to start nor where to end"*** **(M. Prakash)**

Usefulness of hydro?

$$\left.\begin{array}{ll}\bullet\ \textbf{Initial state:} & \textbf{unknown} \\ \bullet\ \textbf{Equation of state:} & \textbf{want to study} \\ \bullet\ \textbf{Transport coefficients:} & \textbf{want to study} \\ \bullet\ \textbf{Freeze-out:} & \textbf{unknown}\end{array}\right\} \Rightarrow \quad \textbf{Predictive power?}$$

$\Longrightarrow$ **Need More Constraints!**

"Hydrodynamical method"

1. Use another model to fix unknowns (and add new assumptions. . .)
   - initial: color glass condensate or pQCD+saturation
   - initial and/or final: hadronic cascade
   - etc.

2. Use data to fix parameters:

Principle		Example @ RHIC
• use one set of data	$\Longleftrightarrow$	$\left.\frac{\mathrm{d}N}{\mathrm{d}y\, p_T\, \mathrm{d}p_T}\right\|_{b=0}$ and $\frac{\mathrm{d}N}{\mathrm{d}y}(b)$
• fix parameters to fit it	$\Longleftrightarrow$	$\epsilon_{0,\mathrm{max}} = 29.6\,\mathbf{GeV/fm}^3$, $\tau_0 = 0.6\,\mathbf{fm}/c$, $T_{\mathrm{fo}} = 130\,\mathbf{MeV}$
• predict another set of data	$\Longleftrightarrow$	HBT, photons & dileptons, elliptic flow. . .

Equations of motion

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$$
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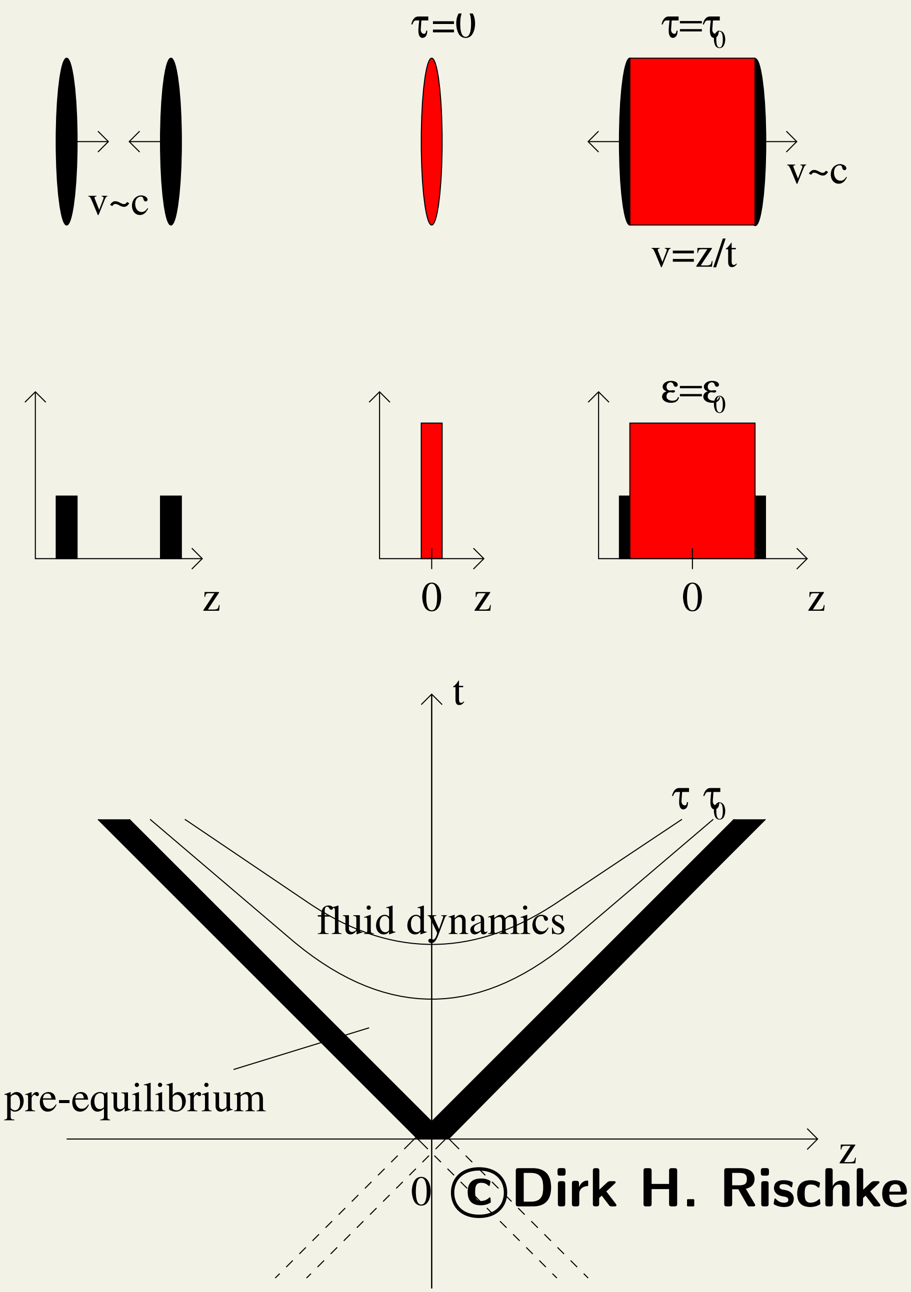
D\epsilon &= -(\epsilon + P)\partial_\mu u^\nu \\
(\epsilon + P)Du^\mu &= \nabla^\mu P,
\end{aligned}
$$

where

$$D = u^\mu \partial_\mu \qquad \text{and} \qquad \nabla^\mu = \Delta^{\mu\nu}\partial_\nu.$$

Bjorken hydrodynamics

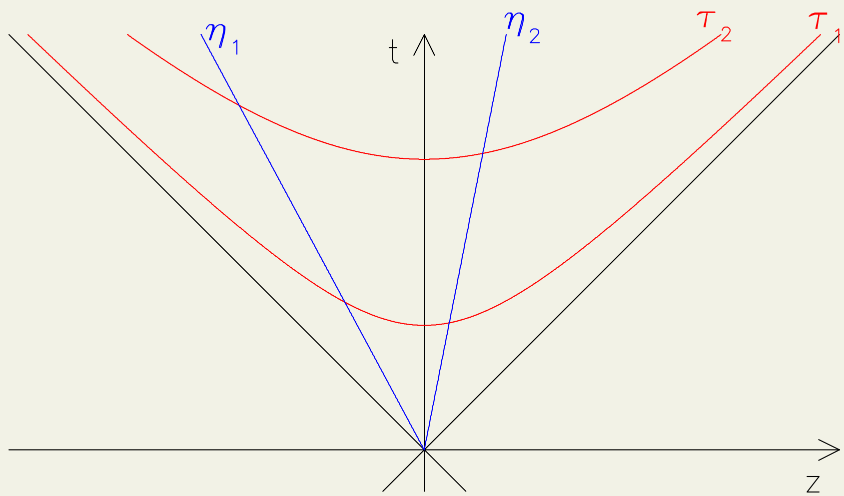


- **At very large energies, $\gamma \to \infty$ and “Landau thickness” $\to 0$**

- **Lack of longitudinal scale**
  $\Rightarrow$ **scaling flow**

$$v = \frac{z}{t}$$

- **Practical coordinates to describe scaling flow expansion are**



  - **Longitudinal proper time τ:**

$$\boxed{\tau \equiv \sqrt{t^2 - z^2} \quad \Leftrightarrow \quad t = \tau \cosh \eta}$$

  - **Space-time rapidity η:**

$$\boxed{\eta = \frac{1}{2} \ln \frac{t+z}{t-z} \quad \Leftrightarrow \quad z = \tau \sinh \eta}$$

- **Scaling flow** $v = z/t \Rightarrow$ **fluid flow rapidity** $y = \eta$**:**

$$y = \frac{1}{2}\ln\frac{1+v}{1-v} = \frac{1}{2}\ln\frac{1+z/t}{1-z/t} = \eta$$

- **Ignore transverse expansion:**
  Hydrodynamic equations turn out to be particularly simple:

$$\left.\frac{\partial \epsilon}{\partial \tau}\right|_{\eta} = -\frac{\epsilon + P}{\tau} \qquad (1)$$

$$\left.\frac{\partial P}{\partial \eta}\right|_{\tau} = 0 \qquad (2)$$

$$\left.\frac{\partial n}{\partial \tau}\right|_{\eta} = -\frac{n}{\tau} \qquad (3)$$

- **Eq. (2)** $\Rightarrow$
  - **No force between fluid elements with different** η**!**
  - $P = P(\tau)$**, no** η**-dependence!**

- **Eq. (2) + thermodynamics:**

$$0 = \left.\frac{\partial P}{\partial \eta}\right|_\tau = s \left.\frac{\partial T}{\partial \eta}\right|_\tau + n \left.\frac{\partial \mu}{\partial \eta}\right|_\tau$$

If $n = 0$, $T = T(\tau) \Rightarrow T =$ const. on $\tau =$ const. surface.

- **In general T and ϵ not constant on $\tau =$ const. surface, but usually they are assumed to be**
  $\Rightarrow$ **boost invariance: the system looks the same in all reference frames!**

$$\epsilon = \epsilon(\tau), \qquad n = n(\tau)$$

- **Note that still**

$$\frac{\partial}{\partial \eta} T^{\mu\nu} \neq 0 \neq \frac{\partial}{\partial \eta} u^\mu$$

Vector and tensor quantities at finite η Lorentz boosted from values at $\eta = 0$

- **Thermodynamics:**

$$\begin{aligned} \mathrm{d}\epsilon &= T\,\mathrm{d}s + \mu\,\mathrm{d}n \\ \epsilon + P &= Ts + \mu n \end{aligned}$$

- **Eq. (1):**

$$\frac{\partial \epsilon}{\partial \tau} + \frac{\epsilon + P}{\tau} = 0$$

$$\Rightarrow \quad T\frac{\partial s}{\partial \tau} + \mu\frac{\partial n}{\partial \tau} + T\frac{s}{\tau} + \mu\frac{n}{\tau} = 0$$

$$\textbf{(Eq. (3))} \quad \Rightarrow \quad \frac{\partial s}{\partial \tau} + \frac{s}{\tau} = 0$$

$$\Rightarrow \quad s(\tau) = s_0\frac{\tau_0}{\tau}$$

$$\Rightarrow \quad s\tau = \textbf{const.} \Rightarrow \mathrm{d}S/\mathrm{d}\eta = \textbf{const}$$

independent of the equation of state!

- **Time evolution of baryon density:**

$$\textbf{Eq. (3)} \Rightarrow n(\tau) = n_0 \frac{\tau_0}{\tau} \Rightarrow \mathrm{d}N/\mathrm{d}\eta = \textbf{const}$$

also independent of the EoS.

- **Time evolution of energy density and temperature depend on the EoS.**

- **Assume ideal gas equation of state,** $P = \frac{1}{3}\epsilon$**,** $\epsilon \propto T^4$**:**

$$\begin{aligned} \textbf{Eq. (1)} \quad &\Rightarrow \quad \frac{\partial \epsilon}{\partial \tau} + \frac{4\epsilon}{3\tau} = 0 \\ &\Rightarrow \quad \epsilon(\tau) = \epsilon_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{4}{3}} \\ &\Rightarrow \quad T(\tau) = T_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{1}{3}} \end{aligned}$$

- **Note:** τ_0**: initial time, thermalization time**

Application: Initial energy density estimate

1. **"Bjorken estimate"**

- **At $y = 0$, $E_T = E$**
- **Thus measuring**

$$\left.\frac{\mathrm{d}E_T}{\mathrm{d}y}\right|_{y=0}$$

  gives total energy at $y = 0$.
- **Estimate the initial volume:**

$$V = A\,\Delta z = \pi R^2 \tau_0\,\Delta\eta$$

- **Thus**

$$\epsilon = \frac{1}{\pi R^2}\frac{E}{\tau_0 \Delta\eta} = \frac{1}{\pi R^2 \tau_0}\frac{\mathrm{d}E_T}{\mathrm{d}y}$$

- **Take $R = 6.3$ fm and $\tau_0 = 1$ fm/c:**

$$\text{@ SPS:}\quad \frac{\mathrm{d}E_T}{\mathrm{d}y} \approx 400 \text{ GeV} \quad \rightarrow \quad \epsilon \sim 3.2 \text{ GeV/fm}^3$$

$$\text{@ RHIC:}\quad \frac{\mathrm{d}E_T}{\mathrm{d}y} \approx 620 \text{ GeV} \quad \rightarrow \quad \epsilon \sim 5.0 \text{ GeV/fm}^3$$

- **Note that in this approach**

$$\epsilon(\tau) = \epsilon_0 \frac{\tau_0}{\tau}$$

No longitudinal work is done.

- **Pressure does work during expansion, $\mathrm{d}E = -P\mathrm{d}t$:**

$$\frac{\partial \epsilon}{\partial \tau} = \frac{\epsilon + P}{\tau} \Rightarrow \mathrm{d}(\epsilon\tau) = P\mathrm{d}\tau$$

Highly nontrivial

2. Entropy conservation

- **Assume ideal gas of massless particles:**

$$
\begin{aligned}
s &= 4n \Rightarrow \frac{\mathrm{d}S}{\mathrm{d}y} = 4\frac{\mathrm{d}N}{\mathrm{d}y} \\
s &= \frac{4g}{\pi^2} T^3 \\
\epsilon &= \frac{3g}{\pi^2} T^4
\end{aligned}
$$

- **With $s\tau =$ const. these give**

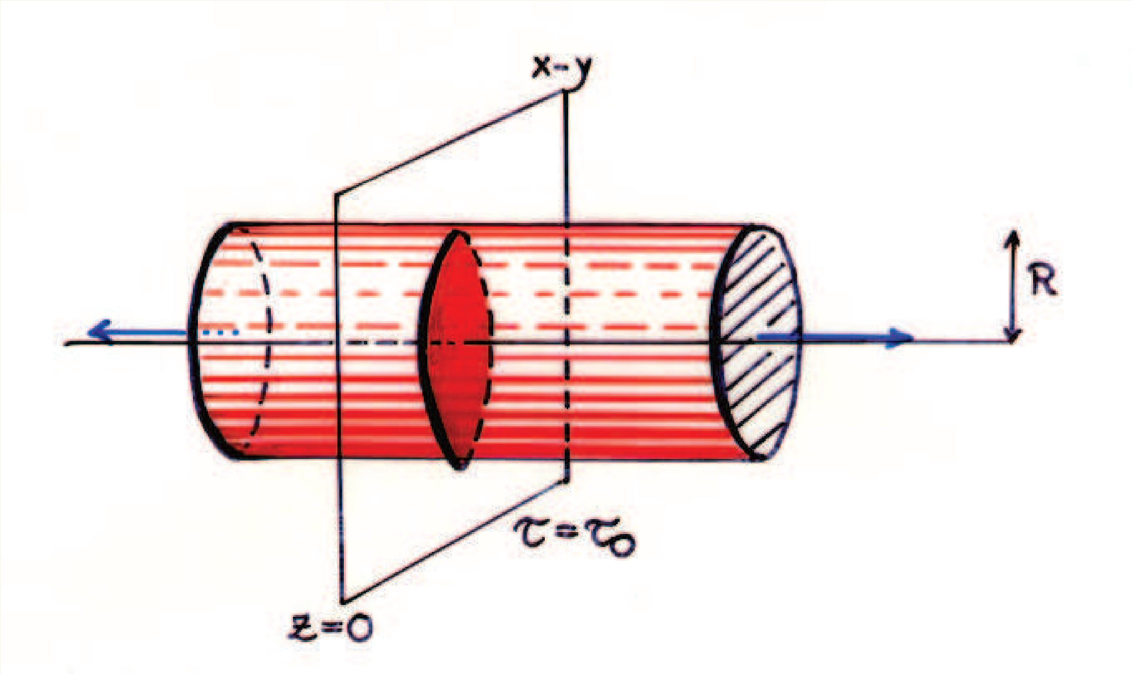
$$
\epsilon_0 = \frac{3}{\pi^{\frac{2}{3}} R^{\frac{8}{3}} \tau_0^{\frac{4}{3}} g^{\frac{1}{3}}} \left(\frac{\mathrm{d}N}{\mathrm{d}y}\right)^{\frac{4}{3}}
$$

@ RHIC: $\frac{\mathrm{d}N}{\mathrm{d}y} \approx 1000$

$g = 40$ **(2 flavours + gluons)**

$\Rightarrow \epsilon_0 \approx 6.0$ **GeV/fm^3**

Transverse expansion and flow



- Transverse expansion will set in latest at $\tau = R/c_s \approx 10$ fm
- Lifetimes in one dimensional expansion ~ 30 fm
- One dimensional expansion an oversimplification

- 2+1D: longitudinal Bjorken, transverse expansion solved numerically
- 3+1D: expansion in all directions solved numerically

Initial conditions

- **Initial time** from early thermalization argument (+finetuning. . .)
- **Total entropy** to fit the multiplicity
- **Density distribution?**
- **Multiplicity is proportional to the number of participants**
- **Glauber model: number of participants/binary collisions**

$$N_{part}(b) = \int \mathrm{d}x\mathrm{d}y\, T_A(x + b/2, y)[\ldots$$

where

$$T_A(x, y) = \int_{-\infty}^{\infty} \mathrm{d}z\, \rho(x, y, z) \qquad \text{and} \qquad \rho(x, y, z) = \frac{\rho_0}{1 + e^{\frac{r-R_0}{a}}}$$

are nuclear thickness function and nuclear density distribution

- **"Differential Optical Glauber:"**
  Number of participants per unit area in transverse plane:

$$n_{\mathrm{WN}}(x,y;b) = T_A(x+b/2,y)\left[1-\left(1-\frac{\sigma}{B}T_B(x-b/2,y)\right)^B\right] + T_B(x-b/2,y)\left[1-\left(1-\frac{\sigma}{A}T_A(x-b/2,y)\right)^A\right]$$

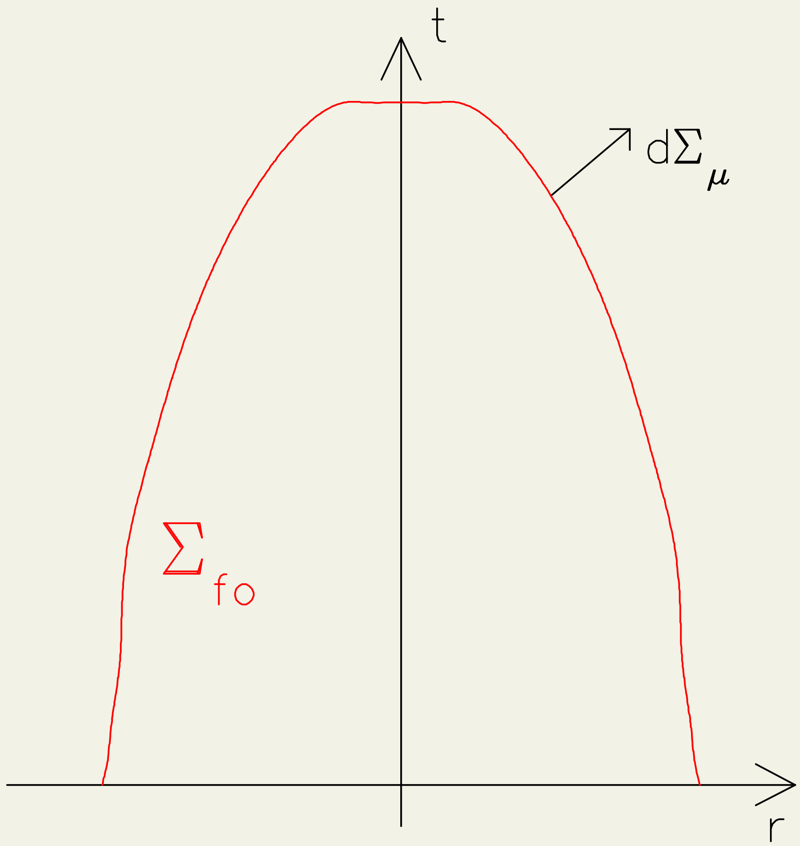
Number of binary collisions per unit area

$$n_{\mathrm{BC}}(x,y;b) = \sigma_{pp}T_A(x+b/2,y)T_B(x-b/2,y)$$

- **MC-Glauber:**
  - **sample $\rho(x,y,z)$ to get the positions of nucleons in 2 nuclei**
  - **count # of nucleons closer than $\sqrt{\sigma_{\mathrm{pp}}/\pi}$ in the collision**
  - **this gives n_{WN} and n_{BC}**
  - **repeat to get enough statistics**

When to end?

- **Particles are observed, not fluid**
- **How and when to convert fluid to particles?**
- **i.e. how far is hydro valid?**



- **Kinetic equilibrium requires scattering rate $\gg$ expansion rate**
- **Scattering rate** $\tau_{\mathrm{sc}}^{-1} \sim \sigma n \propto \sigma T^3$
- **Expansion rate** $\theta = \partial_\mu u^\mu$
- **Fluid description breaks down when** $\tau_{\mathrm{sc}}^{-1} \approx \theta$

$\rightarrow$ **momentum distributions freeze-out**

- $\tau_{\mathrm{sc}}^{-1} \propto T^3 \rightarrow$ **rapid transition to free streaming**
- **Approximation: decoupling takes place on constant temperature hypersurface** Σ_{fo}**, at** $T = T_{\mathrm{fo}}$

Hybrid models

- End hydro when rescatterings still frequent
- Convert fluid to particle ensembles
- Describe evolution of particles using hadronic transport
- Advantages:
  - chemical evolution and dissipation described
  - physical decoupling
- Disadvantages:
  - all the unknowns of hadronic cascade. . .
  - where and how to switch?
- Note: The switch from fluid to cascade is NOT freeze-out
  $\Rightarrow$ particlization

Cooper-Frye

- **Number of particles emitted = Number of particles crossing $\Sigma_{\rm fo}$**

$$\Rightarrow \quad N = \int_{\Sigma_{\rm fo}} \mathrm{d}\Sigma_\mu \, N^\mu$$

- **Frozen-out particles do not interact anymore: kinetic theory**

$$\Rightarrow \quad N^\mu = \int \frac{\mathrm{d}^3\mathbf{p}}{E} p^\mu f(x, p\cdot u)$$

$$\Rightarrow \quad N = \int \frac{\mathrm{d}^3\mathbf{p}}{E} \int_{\Sigma_{\rm fo}} \mathrm{d}\Sigma_\mu \, p^\mu f(x, p\cdot u)$$

- **Invariant single inclusive momentum spectrum: (Cooper-Frye formula)**

$$\boxed{E\frac{\mathrm{d}N}{\mathrm{d}\mathbf{p}^3} = \int_{\Sigma_{\rm fo}} \mathrm{d}\Sigma_\mu \, p^\mu f(x, p\cdot u)}$$

Cooper and Frye, PRD 10, 186 (1974)

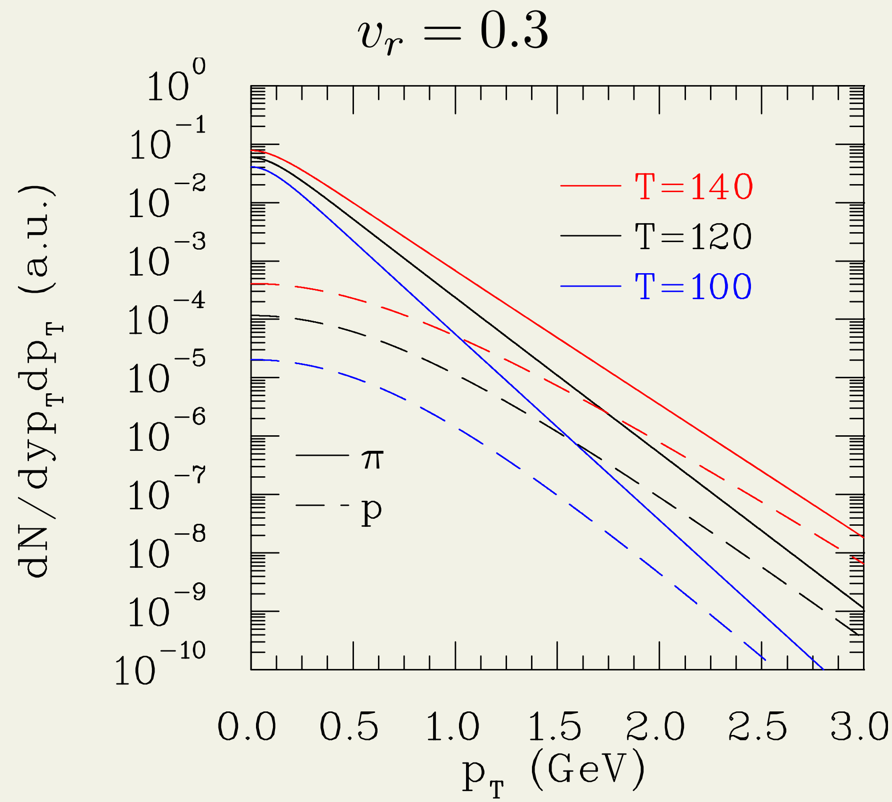
Blast wave

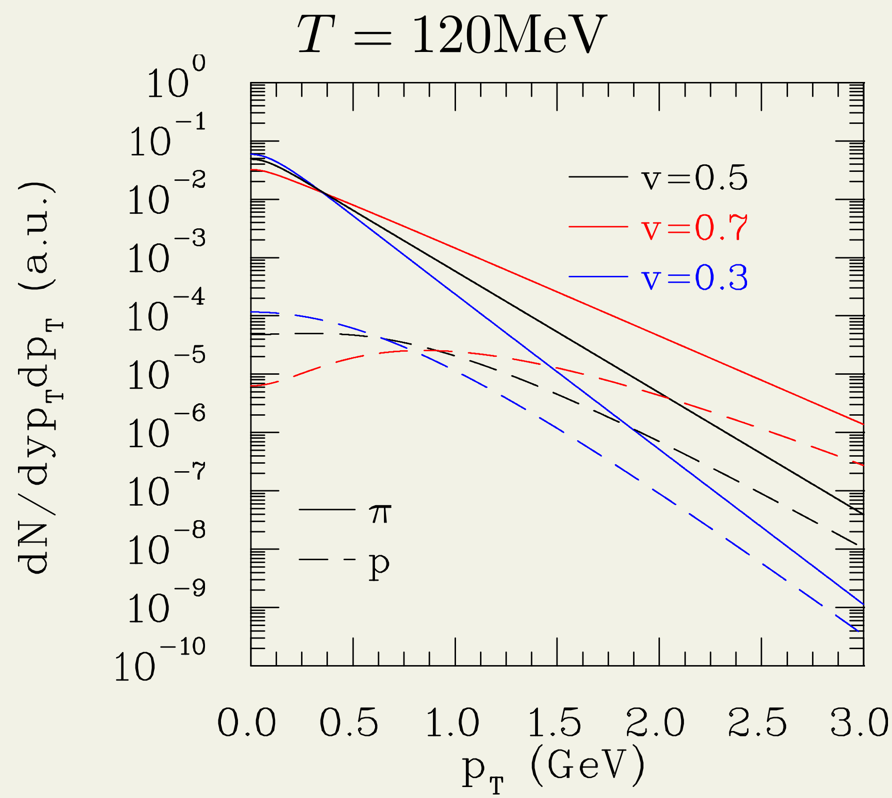
(Siemens and Rasmussen, PRL 42, 880 (1979))

- **Freeze-out surface a thin cylindrical shell radius r, thickness $\mathrm{d}r$, expansion velocity v_r, decoupling time τ_{fo}, boost invariant**
- **Cooper-Frye for Boltzmannions**

$$\frac{\mathrm{d}N}{\mathrm{d}y\, p_T\, \mathrm{d}p_T} = \frac{g}{\pi}\, \tau_{\mathrm{fo}}\, r\, m_T\, \mathrm{I}_0\left(\frac{v_r \gamma_r p_T}{T}\right) \mathrm{K}_1\left(\frac{\gamma_r m_T}{T}\right)$$

effect of temperature and flow velocity





- **The larger the temperature, the flatter the spectra**
- **The larger the velocity, the flatter the spectra $\Rightarrow$ blueshift**
- **The heavier the particle, the more sensitive it is to flow (shape and slope)**

- **Define speed of sound c_s:**

$$c_s^2 = \left.\frac{\partial P}{\partial \epsilon}\right|_{s/n_b}$$

- **large $c_s \Rightarrow$ "stiff EoS"**

- **small $c_s \Rightarrow$ "soft EoS"**

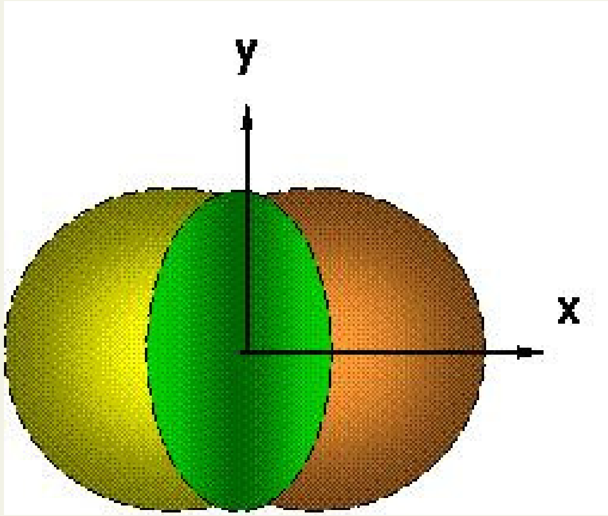
- **For baryon-free matter in rest frame**

$$(\epsilon + P) D u^\mu = \nabla^\mu P \qquad \Longleftrightarrow \qquad \frac{\partial}{\partial \tau} u_\mu = -\frac{c_s^2}{s} \partial_\mu s$$

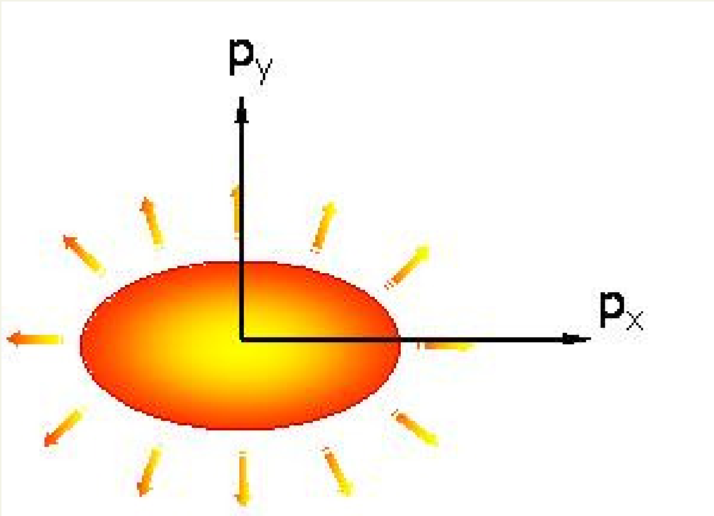
$\Rightarrow$ **The stiffer the EoS, the larger the acceleration**

Elliptic flow v_2

spatial anisotropy $\rightarrow$ final azimuthal momentum anisotropy



$\Rightarrow$



- **Anisotropy in coordinate space + rescattering**

$\Rightarrow$ **Anisotropy in momentum space**

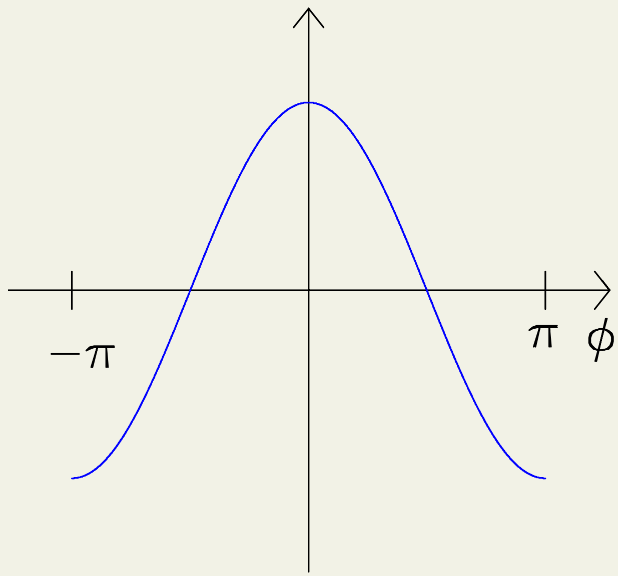
$$\frac{\partial}{\partial\tau}u_x = -\frac{c_s^2}{s}\frac{\partial}{\partial x}s \qquad \text{and} \qquad \frac{\partial}{\partial\tau}u_y = -\frac{c_s^2}{s}\frac{\partial}{\partial y}s$$

Elliptic flow v_2

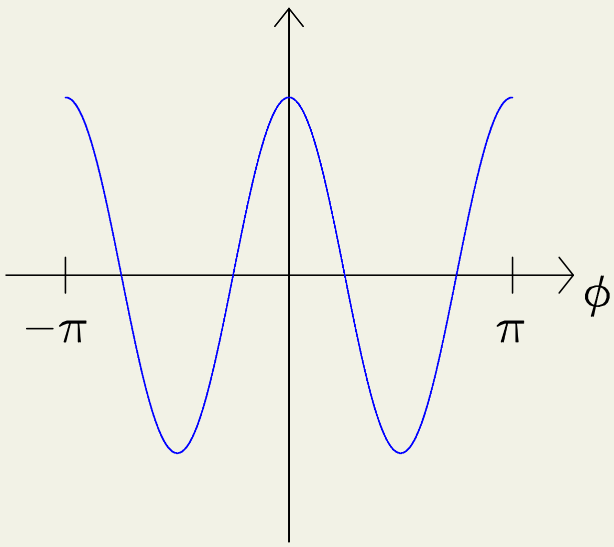
- **Fourier expansion of momentum distribution:**

$$\frac{\mathrm{d}N}{\mathrm{d}y\, p_T \mathrm{d}p_T\, \mathrm{d}\phi} = \frac{1}{2\pi}\frac{\mathrm{d}N}{\mathrm{d}y\, p_T \mathrm{d}p_T}(1 + 2v_1(y,p_T)\cos\phi + 2v_2(y,p_T)\cos 2\phi + \cdots)$$

$\mathbf{v_1}$: Directed flow: preferred direction

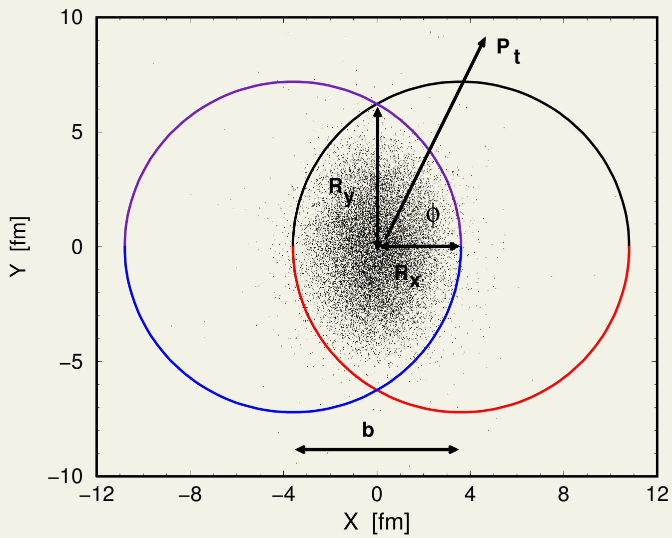


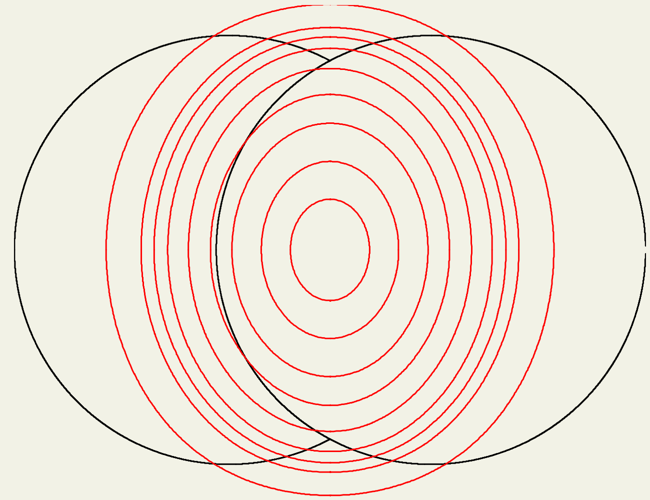
$\mathbf{v_2}$: Elliptic flow: preferred plane



sensitive to speed of sound $c_s^2 = \partial p/\partial e$ and shear viscosity η

Measures of anisotropy





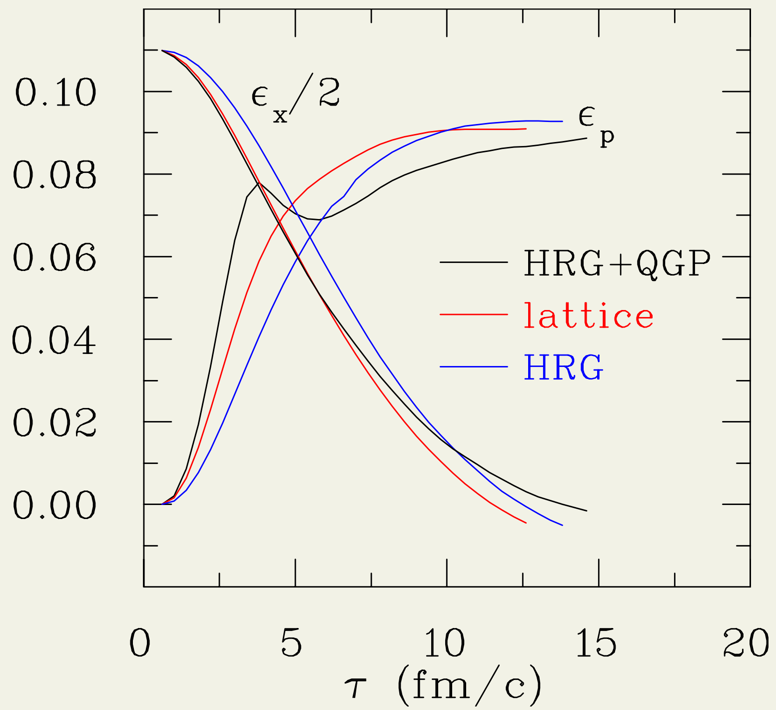
- **Spatial eccentricity**

$$\epsilon_x = \frac{\langle\langle y^2 - x^2 \rangle\rangle}{\langle\langle y^2 + x^2 \rangle\rangle} = \frac{\int \mathrm{d}x\,\mathrm{d}y\,\epsilon \cdot (y^2 - x^2)}{\int \mathrm{d}x\,\mathrm{d}y\,\epsilon \cdot (y^2 - x^2)}$$

- **Momentum anisotropy**

$$\epsilon_p = \frac{\langle T^{xx} - T^{yy} \rangle}{\langle T^{xx} + T^{yy} \rangle} = \frac{\int \mathrm{d}x\,\mathrm{d}y\,T^{xx} - T^{yy}}{\int \mathrm{d}x\,\mathrm{d}y\,T^{xx} - T^{yy}}$$

- **Au+Au @ RHIC, $b = 6$ fm:**

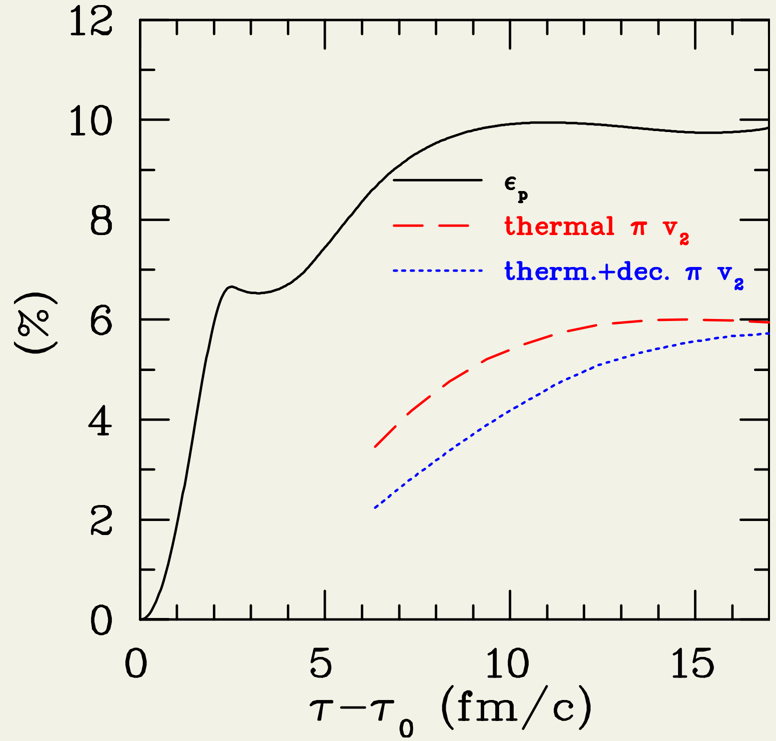


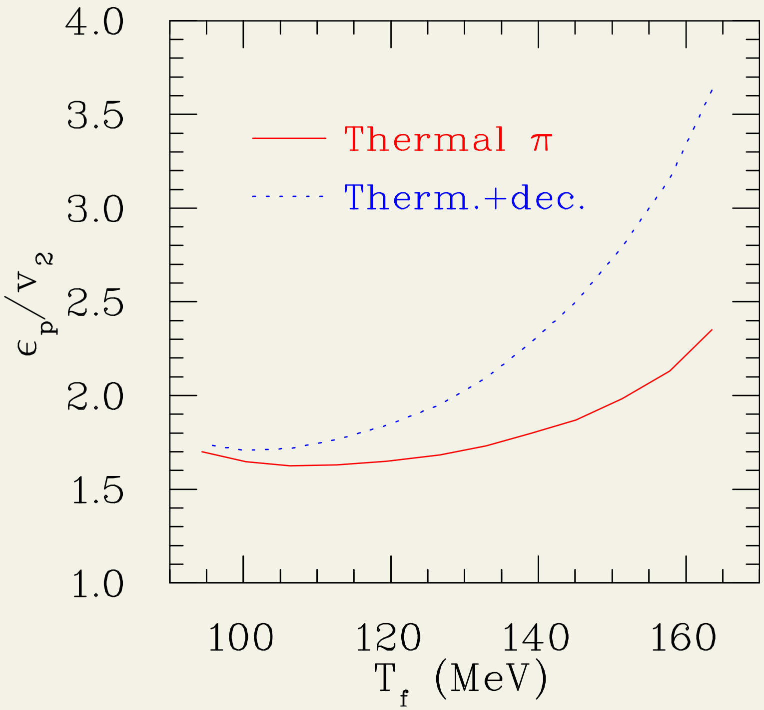
- **ϵ_x decreases during the evolution $\Rightarrow$ elliptic flow is self-quenching**

- **Most of ϵ_p is built up early in the evolution**

ϵ_p **vs.** v_2

- v_2: **not only collective but also thermal motion**
- **Au+Au @ RHIC,** $b = 7$ **fm:**





- **NO clear correspondence**
- **especially if one includes resonance decays**

v_2

- **Elliptic flow v_2 a.k.a. p_T-averaged v_2:**

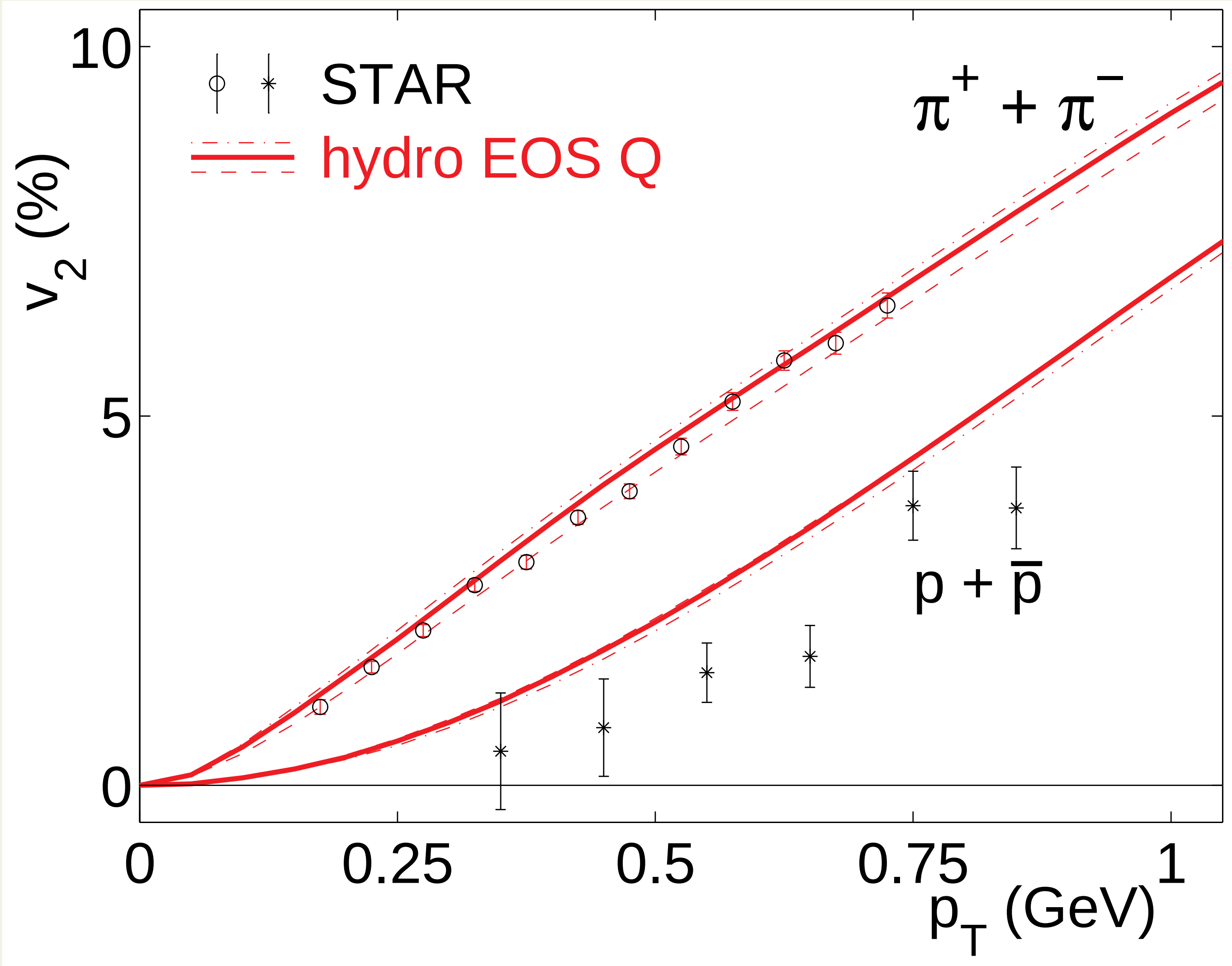
$$v_2 = \frac{\langle p_x^2 - p_y^2 \rangle}{\langle p_x^2 + p_y^2 \rangle} = \frac{\int \mathrm{d}\phi\, \cos(2\phi) \frac{\mathrm{d}N}{\mathrm{d}y\, \mathrm{d}\phi}}{\mathrm{d}N/\mathrm{d}y}$$

- **p_T-differential v_2**

$$v_2(p_T) = \frac{\int \mathrm{d}\phi\, \cos(2\phi) \frac{\mathrm{d}N}{\mathrm{d}y\, p_T \mathrm{d}p_T\, \mathrm{d}\phi}}{\int \mathrm{d}\phi \frac{\mathrm{d}N}{\mathrm{d}y\, p_T \mathrm{d}p_T\, \mathrm{d}\phi}}$$

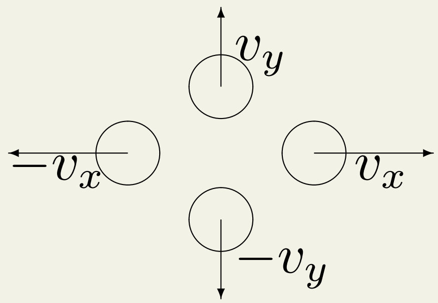
- **If $m_1 > m_2$, $v_2(m_1) > v_2(m_1)$, but $v_2(p_T, m_1) < v_2(p_T, m_2)$!**
- **No contradiction, since**

$$v_2 = \frac{\int \mathrm{d}p_T\, v_2(p_T) \frac{\mathrm{d}N}{\mathrm{d}p_T}}{\int \mathrm{d}p_T\, \frac{\mathrm{d}N}{\mathrm{d}p_T}}$$


10
5
0
STAR
hydro EOS Q
$\pi^+ + \pi^-$
$p + \bar{p}$
v_2 (%)
0
0.25
0.5
0.75
1
p_T (GeV)

Why $m_1 < m_2 \Rightarrow v_2(p_T, m_1) > v_2(p_T, m_2)$?

Simple source:



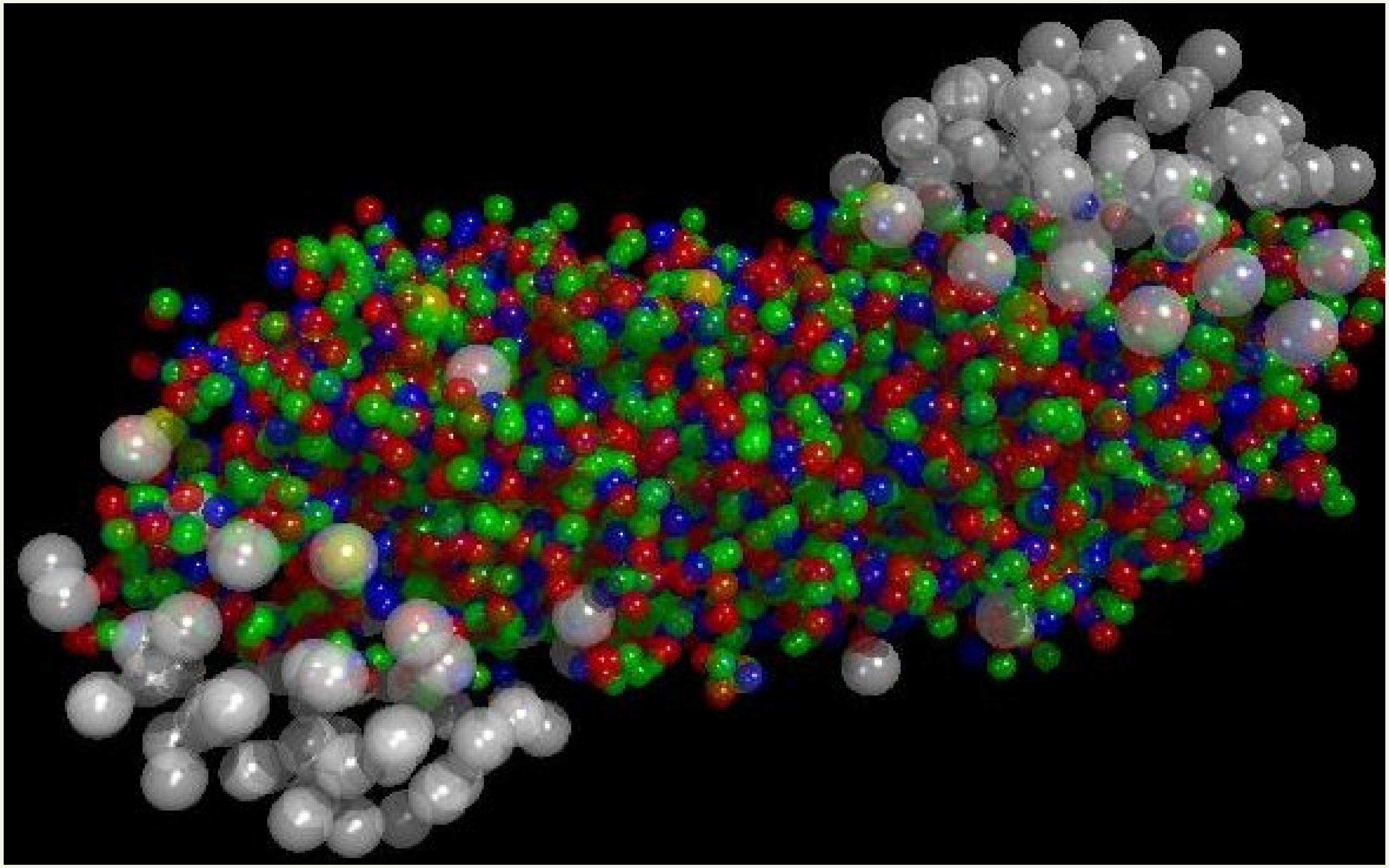
Each element has unit volume, decouples at the same time at the same temperature. Flow velocity in plane is larger than out of plane, $|v_x| > |v_y|$.

Boltzmann distribution and Cooper-Frye formula:

$$v_2(p_T) = \frac{\mathrm{I}_2\left(\frac{\gamma_x v_x p}{T}\right) - e^{\frac{E}{T}(\gamma_x - \gamma_y)}\mathrm{I}_2\left(\frac{\gamma_y v_y p}{T}\right)}{\mathrm{I}_0\left(\frac{\gamma_x v_x p}{T}\right) + e^{\frac{E}{T}(\gamma_x - \gamma_y)}\mathrm{I}_0\left(\frac{\gamma_y v_y p}{T}\right)}$$

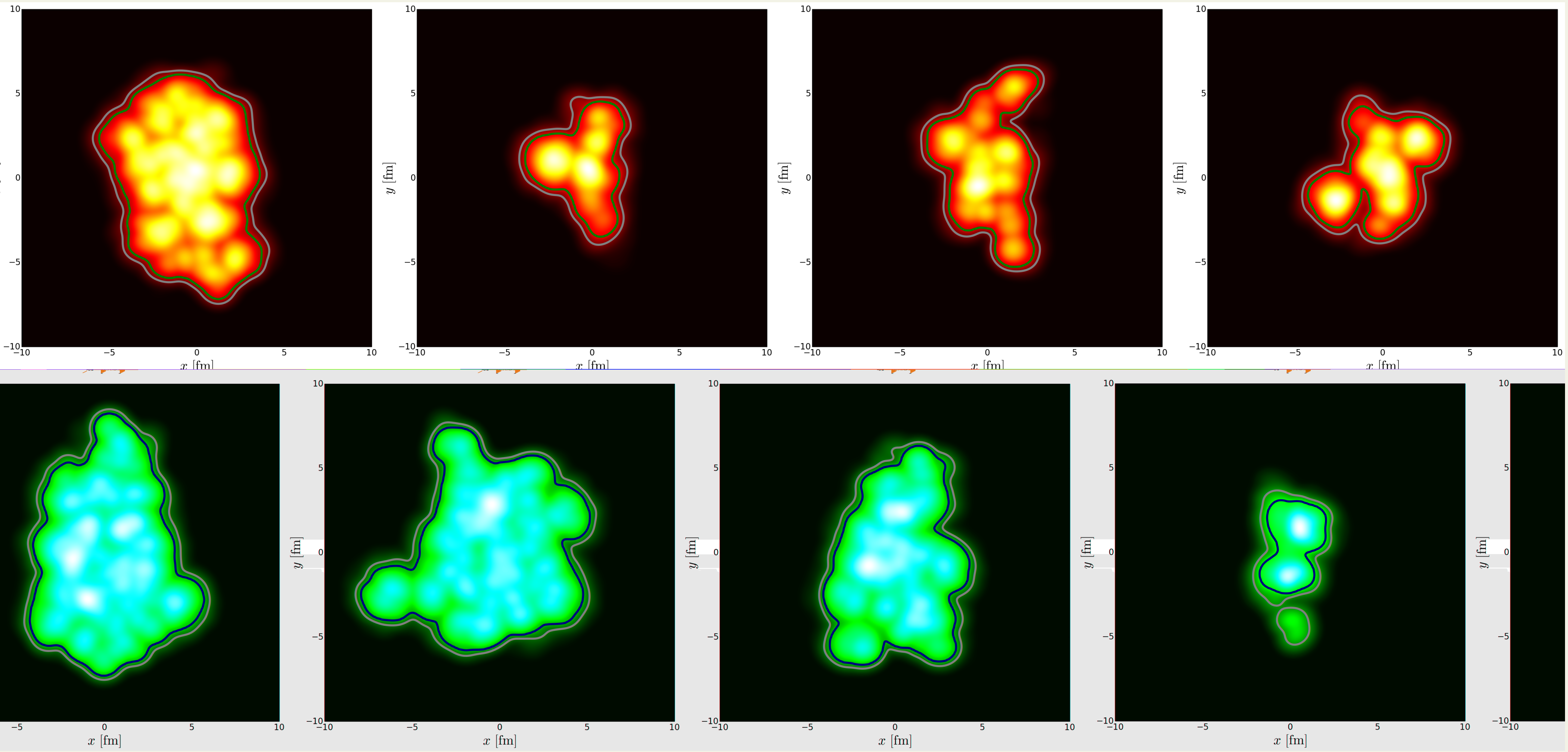
$$= \frac{C_1 - e^{\lambda\sqrt{m^2+p^2}}C_2}{C_3 + e^{\lambda\sqrt{m^2+p^2}}C_4}$$

mass increases, numerator decreases and denominator increases
$\rightarrow v_2$ decreases



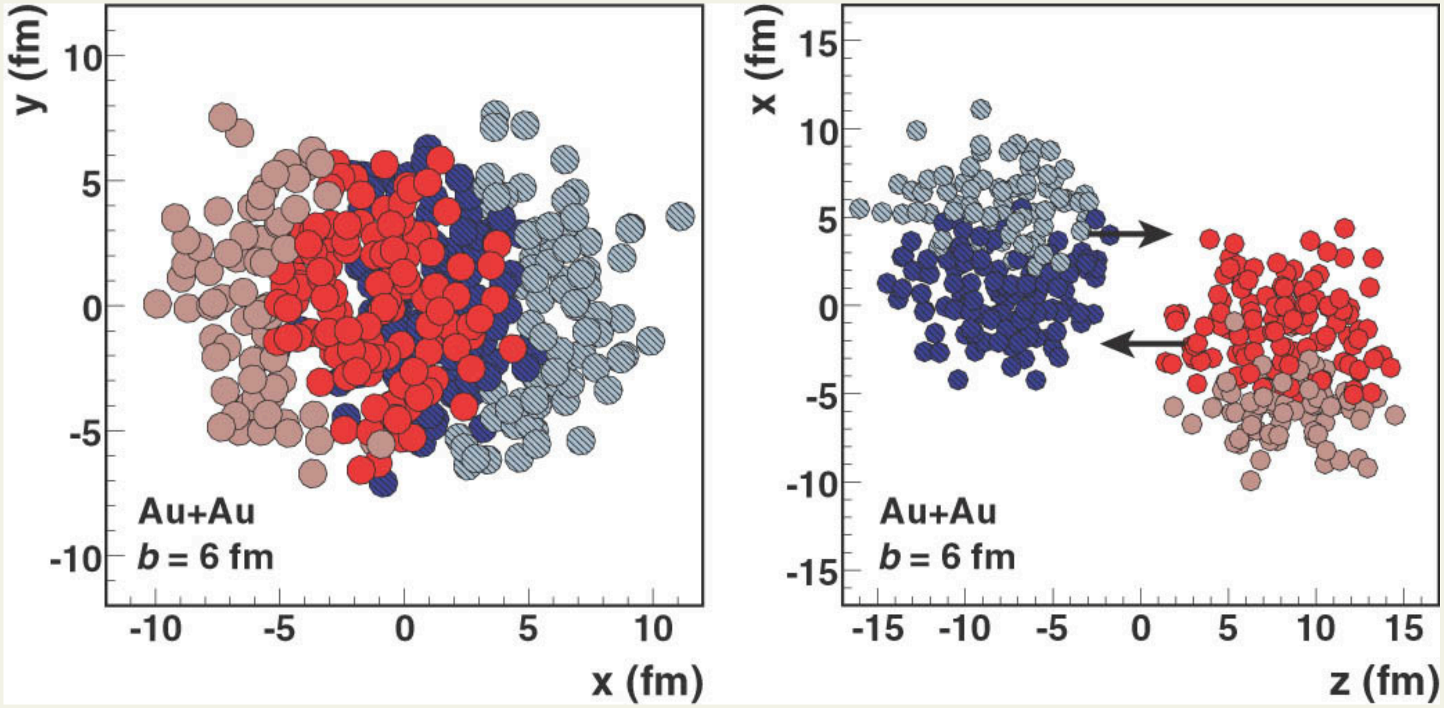
Initial state fluctuates

temperature profiles in transverse plane



©Harri Niemi

event-by-event



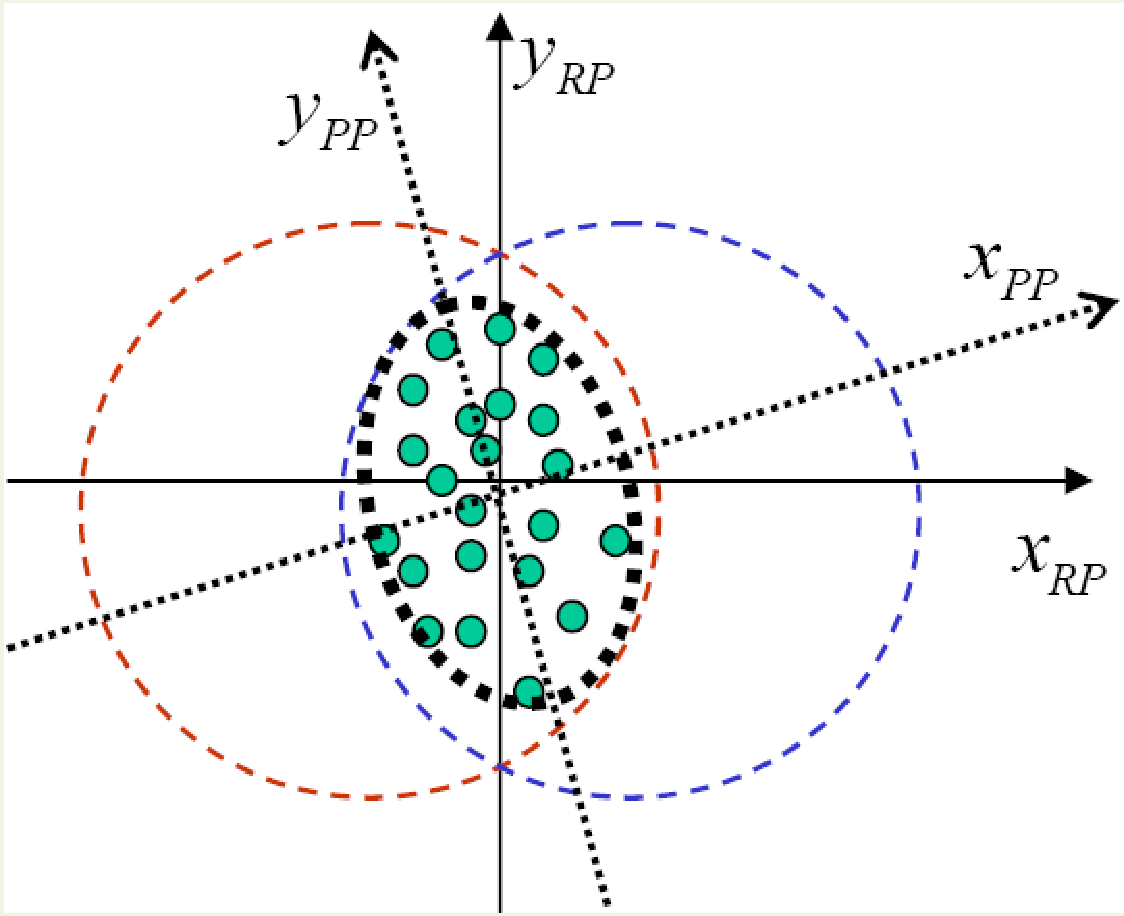
Miller *et al.*, Ann.Rev.Nucl.Part.Sci. 57, 205 (2007)

- **shape fluctuates event-by-event**
- **all coefficients v_n finite**

$$\frac{\mathrm{d}N}{\mathrm{d}y\mathrm{d}\phi} = \frac{\mathrm{d}N}{\mathrm{d}y}\left[1 + \sum_n 2v_n \cos(2(\phi - \Psi_n))\right]$$

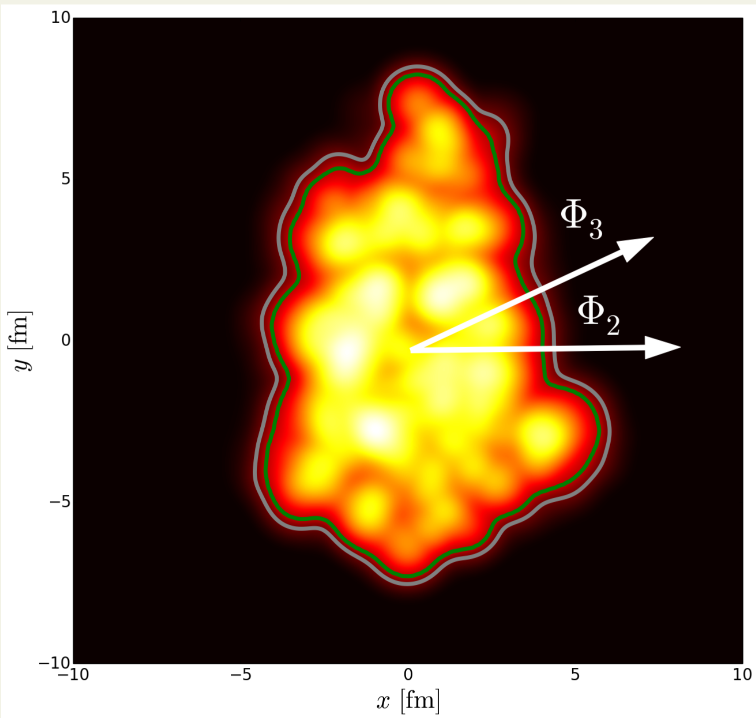
All the planes. . .

Voloshin *et al.* Phys. Lett. B 659, 537 (2008)

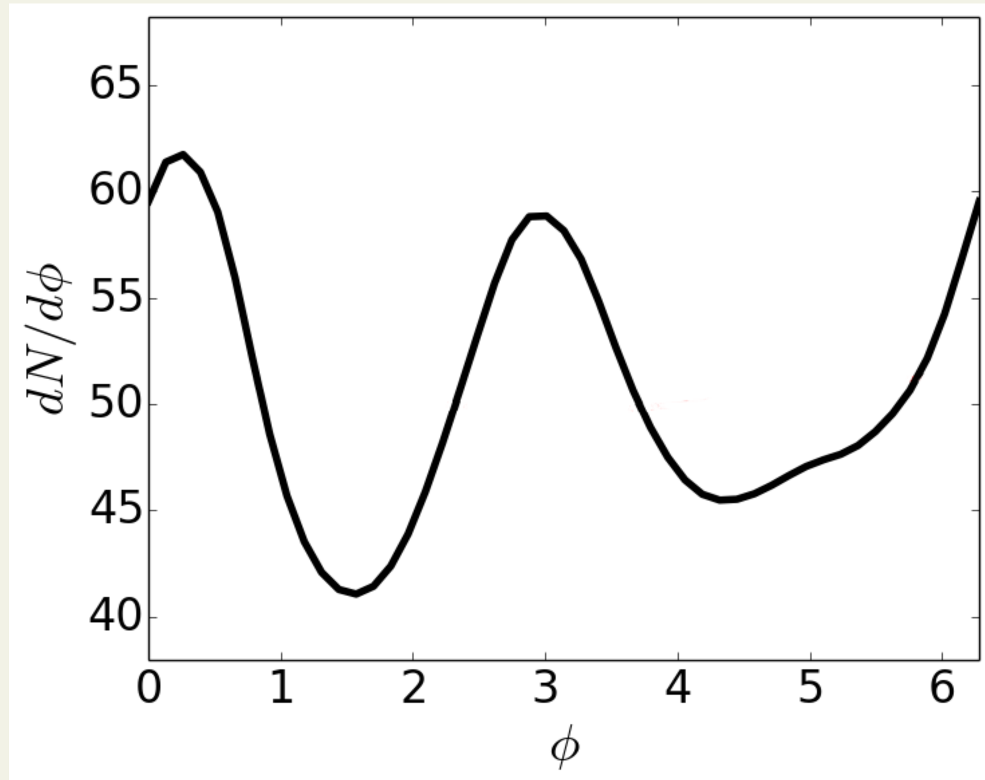


- X_{RP}: **Reaction plane, spanned by beam and impact parameter**
- X_{PP}: **Participant plane, maximises spatial anisotropy** ϵ_n
- Ψ_n: **Event plane, maximises anisotropy** v_n

From fluid to distribution



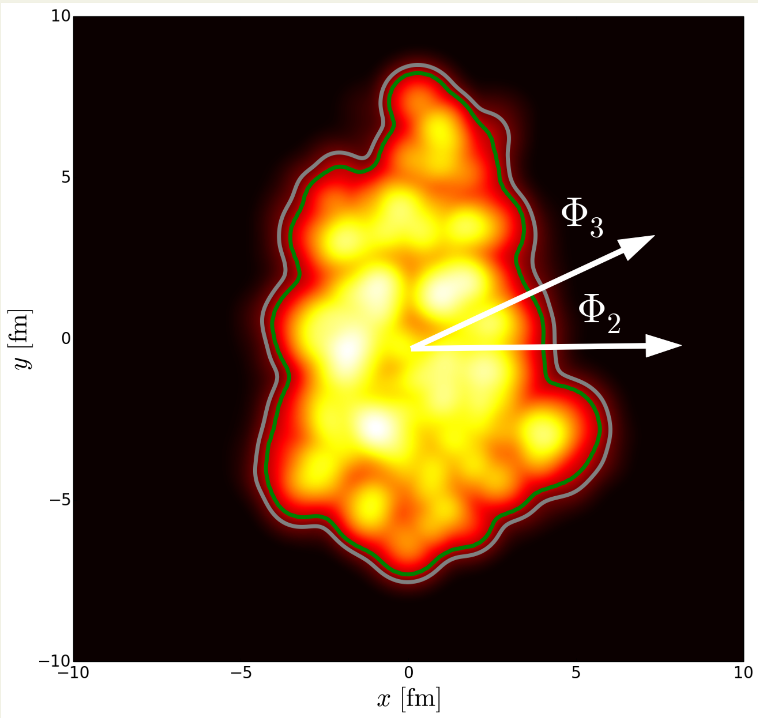
$\Longrightarrow$



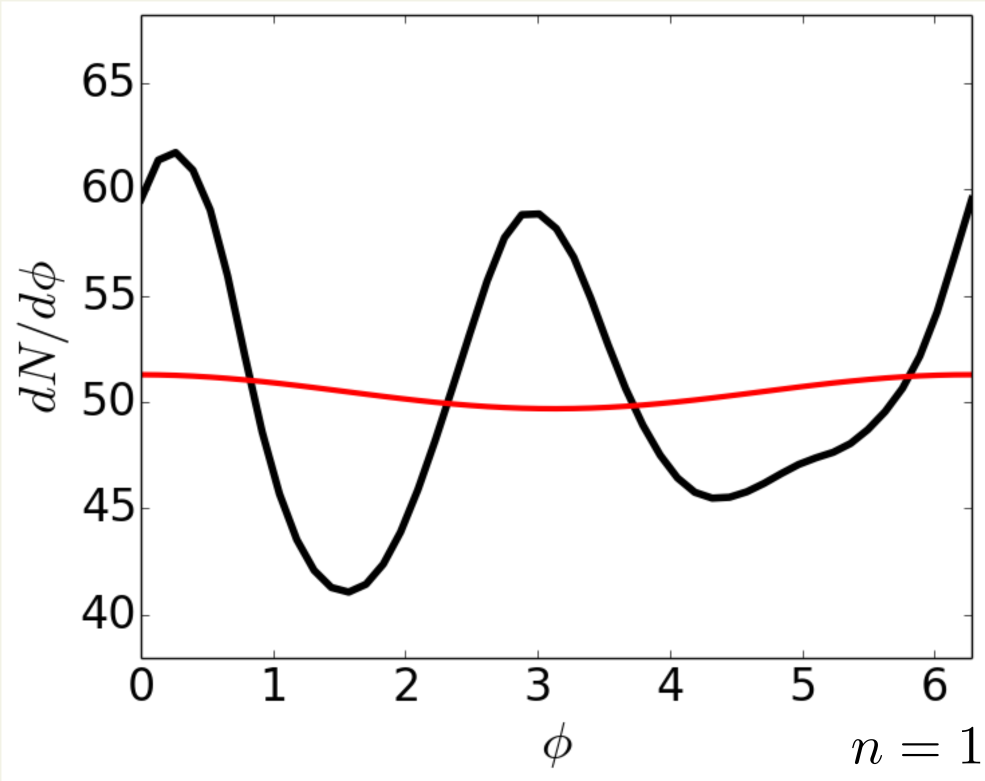
$$\epsilon_n, \Phi_n \Longrightarrow v_n, \Psi_n$$

$$\frac{\mathrm{d}N}{\mathrm{d}y\mathrm{d}\phi} = \frac{\mathrm{d}N}{\mathrm{d}y}\left[1 + \sum_n 2v_n \cos(n(\phi - \Psi_n))\right]$$

From fluid to distribution



$\Longrightarrow$



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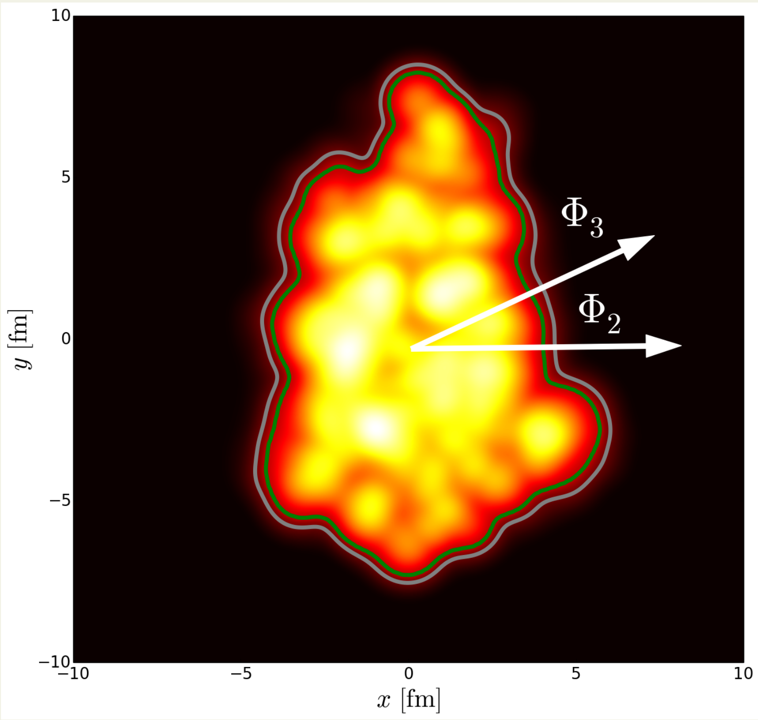
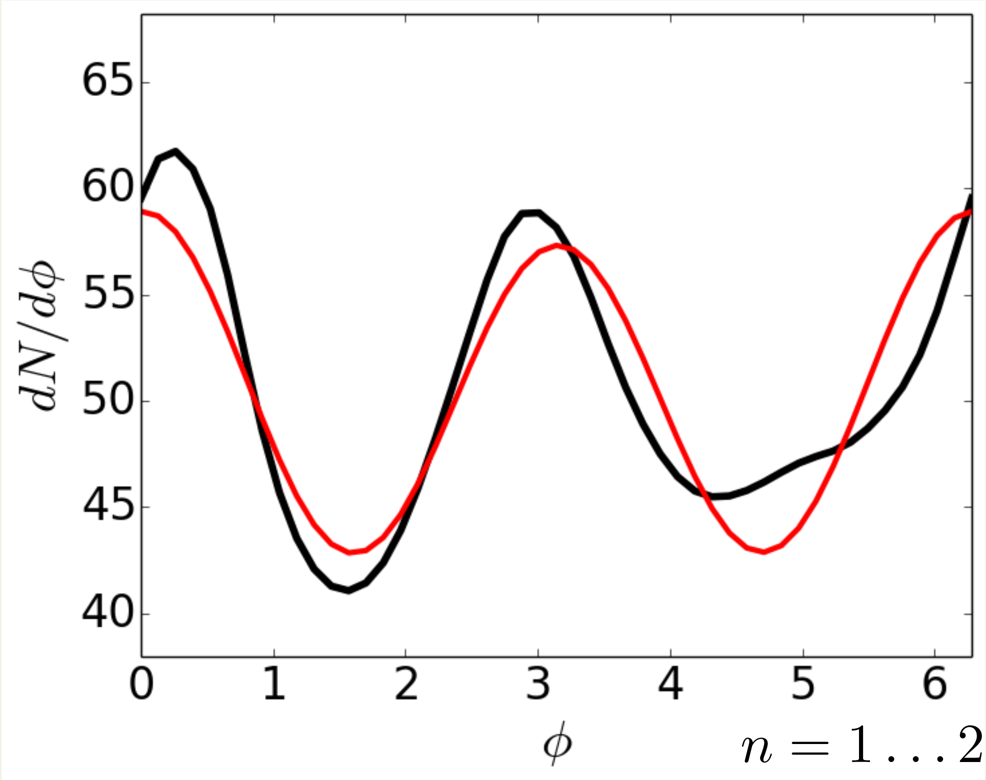
From fluid to distribution



$\Longrightarrow$



$$\epsilon_n, \Phi_n \Longrightarrow v_n, \Psi_n$$

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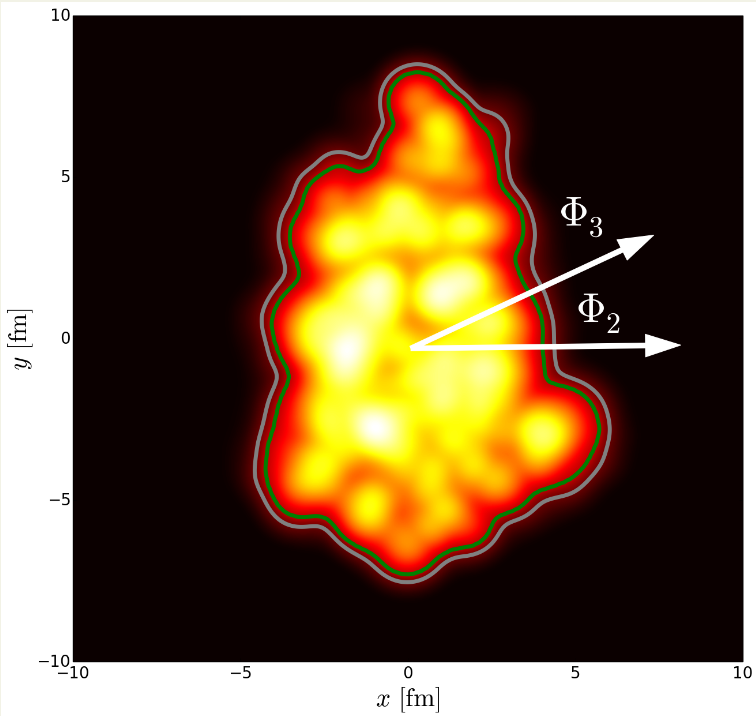
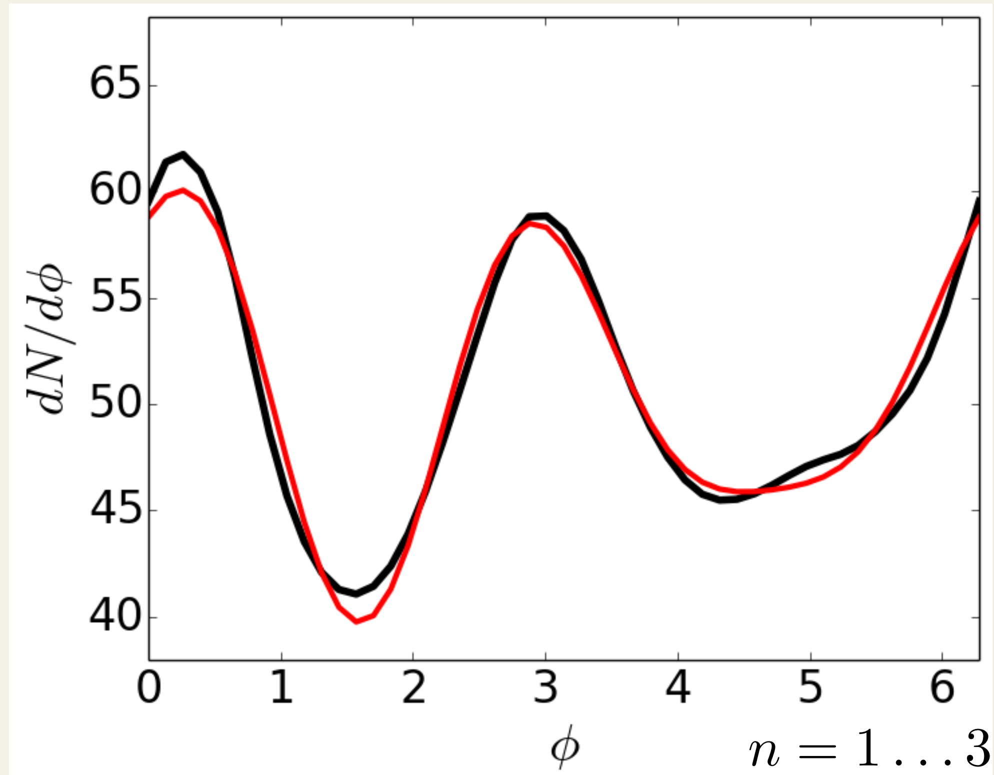
From fluid to distribution



$\Longrightarrow$



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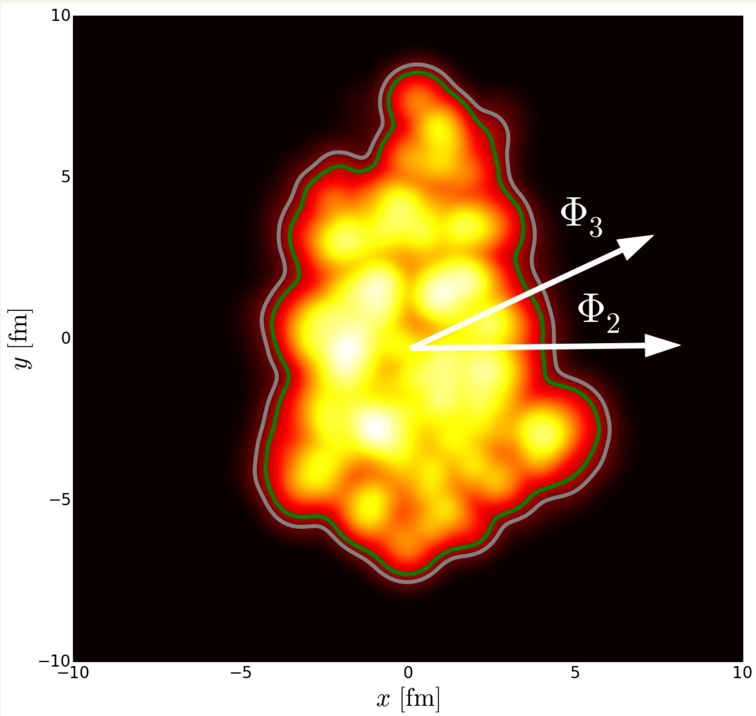
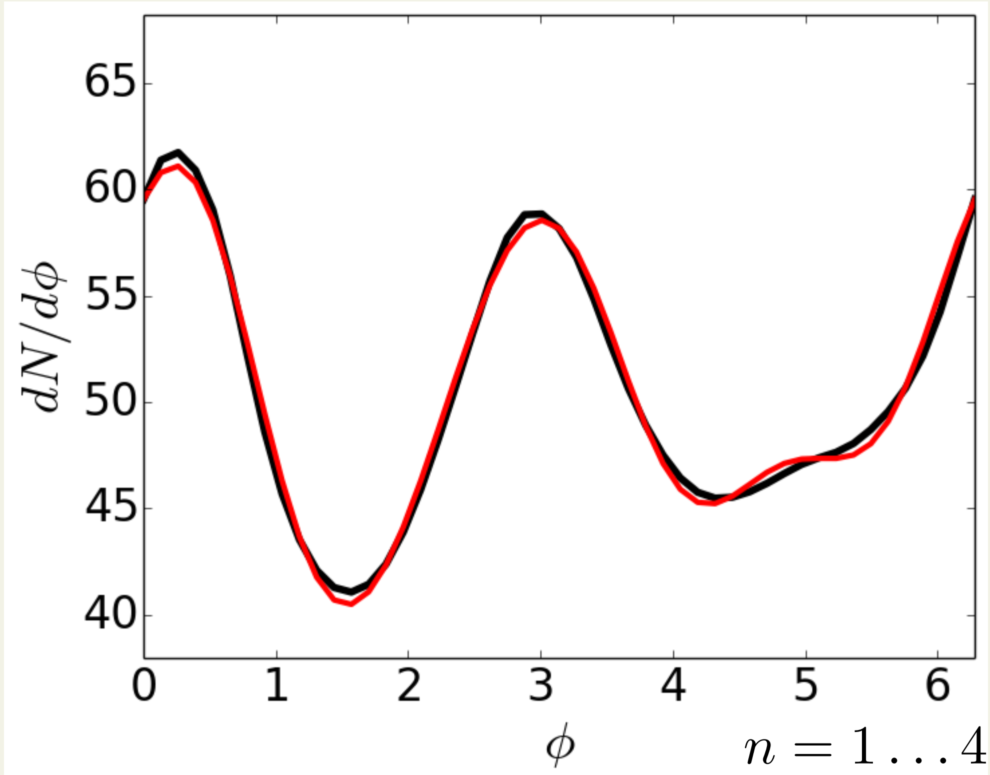
From fluid to distribution



$\Longrightarrow$



$\epsilon_n, \Phi_n \Longrightarrow v_n, \Psi_n$

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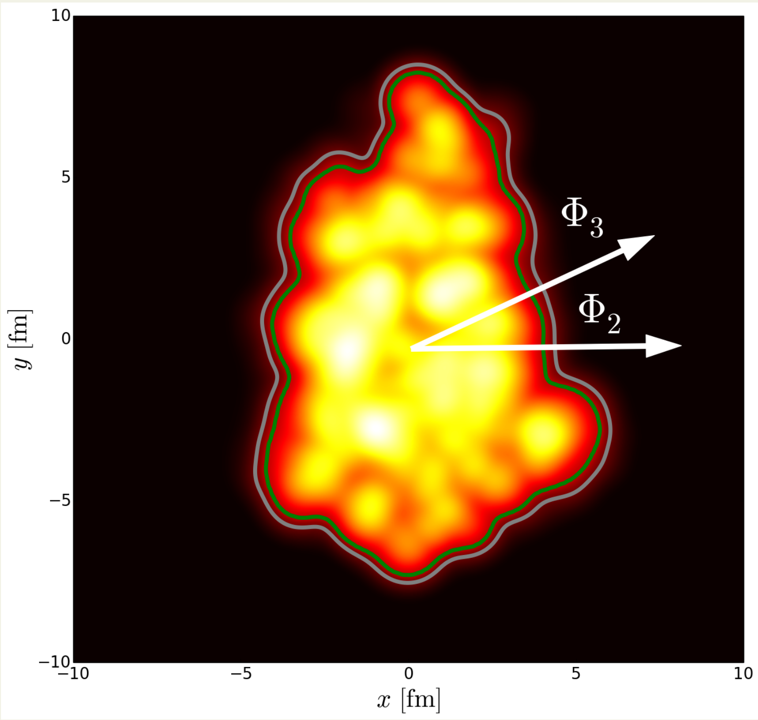
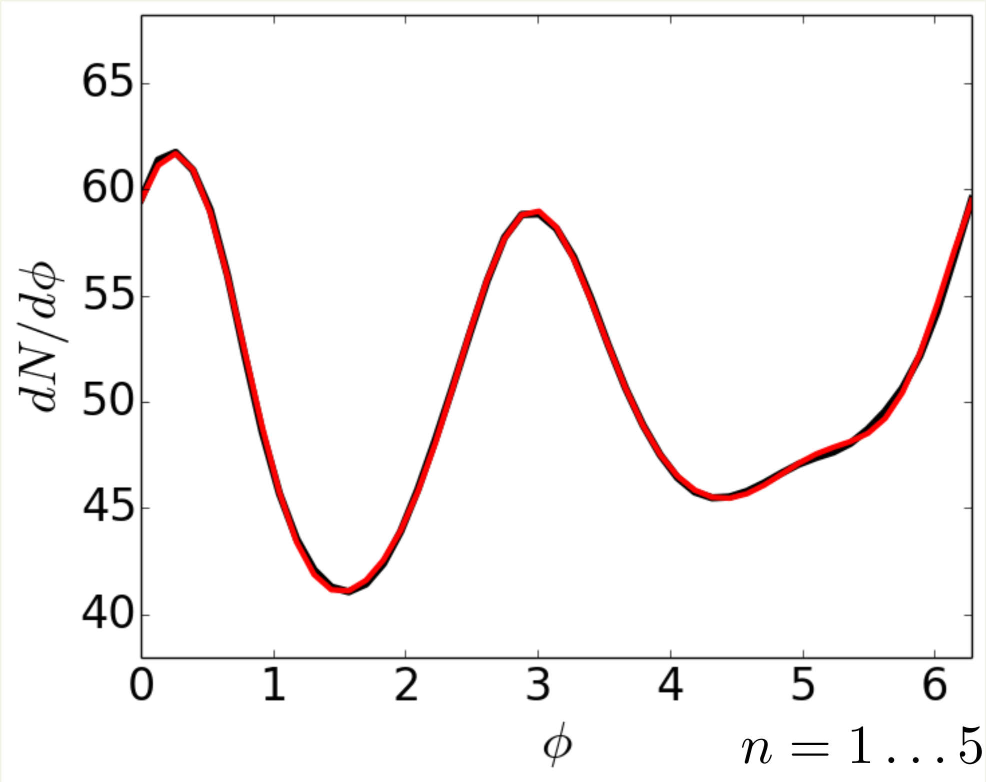
From fluid to distribution



$\Longrightarrow$



$\epsilon_n, \Phi_n \Longrightarrow v_n, \Psi_n$

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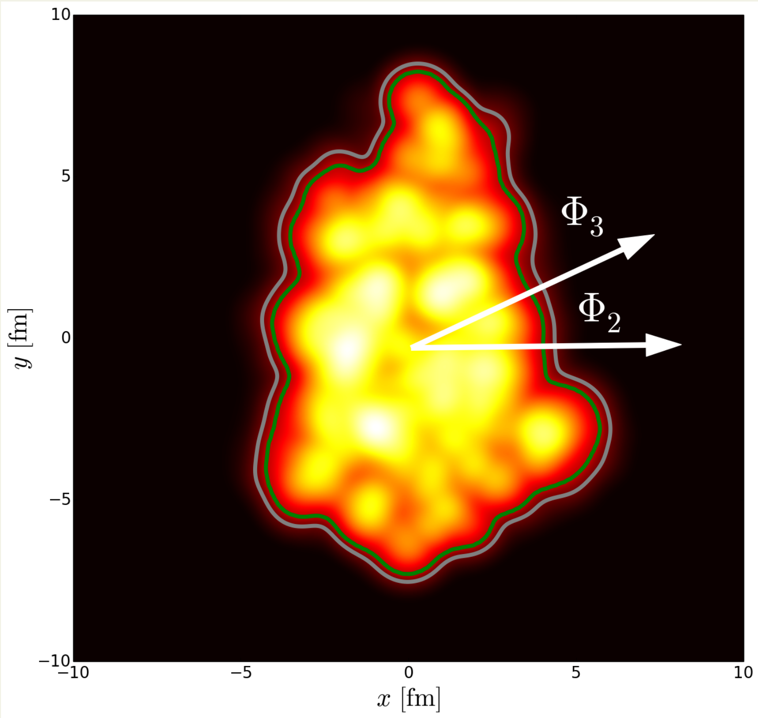
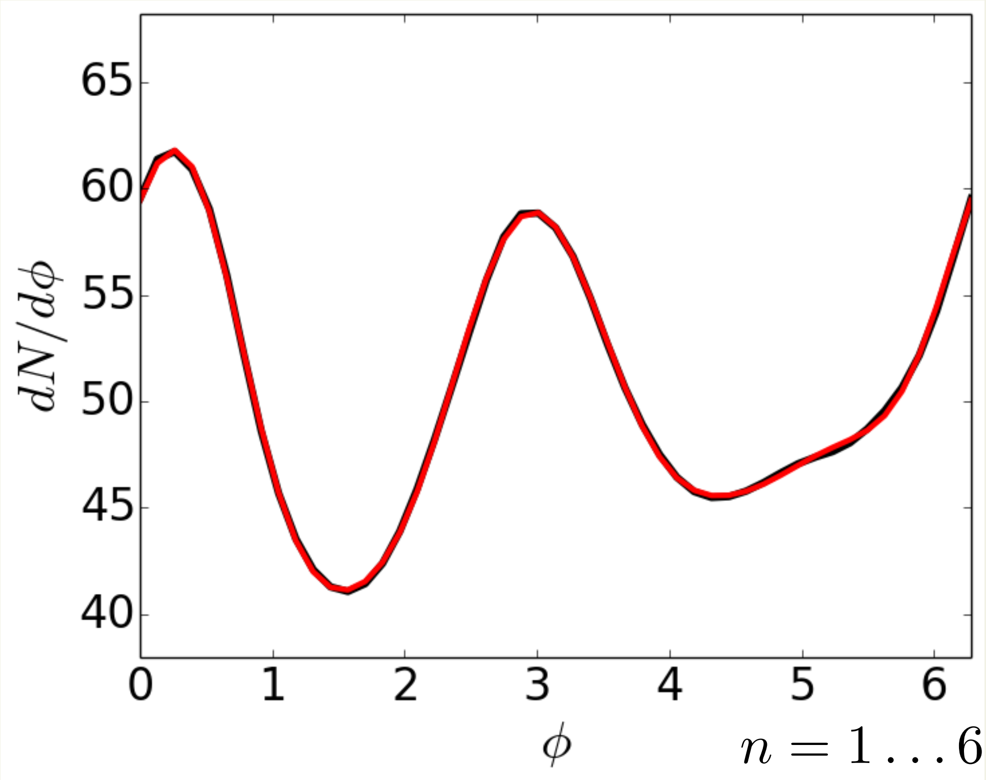
From fluid to distribution



$\Longrightarrow$



$$\epsilon_n, \Phi_n \Longrightarrow v_n, \Psi_n$$

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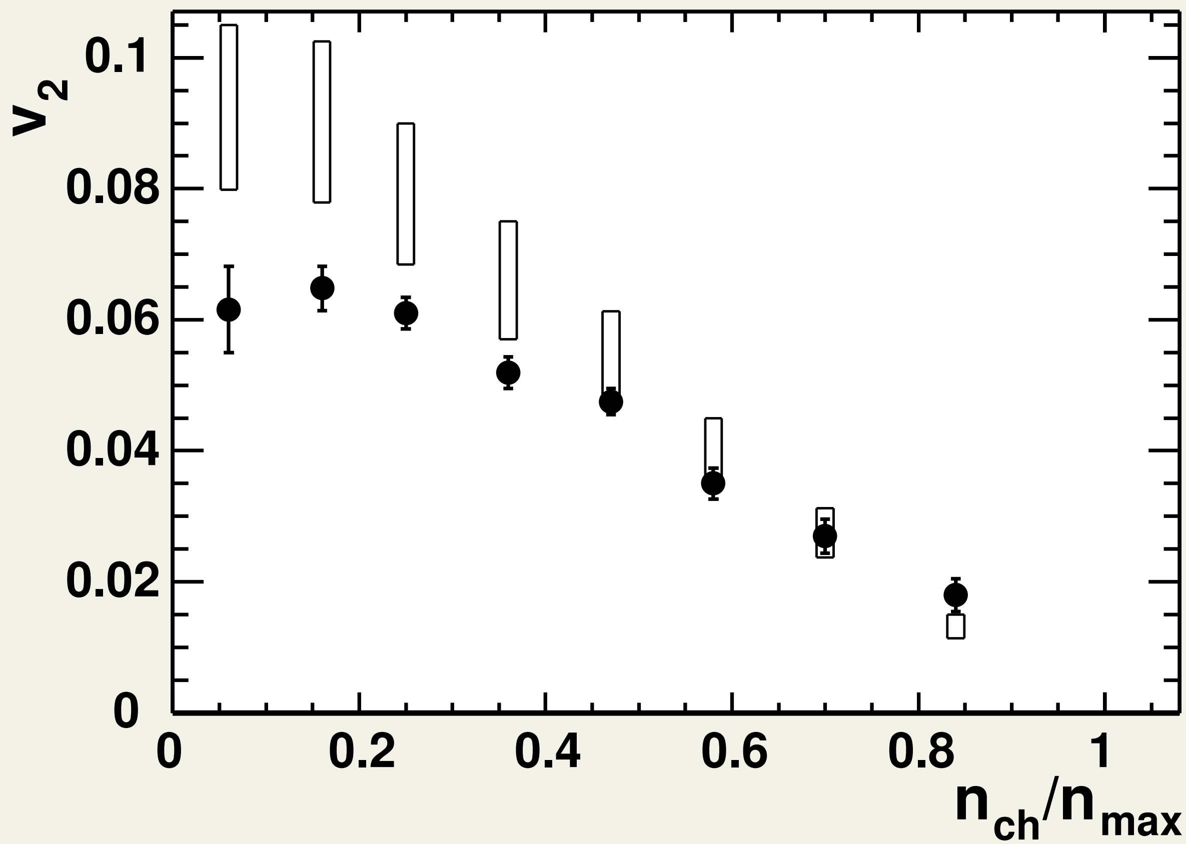
Success of ideal hydrodynamics

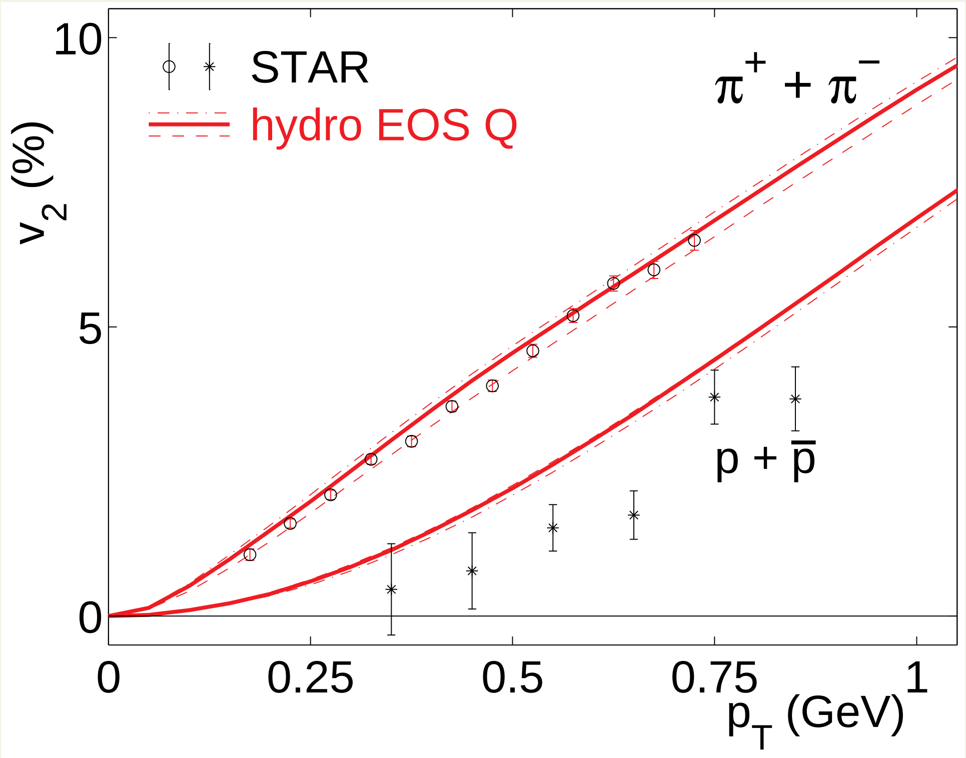
- p_T-averaged v_2 of charged hadrons:

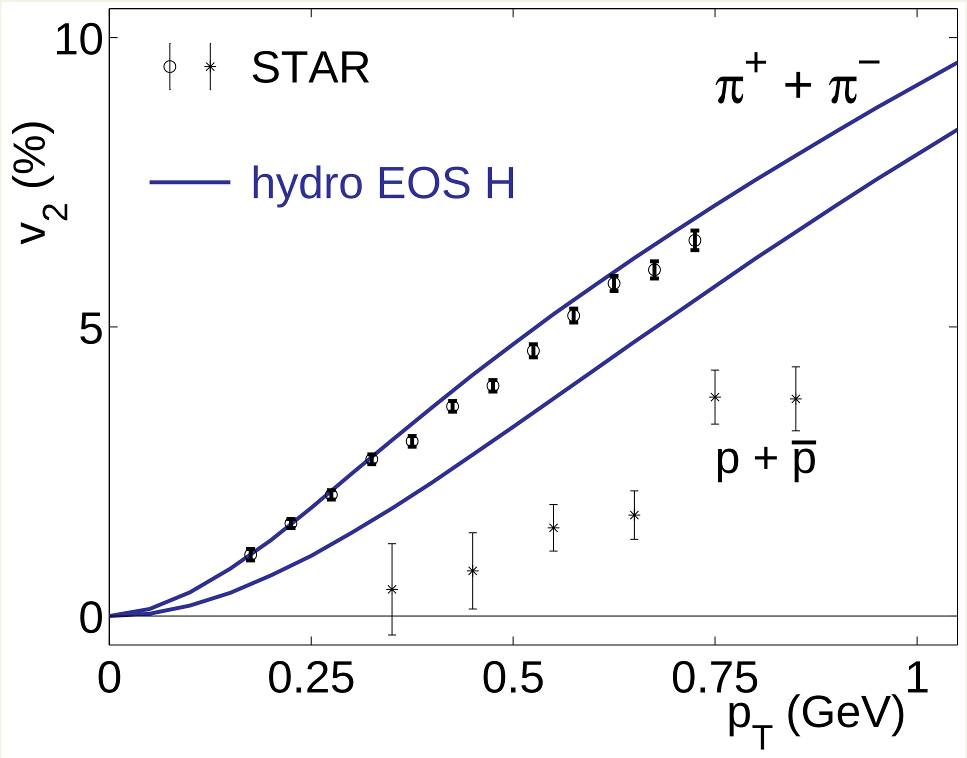


- works beautifully in central and semi-central collisions
- but why is $v_{2,\mathrm{obs}} > v_{2,hydro}$ in most central collisions?

Success of ideal hydrodynamics

Kolb, Heinz, Huovinen et al ('01) **minbias Au+Au at RHIC**

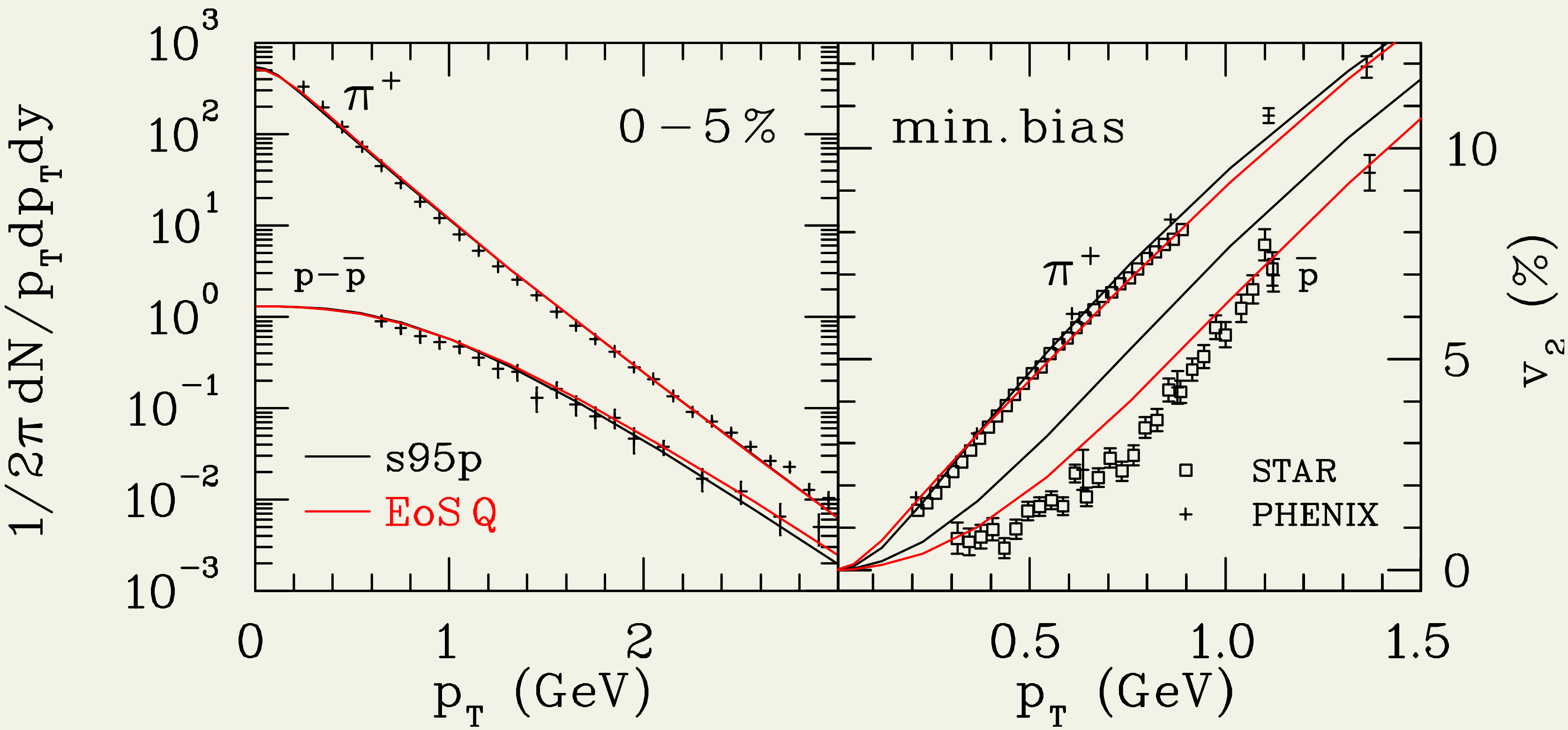




not perfect agreement but plasma EoS favored

Lattice EoS

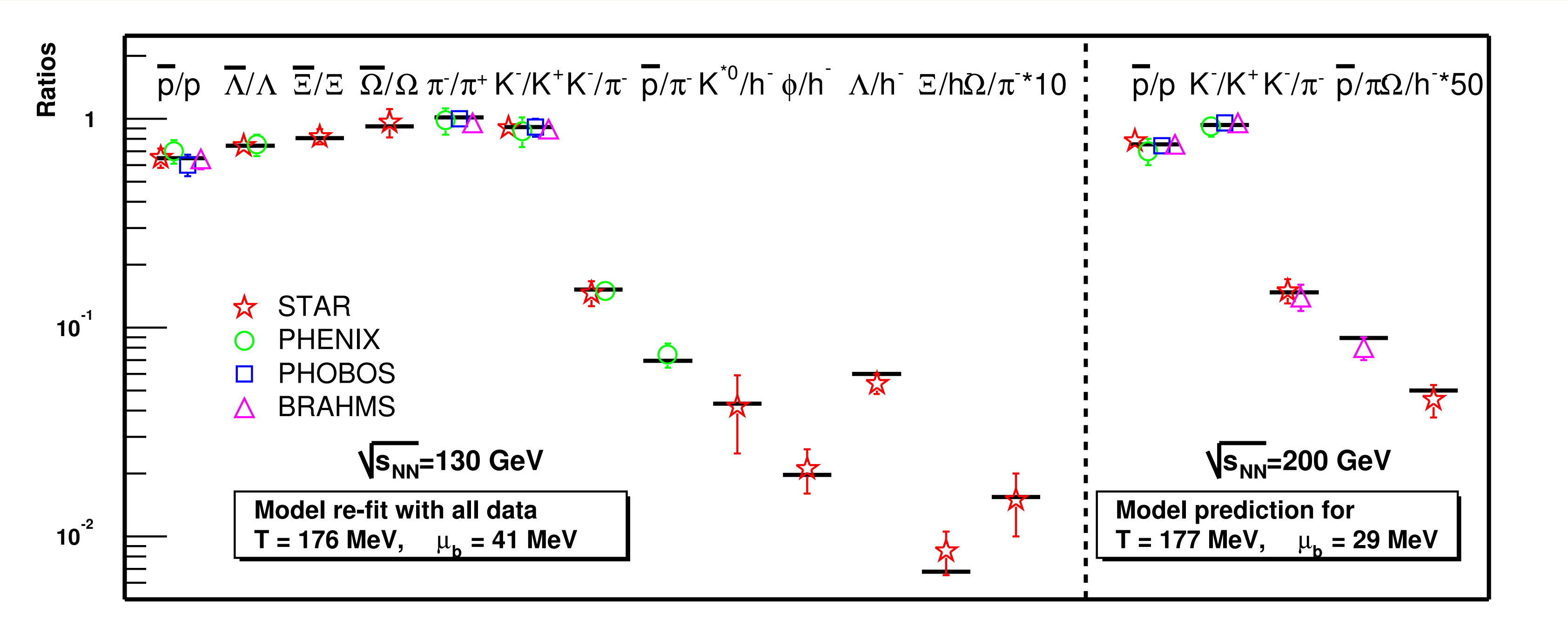
- **ideal hydro, Au+Au at $\sqrt{s_{NN}} = 200$ GeV**
- **chemical equilibrium**



- **s95p:** $T_{dec} = 140$ **MeV**
- **EoS Q: first order phase transition at $T_c = 170$ MeV, $T_{dec} = 125$ MeV**

Thermal models

- Hadronic phase: ideal gas of massive hadrons and resonances
- in chemical equilibrium



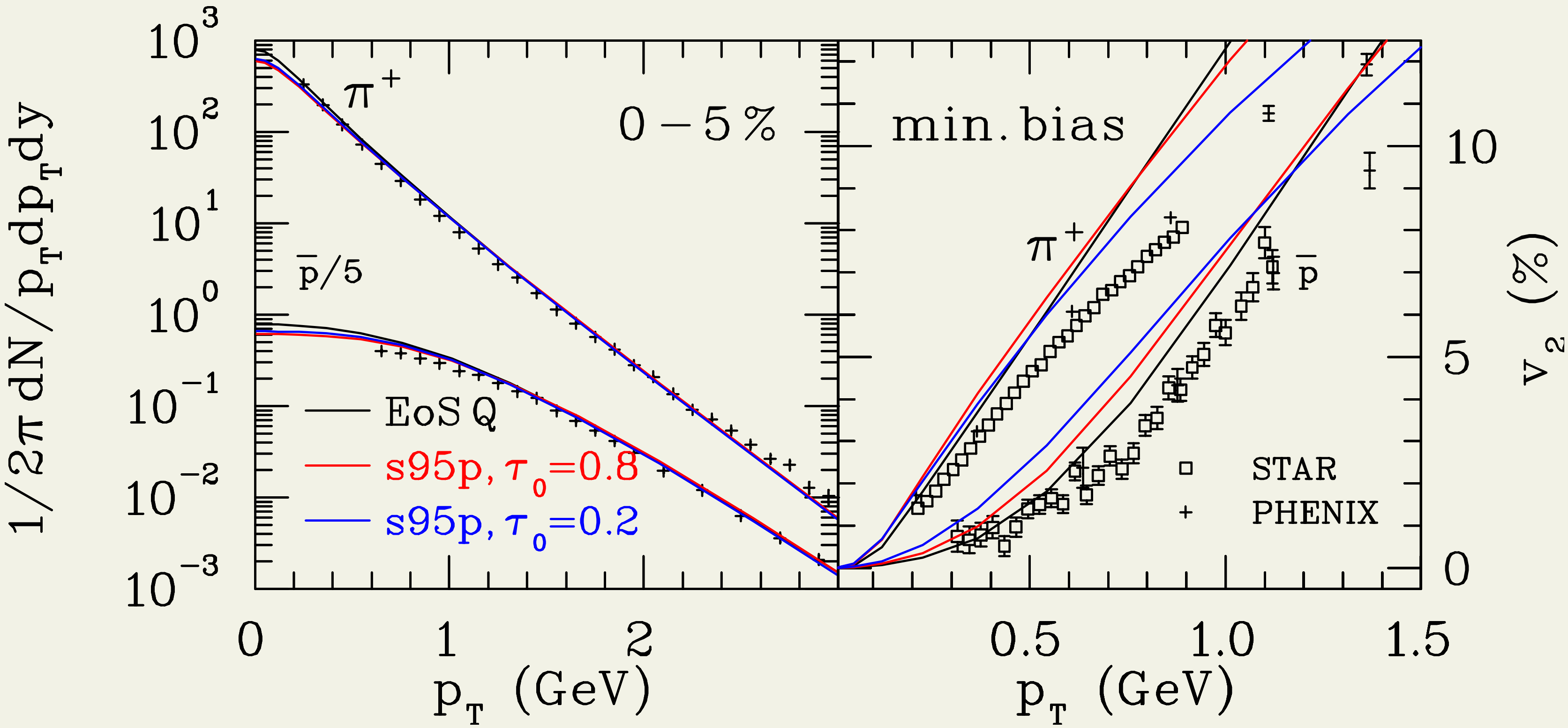
- **Particle ratios** $\Longleftrightarrow$ $T \approx 160$–170 **MeV temperature**

- **Evolution to** $T \approx 100$–140 **MeV temperature**

$\Rightarrow$ **In hydro particle ratios become wrong**

More realistic EoS

- **ideal hydro, Au+Au at** $\sqrt{s_{NN}} = 200$ **GeV**
- $T_{\rm chem} = 150$ **MeV**



- **EoS Q:** $T_{dec} = 120$ **MeV**, $s_{\rm ini} \propto N_{bin}$ $\tau_0 = 0.2$ **fm/**c
- **s95p,** $\tau_0 = 0.8$**:** $T_{dec} = 120$ **MeV**, $s_{\rm ini} \propto N_{bin}$, $\tau_0 = 0.8$ **fm/**c
- **s95p,** $\tau_0 = 0.2$**:** $T_{dec} = 120$ **MeV**, $s_{\rm ini} \propto N_{bin} + N_{part}$, $\tau_0 = 0.2$ **fm/**c

Dissipative hydrodynamics

In general

$$\begin{aligned} N^\mu &= nu^\mu + \nu^\mu \\ T^{\mu\nu} &= \epsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + W^\mu u^\nu + W^\nu u^\mu + \pi^{\mu\nu} \end{aligned}$$

In Landau frame,

$$W^\mu \equiv 0, \qquad \nu^\mu = -\frac{q^\mu}{h} = -\frac{n}{\epsilon + P} q^\mu \tag{4}$$

and thus

$$\begin{aligned} N^\mu &= nu^\mu + \nu^\mu \\ T^{\mu\nu} &= \epsilon u^\mu u^\nu - (P_{\mathrm{eq}} + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} \end{aligned}$$

Dissipative hydrodynamics

In Landau frame,

$$\begin{aligned} N^\mu &= nu^\mu + \nu^\mu \\ T^{\mu\nu} &= \epsilon u^\mu u^\nu - (P_{\rm eq} + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} \end{aligned}$$

Need 9 additional equations to determine

$$\Pi,\ \pi^{\mu\nu},\ q^\mu,\ P_{\rm eq}$$

Equation of state

$$P_{\rm eq} = P(T,\mu)$$

Matching conditions

ideal fluid $\Longleftrightarrow$ exact local kinetic equilibrium

dissipation $\Longleftrightarrow$ deviations from thermal distribution

Non-equilibrium thermodynamics?

- **What are entropy and pressure?**
- **EoS? Temperature?**

Matching conditions

ideal fluid $\Longleftrightarrow$ exact local kinetic equilibrium

dissipation $\Longleftrightarrow$ deviations from thermal distribution

Non-equilibrium thermodynamics?

- What are entropy and pressure?
- EoS? Temperature?

Energy and particle number defined for arbitrary system:

$$\epsilon = u_\mu T^{\mu\nu} u_\nu \qquad \text{and} \qquad n = N^\mu u_\mu$$

apply equilibrium EoS:

$$s = s_0(\epsilon, n) \qquad \text{and} \qquad P = P_0(\epsilon, n)$$

i.e. we match the system to an equilibrium system of *the same* ϵ and n

relativistic Navier-Stokes

Entropy four-current:

$$S^\mu = su^\mu + \frac{\mu}{T}\frac{q^\mu}{h}$$

where

$$h = \frac{\epsilon + P}{n}$$

Require non-decrease of entropy:

$$0 \leq \partial_\mu S^\mu = -\Pi \nabla^\mu u_\mu - q_\mu \frac{T}{e+p}\nabla^\mu \frac{\mu}{T} + \pi_{\mu\nu}\nabla^{\langle\mu}u^{\nu\rangle}$$

where

$$A^{\langle\mu\nu\rangle} = \left[\frac{1}{2}(\Delta^\mu_\sigma \Delta^\nu_\tau + \Delta^\nu_\tau \Delta^\mu_\sigma) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\sigma\tau}\right] A^{\sigma\tau}$$

and

$$\nabla^\mu = \Delta^{\mu\nu}\partial_\nu$$

relativistic Navier-Stokes

$$0 \leq \partial_\mu S^\mu = \Pi X + q_\mu X^\mu + \pi_{\mu\nu} X^{\mu\nu}$$

is always valid if we identify

$$\Pi \propto X, \qquad q^\mu \propto X^\mu, \qquad \pi^{\mu\nu} \propto X^{\mu\nu}$$

dissipative currents small corrections linear in gradients

$$\begin{aligned} \Pi &= -\zeta \nabla^\mu u_\mu \\ q^\mu &= -\kappa \frac{T}{e+p} \nabla^\mu \frac{\mu}{T} \\ \pi^{\mu\nu} &= 2\eta \nabla^{\langle\mu} u^{\nu\rangle} \end{aligned}$$

η, ζ **shear and bulk viscosities,** κ **heat conductivity**

Navier-Stokes equations of motion

$$
\begin{aligned}
Dn &= -n\partial_\mu u^\mu - \partial_\mu\left(\kappa\frac{Tn}{h^2}\nabla^\mu\frac{\mu}{T}\right)\\
D\epsilon &= -(\epsilon + P - \zeta\nabla^\alpha u_\alpha)\partial_\mu u^\nu + 2\eta\nabla^{\langle\alpha}u^{\beta\rangle}\nabla_{\langle\alpha}u_{\beta\rangle}\\
(\epsilon + P - \zeta\nabla^\alpha u_\alpha)Du^\mu &= \nabla^\mu(P - \zeta\nabla^\alpha u_\alpha) - 2\Delta^\mu_\alpha\partial_\beta(\eta\nabla^{\langle\alpha}u^{\beta\rangle})
\end{aligned}
$$

where

$$D = u^\mu\partial_\mu \qquad \text{and} \qquad \nabla^\mu = \Delta^{\mu\nu}\partial_\nu$$

Navier-Stokes equations of motion

$$
\begin{aligned}
Dn &= -n\partial_\mu u^\mu - \partial_\mu\left(\kappa\frac{Tn}{h^2}\nabla^\mu\frac{\mu}{T}\right)\\
D\epsilon &= -(\epsilon + P - \zeta\nabla^\alpha u_\alpha)\partial_\mu u^\nu + 2\eta\nabla^{\langle\alpha}u^{\beta\rangle}\nabla_{\langle\alpha}u_{\beta\rangle}\\
(\epsilon + P - \zeta\nabla^\alpha u_\alpha)Du^\mu &= \nabla^\mu(P - \zeta\nabla^\alpha u_\alpha) - 2\Delta^\mu_\alpha\partial_\beta(\eta\nabla^{\langle\alpha}u^{\beta\rangle})
\end{aligned}
$$

where

$$D = u^\mu\partial_\mu \qquad \text{and} \qquad \nabla^\mu = \Delta^{\mu\nu}\partial_\nu$$

but these are parabolic. . .

Parabolic partial differential equations

PDE of the form

$$A\frac{\partial^2}{\partial x^2}u + B\frac{\partial^2}{\partial x\partial y}u + C\frac{\partial^2}{\partial y^2}u + D\frac{\partial}{\partial x}u + E\frac{\partial}{\partial y}u + F = 0$$

is parabolic if

$$B^2 - AC = 0$$

Such equations provide infinite speed for signal propagation

Müller ('76), Israel & Stewart ('79) ...

Solutions are unstable

Hiscock & Lindblom, PRD31, 725 (1985) ...

Hyperbolic partial differential equations

PDE of the form

$$A\frac{\partial^2}{\partial x^2}u + B\frac{\partial^2}{\partial x \partial y}u + C\frac{\partial^2}{\partial y^2}u + D\frac{\partial}{\partial x}u + E\frac{\partial}{\partial y}u + F = 0$$

is hyperbolic if

$$B^2 - AC > 0$$

For example one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

Solutions stable and with finite propagation speed.

Causal viscous hydro

To obtain causal equations we have to replace

$$\Pi = -\zeta \nabla^\mu u_\mu$$

by

$$\tau_\Pi D\Pi + \Pi = -\zeta \nabla^\mu u_\mu + \cdots$$

or something similar.

Causal viscous hydro

Israel & Stewart:

Entropy four-flow including terms second order in dissipative fluxes:

$$S^\mu = \quad su^\mu + \frac{\mu}{T}\frac{q^\mu}{h} \quad -\left(\beta_0\Pi^2 - \beta_1 q_\nu q^\nu + \beta_2 \pi_{\lambda\nu}\pi^{\lambda\nu}\right)\frac{u^\mu}{2T}$$
$$-\frac{\alpha_0 q^\mu \Pi}{T} + \frac{\alpha_1 q_\nu \pi^{\nu\mu}}{T}$$

$\Rightarrow$ **“Second order theory”**

or, rather, Transient fluid dynamics

Evolution equation for shear

Require non-decrease of entropy:

$$0 \leq \partial_\mu S^\mu = \Pi X + q_\mu X^\mu + \pi_{\mu\nu} X^{\mu\nu}$$

Identify $\pi^{\mu\nu} = 2\eta X^{\langle\mu\nu\rangle}$**:**

$$\begin{aligned} \pi^{\mu\nu} &= 2\eta \left[\nabla^{\langle\mu} u^{\nu\rangle} - \beta_2 \langle u^\lambda \partial_\lambda \pi^{\mu\nu} \rangle - \frac{1}{2} \pi^{\mu\nu} T \partial_\lambda \left(\frac{\tau_\pi u^\lambda}{2\eta T} \right) \right] \\ &\quad + 2\eta \left[\alpha_1 \nabla^{\langle\mu} q^{\nu\rangle} + a_1' q^{\langle\mu} u^\lambda \partial_\lambda u^{\nu\rangle} \right] \end{aligned}$$

where

$$A^{\langle\mu\nu\rangle} = \left[\frac{1}{2} (\Delta^\mu_\sigma \Delta^\nu_\tau + \Delta^\nu_\tau \Delta^\mu_\sigma) - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\sigma\tau} \right] A^{\sigma\tau}$$

and

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu.$$

Israel-Stewart evolution equations

$$
\begin{aligned}
D\Pi &= -\frac{1}{\tau_\Pi}\left(\Pi + \zeta\nabla_\mu u^\mu\right) - \frac{1}{2}\Pi\left(\nabla_\mu u^\mu + D\ln\frac{\beta_0}{T}\right) \\
&\quad + \frac{\alpha_0}{\beta_0}\partial_\mu q^\mu - \frac{a_0'}{\beta_0} q^\mu D u_\mu \\
Dq^\mu &= -\frac{1}{\tau_q}\left[q^\mu + \kappa_q \frac{T^2 n}{\varepsilon + p}\nabla^\mu\left(\frac{\mu}{T}\right)\right] - u^\mu q_\nu D u^\nu \\
&\quad -\frac{1}{2} q^\mu\left(\nabla_\lambda u^\lambda + D\ln\frac{\beta_1}{T}\right) - \omega^{\mu\lambda} q_\lambda \\
&\quad -\frac{\alpha_0}{\beta_1}\nabla^\mu\Pi + \frac{\alpha_1}{\beta_1}\left(\partial_\lambda \pi^{\lambda\mu} + u^\mu \pi^{\lambda\nu}\partial_\lambda u_\nu\right) + \frac{a_0}{\beta_1}\Pi D u^\mu - \frac{a_1}{\beta_1}\pi^{\lambda\mu} D u_\lambda \\
D\pi^{\mu\nu} &= -\frac{1}{\tau_\pi}\left(\pi^{\mu\nu} - 2\eta\nabla^{\langle\mu} u^{\nu\rangle}\right) - \left(\pi^{\lambda\mu} u^\nu + \pi^{\lambda\nu} u^\mu\right) D u_\lambda \\
&\quad -\frac{1}{2}\pi^{\mu\nu}\left(\nabla_\lambda u^\lambda + D\ln\frac{\beta_2}{T}\right) - 2\pi_\lambda{}^{\langle\mu}\omega^{\nu\rangle\lambda} \\
&\quad -\frac{\alpha_1}{\beta_2}\nabla^{\langle\mu} q^{\nu\rangle} + \frac{a_1'}{\beta_2} q^{\langle\mu} D u^{\nu\rangle}
\end{aligned}
$$

Israel-Stewart evolution. . .

bulk pressure Π, shear stress $\pi^{\mu\nu}$ heat flow q^{μ} treated as independent dynamical quantities that relax to their Navier-Stokes value on time scales $\tau_{\Pi}(e,n)$, $\tau_{\pi}(e,n)$, $\tau_{q}(e,n)$

Equations of motion	5 equations
evolution of bulk	1 equation
evolution of heat flow	3 equations
evolution of shear stress	5 equations
14 equations, 14 unknowns	

These equations are causal and stable

But what are the parameters α_0, α_1, $\beta_0, \beta_1, \beta_2$?

Or how to obtain ζ, κ, η?

$\Longrightarrow$ use kinetic theory

Or some other microscopic theory

more terms. . .

- **Kinetic theory derivation (see Denicol et al., PRD85, 114047 (2012)) or gradient expansion (see Romatschke et al., JHEP 0804, 100 (2008)) lead to even more terms (all possible in second order in products of gradients)**

- **Do not contribute to entropy, may affect the evolution**

- **What is usually solved is**

$$\pi^{\mu\nu} + \tau_\pi \left[\Delta^\mu_\alpha \Delta^\nu_\beta D\pi^{\alpha\beta} + \frac{4}{3}\pi^{\mu\nu}\nabla_\alpha u^\alpha \right] = \eta \nabla^{\langle\mu} u^{\nu\rangle}$$

η/s

Ideal:

$$(\epsilon + P)Du^\mu = \nabla^\mu P$$

c.f.

$$ma = F$$

η/s

Ideal:

$$(\epsilon + P)Du^\mu = \nabla^\mu P$$

Viscous:

$$\begin{aligned}(\epsilon + P)Du^\mu &= \nabla^\mu P - \Delta^\mu{}_\alpha \partial_\beta \pi^{\alpha\beta} \\ Du^\mu &= \frac{1}{\epsilon + P}\nabla^\mu P - \frac{2\eta}{\epsilon + P}\Delta^\mu{}_\alpha \partial_\beta \left[\nabla^{\langle\alpha} u^{\beta\rangle} + \cdots\right] + \cdots\end{aligned}$$

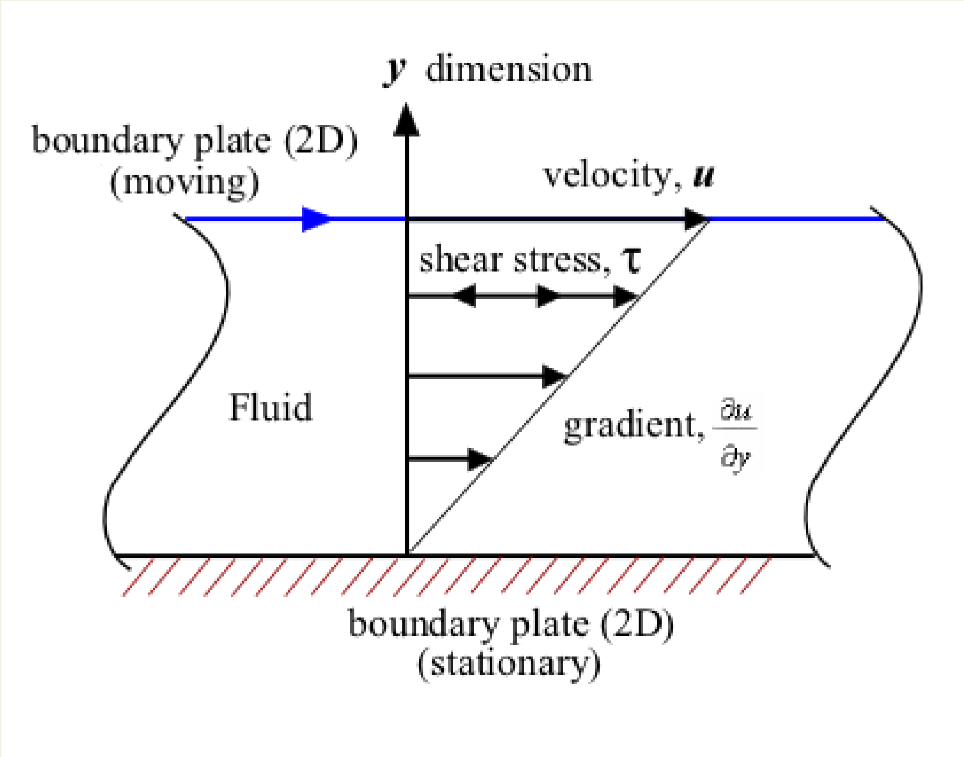
$$\mu = 0 \Longrightarrow Ts = \epsilon + P :$$

$$Du^\mu = \frac{1}{\epsilon + P}\nabla^\mu P - \frac{2}{T}\frac{\eta}{s}\Delta^\mu{}_\alpha \partial_\beta \left[\nabla^{\langle\alpha} u^{\beta\rangle} + \cdots\right] + \cdots$$

Shear viscosity

Newton:

$$T_{xy} = -\eta \frac{\partial u_x}{\partial y}$$



acts to reduce velocity gradients

in closed system: **energy conserved**

kinetic energy gets converted to internal energy

$\Rightarrow$ **dissipation**

Shear in 1D-bjorken

Navier-Stokes stress

$$\pi^{\mu\nu} = 2\eta\nabla^{\langle\mu}u^{\nu\rangle} = \mathrm{diag}(0, \frac{2\eta}{3\tau}, \frac{2\eta}{3\tau}, -\frac{4\eta}{3\tau})$$

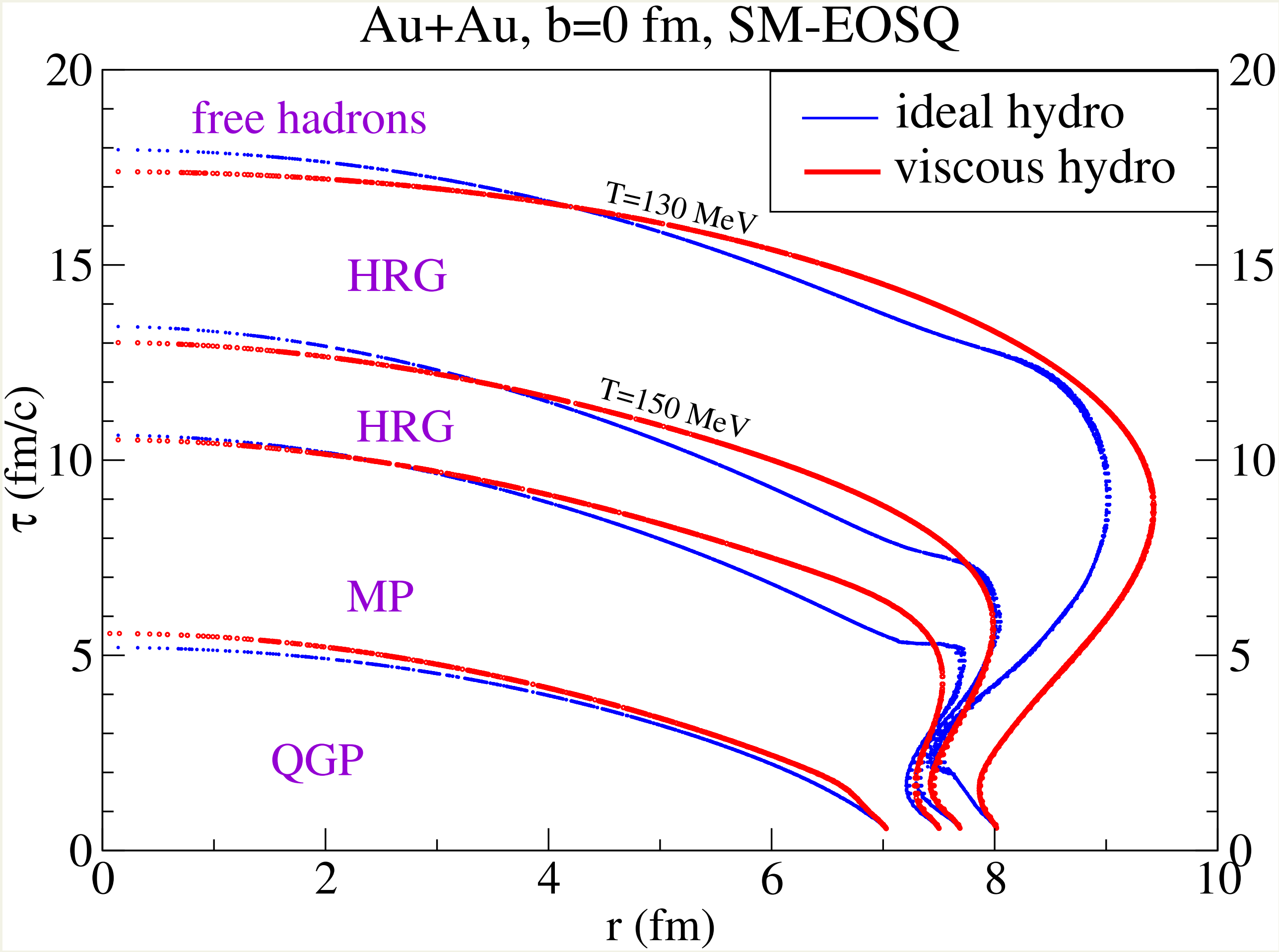
$$T^{\mu\nu} = \mathrm{diag}(\epsilon, P - \frac{\pi_L}{2}, P - \frac{\pi_L}{2}, P + \pi_L)$$

where $\pi_L = \pi^{\eta\eta} = -\frac{4\eta}{3\tau}$

Effective longitudinal pressure $P + \pi_L < P$
Effective transverse pressure $P - \pi_L/2 > P$

Shear slows down longitudinal expansion and accelerates transverse expansion

Effect on temperature



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- **Edges expand further and stay hotter**
- **At first core cools slower, later faster**