Transverse momentum distributions of hadrons in the Tsallis nonextensive statistics

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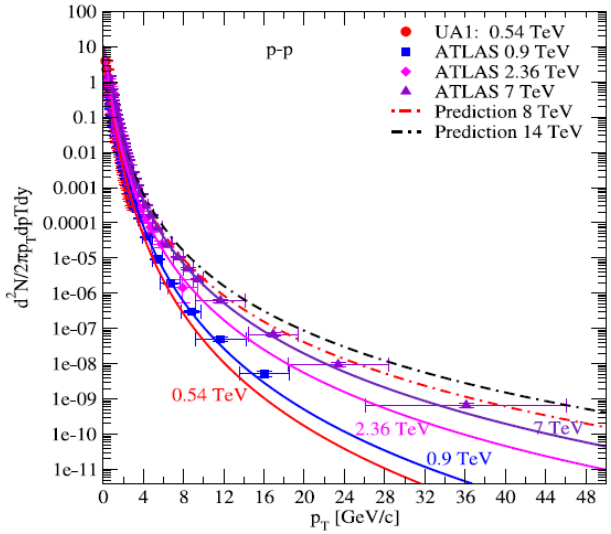
Phenomenological Tsallis distribution: Proton-Proton and Heavy-Ion Collisions

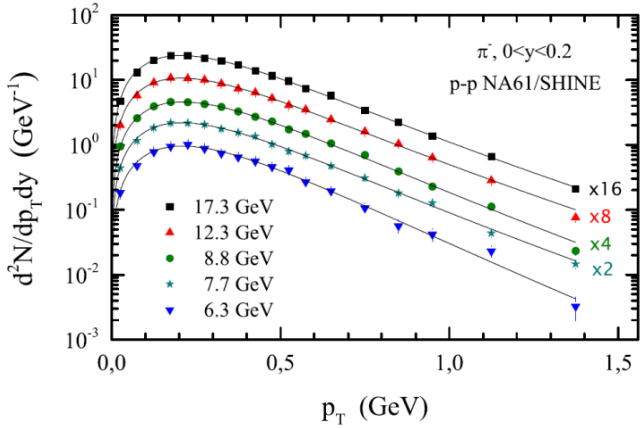
Proton-Proton Collisions

✓ **Charged hadron yields:**



J.Cleymans, G.I.Lykasov, A.S.P., A.S.Sorin, O.V.Teryaev, D.Worku, Phys. Lett. B 723 (2013) 351

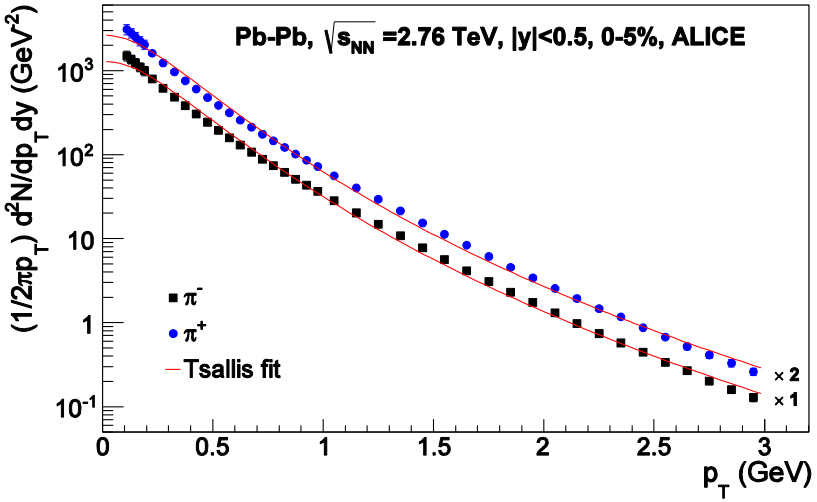
✓ **Identified hadrons:**



A.S.P., O.V.Teryaev, J.Cleymans, Eur. Phys. J. A 53 (2017) 102

Central Pb-Pb collisions at 2.76 TeV

Exp.: ALICE, Phys. Rev. C 88 (2013) 044910



The data of $\pi^{\pm}$ at 2.76 TeV are not described by the phenomenological Tsallis distributions at low pT momenta

✓ **Phenomenological Tsallis distribution:**

$$\left.\frac{d^2N}{dp_T dy}\right|_{y_0}^{y_1} = gV\int_{y_0}^{y_1} dy\,\frac{p_T m_T \cosh y}{(2\pi)^2}\left[1-(1-q)\frac{m_T\cosh y-\mu}{T}\right]^{\frac{q}{1-q}}$$

J. Cleymans, D. Worku, Eur. Phys. J. A 48 (2012) 160

It corresponds to Maxwell-Boltzmann Ideal Gas in volume *V* at temperature *T* and chemical potential μ

✓ **Distributions of Boltzmann-Gibbs statistics:**

$$\cancel{\left\langle n_{\vec{p}}\right\rangle = \frac{1}{e^{\frac{m_T\cosh y-\mu}{T}}+\eta}}$$

$\eta = 1$ Fermi-Dirac (F-D)

$\eta = -1$ Bose-Einstein (B-E)

$\eta = 0$ Maxwell-Boltzmann (M-B)

Generalized Statistical Mechanics

- **Fundamental statistical ensemble** $(x^1,\ldots,x^n)$

$$E = E(x^1,\ldots,x^n), \qquad x^1 = S,\ x^2 = N,\ x^3 = V,\ \ldots$$

$$dE = \sum_{k=1}^{n} u^k dx^k,$$

$$u^k = \frac{\partial E}{\partial x^k}, \qquad u^1 = T,\ u^2 = \mu,\ u^3 = -p,\ \ldots$$

- **Statistical ensemble** $(u^1,\ldots,u^m,x^{m+1},\ldots,x^n)$

A.S.P., in Recent Advances in Thermo and Fluid Dynamics, ed. Mod Gorji-Bandpy, InTech, Chapter 11, 2015, pp.303-331

✓ **Legendre transform** for the thermodynamic potential:

$$Y = Y(u^1,\ldots,u^m,x^{m+1},\ldots,x^n) = E - \sum_{k=1}^{m} u^k x^k$$

✓ Statistical averages for the fluctuating quantities:

$$x^k = \sum_i p_i x_i^k, \qquad x_i^1 = S_i,\ x_i^2 = N_i,\ x_i^3 = V_i,\ \ldots,\ \ k = 1,\ldots,m,$$

$$E = \sum_i p_i E_i,$$

$$Y = \sum_i p_i Y_i, \qquad Y_i = E_i - \sum_{k=1}^{m} u^k x_i^k$$

$$S_i = -\ln p_i$$ -Boltzmann-Gibbs entropy

$$S_i = \frac{p_i^{q-1} - 1}{1-q}$$ -Tsallis entropy

Generalized Statistical Mechanics: Probability Distribution of Microstates

- ✓ **Principle of the maximum entropy** from the second law of thermodynamics (constrained local extrema of the thermodynamic potential in the method of Lagrange multipliers) :

$$\Phi = Y - \lambda\varphi, \qquad \varphi = \sum_i p_i - 1 = 0, \qquad \frac{\partial \Phi}{\partial p_i} = 0, \qquad Y = \sum_i p_i \left[E_i - \sum_{k=1}^{m} u^k x_i^k \right]$$

- ✓ **Probabilities of microstates and norm function**:

$$p_i = F\left(\frac{1}{u^1} \left(\Lambda - E_i + \sum_{k=2}^{m} u^k x_i^k \right) \right),$$

$$u^1 = T,\ u^2 = \mu,\ u^3 = -p,\ \ldots$$

$$x^1 = S,\ x^2 = N,\ x^3 = V,\ \ldots$$

$$\sum_i \delta_{x^{m+1}, x_i^{m+1}} \cdots \delta_{x^n, x_i^n} F\left(\frac{1}{u^1} \left(\Lambda - E_i + \sum_{k=2}^{m} u^k x_i^k \right) \right) = 1 \quad \rightarrow \quad \Lambda = \Lambda(u^1, \ldots, u^m, x^{m+1}, \ldots, x^n)$$

- ✓ Thermodynamic quantities are partial derivatives of thermodynamic potential :

$$dY(u^1, \ldots, u^m, x^{m+1}, \ldots, x^n) = -\sum_{k=1}^{m} x^k du^k + \sum_{k=m+1}^{n} u^k dx^k,$$

$$x^k = -\frac{\partial Y}{\partial u^k} = \sum_i p_i x_i^k \qquad (k = 1, \ldots, m)$$

$$u^k = \frac{\partial Y}{\partial x^k} = \sum_i p_i \left[\frac{\partial E_i}{\partial x^k} - \sum_{j=2}^{m} u^j \frac{\partial x_i^j}{\partial x^k} \right] \qquad (k = m+1, \ldots, n)$$

A.S.P., in Recent Advances in Thermo and Fluid Dynamics, ed. Mod Gorji-Bandpy, InTech, Chapter 11, 2015, pp.303-331

Tsallis Generalized Statistical Mechanics

Boltzmann-Gibbs Statistics	Tsallis-1 Statistics	Tsallis-2 Statistics	Tsallis-3 Statistics
$q=1$	$0<q<\infty$	$0<q<\infty$	$0<q<\infty$
$S=-\sum_i p_i \ln p_i$	$S=-\sum_i \frac{p_i-p_i^q}{1-q}$	$S=-\sum_i \frac{p_i-p_i^q}{1-q}$	$S=-\sum_i \frac{p_i-p_i^q}{1-q}$
$\sum_i p_i=1$	$\sum_i p_i=1$	$\sum_i p_i=1$	$\sum_i p_i=1$
$\langle A\rangle=\sum_i p_i A_i$	$\langle A\rangle=\sum_i p_i A_i$	$\langle A\rangle=\sum_i p_i^q A_i$	$\langle A\rangle=\frac{\sum_i p_i^q A_i}{\sum_i p_i^q}$

p_i-probability of i-th microstate of system

Expectation values of Tsallis-2 statistics are unnormalized

Standard expectation values

Generalized expectation values

C. Tsallis, J. Stat. Phys. 52 (1988) 479
C. Tsallis, R.S. Mendes, A.R. Plastino, Physica A 261 (1998) 534

Tsallis Generalized Statistical Mechanics: Isolated Systems

- ✓ Variables of state: $\left(E,\ x^2 = N,\ x^3 = V \right)$
- ✓ Fluctuating variables: $\left(u^1 = T,\ u^2 = \mu,\ u^3 = -p \right)$

C. Tsallis, J. Stat. Phys. 52 (1988) 479
C. Tsallis, R.S. Mendes, A.R. Plastino, Physica A 261 (1998) 534
A.S.P., Phys. Lett. A 350 (2006) 331

- ✓ **Second law of thermodynamics** (constrained local extrema of thermodynamic potential):

$$\Phi = S - \lambda\varphi, \qquad \varphi = \sum_i p_i - 1 = 0, \qquad \frac{\partial \Phi}{\partial p_i} = 0 \qquad [Y = S]$$

	Boltzmann-Gibbs Statistics	Tsallis-1 Statistics	Tsallis-2 Statistics
	$q = 1$	$0 < q < \infty$	$0 < q < \infty$
✓ Thermodynamic potential:	$Y = S = -\sum_i p_i \ln p_i$	$Y = S = -\sum_i \frac{p_i - p_i^q}{1-q}$	$Y = S = -\sum_i \frac{p_i - p_i^q}{1-q}$
	$S = \ln W$	$S = \frac{W^{1-q} - 1}{1-q}$	$S = \frac{W^{1-q} - 1}{1-q}$
✓ Probability distribution:	$p_i = \frac{1}{W}$	$p_i = \frac{1}{W}$	$p_i = \frac{1}{W}$
✓ Statistical weight:	$W = \sum_i \delta_{E_i,E} \delta_{N_i,N} \delta_{V_i,V}$	$W = \sum_i \delta_{E_i,E} \delta_{N_i,N} \delta_{V_i,V}$	$W = \sum_i \delta_{E_i,E} \delta_{N_i,N} \delta_{V_i,V}$

Tsallis Generalized Statistical Mechanics: Closed Systems

C. Tsallis, J. Stat. Phys. 52 (1988) 479
C. Tsallis, R.S. Mendes, A.R. Plastino, Physica A 261 (1998) 534
A.S.P., Phys. Lett. A 360 (2006) 26

- ✓ Variables of state: $\left(u^1 = T,\ \ x^2 = N,\ \ x^3 = V\right)$
- ✓ Fluctuating variables: $\left(x^1 = S,\ \ u^2 = \mu,\ \ u^3 = -p\right)$
- ✓ **Second law of thermodynamics** (constrained local extrema of thermodynamic potential):

$$\Phi = F - \lambda\varphi, \qquad \varphi = \sum_i p_i - 1 = 0, \qquad \frac{\partial \Phi}{\partial p_i} = 0 \qquad \left[Y = F = E - TS\right]$$

	Boltzmann-Gibbs Statistics	Tsallis-1 Statistics	Tsallis-2 Statistics
	$q = 1$	$0 < q < \infty$	$0 < q < \infty$
✓ Thermodynamic potential:	$F = T\sum_i p_i\left[\ln p_i + \frac{E_i}{T}\right]$	$F = T\sum_i p_i\left[\frac{1 - p_i^{q-1}}{1-q} + \frac{E_i}{T}\right]$	$F = T\sum_i p_i^q\left[\frac{p_i^{1-q} - 1}{1-q} + \frac{E_i}{T}\right]$
✓ Probability distribution:	$p_i = \frac{1}{Z}\exp\left(-\frac{E_i}{T}\right)$	$p_i = \left[1 + \frac{q-1}{q}\frac{\Lambda - E_i}{T}\right]^{\frac{1}{q-1}}$	$p_i = \frac{1}{Z}\left[1 - (1-q)\frac{E_i}{T}\right]^{\frac{1}{1-q}}$
✓ Norm function:	$Z = \sum_i \exp\left(-\frac{E_i}{T}\right)$	$1 = \sum_i\left[1 + \frac{q-1}{q}\frac{\Lambda - E_i}{T}\right]^{\frac{1}{q-1}}$	$Z = \sum_i\left[1 - (1-q)\frac{E_i}{T}\right]^{\frac{1}{1-q}}$
✓ Expectation values:	$\langle A\rangle = \frac{1}{Z}\sum_i A_i \exp\left(-\frac{E_i}{T}\right)$	$\langle A\rangle = \sum_i A_i\left[1 + \frac{q-1}{q}\frac{\Lambda - E_i}{T}\right]^{\frac{1}{q-1}}$	$\langle A\rangle = \frac{1}{Z^q}\sum_i A_i\left[1 - (1-q)\frac{E_i}{T}\right]^{\frac{q}{1-q}}$

Expectation values of Tsallis-2 statistics are unnormalized

Tsallis Generalized Statistical Mechanics: Open Systems

A.S.P., Eur. Phys. J. A 51 (2015) 108;
Eur. Phys. J. A 53 (2017) 53

- ✓ Variables of state: $\left(u^1 = T,\ \ u^2 = \mu,\ \ x^3 = V\right)$
- ✓ Fluctuating variables: $\left(x^1 = S,\ \ x^2 = N,\ \ u^3 = -p\right)$
- ✓ **Second law of thermodynamics** (constrained local extrema of thermodynamic potential):

$$\Phi = \Omega - \lambda\varphi, \qquad \varphi = \sum_i p_i - 1 = 0, \qquad \frac{\partial \Phi}{\partial p_i} = 0 \qquad \left[Y = \Omega = E - TS - \mu N\right]$$

	Boltzmann-Gibbs Statistics	Tsallis-1 Statistics	Tsallis-2 Statistics
	$q = 1$	$0 < q < \infty$	$0 < q < \infty$
✓ Thermodynamic potential:	$\Omega = T\sum_i p_i\left[\ln p_i + \frac{E_i - \mu N_i}{T}\right]$	$\Omega = T\sum_i p_i\left[\frac{1 - p_i^{q-1}}{1-q} + \frac{E_i - \mu N_i}{T}\right]$	$\Omega = T\sum_i p_i^q\left[\frac{p_i^{1-q} - 1}{1-q} + \frac{E_i - \mu N_i}{T}\right]$
✓ Probability distribution:	$p_i = \frac{1}{Z}\exp\left(-\frac{E_i - \mu N_i}{T}\right)$	$p_i = \left[1 + \frac{q-1}{q}\frac{\Lambda - E_i + \mu N_i}{T}\right]^{\frac{1}{q-1}}$	$p_i = \frac{1}{Z}\left[1 - (1-q)\frac{E_i - \mu N_i}{T}\right]^{\frac{1}{1-q}}$
✓ Norm function:	$Z = \sum_i \exp\left(-\frac{E_i - \mu N_i}{T}\right)$	$1 = \sum_i\left[1 + \frac{q-1}{q}\frac{\Lambda - E_i + \mu N_i}{T}\right]^{\frac{1}{q-1}}$	$Z = \sum_i\left[1 - (1-q)\frac{E_i - \mu N_i}{T}\right]^{\frac{1}{1-q}}$
✓ Expectation values:	$\langle A\rangle = \sum_i p_i A_i$	$\langle A\rangle = \sum_i p_i A_i$	$\langle A\rangle = \sum_i p_i^q A_i$

Expectation values of Tsallis-2 statistics are unnormalized

Ultrarelativistic Transverse Momentum Distribution: Tsallis-2 Statistics

Ultrarelativistic Maxwell-Boltzmann Ideal Gas in the Grand Canonical Ensemble $\left(m=0,\quad q>1\right)$

✓ **Mean occupation numbers:**

$$\langle n_{\vec{p}\sigma}\rangle = \frac{1}{Z^q}\sum_{\{n_{\vec{p}\sigma}\}} n_{\vec{p}\sigma}\frac{1}{\prod_{\vec{p}\sigma} n_{\vec{p}\sigma}!}\left[1-(1-q)\frac{\sum_{\vec{p}\sigma} n_{\vec{p}\sigma}(\varepsilon_{\vec{p}}-\mu)}{T}\right]^{\frac{q}{1-q}} = \frac{1}{Z^q}\sum_{N=0}^{N_0}\frac{\tilde{\omega}^N}{N!}a_0(1)\left[1+(q-1)\frac{\varepsilon_{\vec{p}}-\mu(N+1)}{T}\right]^{\frac{q}{1-q}+3N}$$

✓ **Norm function:**

A.S.P., Eur. Phys. J. A 53 (2017) 53

$$Z = \sum_{\{n_{\vec{p}\sigma}\}}\frac{1}{\prod_{\vec{p}\sigma} n_{\vec{p}\sigma}!}\left[1-(1-q)\frac{\sum_{\vec{p}\sigma} n_{\vec{p}\sigma}(\varepsilon_{\vec{p}}-\mu)}{T}\right]^{\frac{1}{1-q}} = \sum_{N=0}^{N_0}\frac{\tilde{\omega}^N}{N!}a_0(0)\left[1+(1-q)\frac{\mu N}{T}\right]^{\frac{1}{1-q}+3N},$$

where

$$a_\eta(\xi) = \frac{\Gamma\left(\frac{1}{q-1}+\xi-3(N+\eta)\right)}{(q-1)^{3(N+\eta)}\Gamma\left(\frac{1}{q-1}+\xi\right)},\qquad \tilde{\omega} = \frac{gVT^3}{\pi^2}$$

- Expectation values of Tsallis-2 statistics are unnormalized
- N_0 is a cut-off parameter
- Terms with $N > N_0$ in Tsallis-2 statistics increase and became divergent for q>1

Ultrarelativistic Transverse Momentum Distribution: Tsallis-2 Statistics (continuation)

✓ **Ultrarelativistic transverse momentum distribution:**

A.S.P., Eur. Phys. J. A 53 (2017) 53

$$\frac{d^2N}{dp_T dy} = \frac{V}{(2\pi)^3}\sum_{\sigma}\int_0^{2\pi} d\varphi\, p_T \varepsilon_{\vec{p}}\,\langle n_{\vec{p}\sigma}\rangle, \qquad \varepsilon_{\vec{p}} = p_T \cosh y \;\text{ for }\; m=0$$

✓ **Exact Solution:** $(m=0,\;\; q>1)$

$$\boxed{\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2}\, p_T^2 \cosh y\, \frac{1}{Z^q}\sum_{N=0}^{N_0}\frac{\tilde{\omega}^N}{N!}\,\frac{\Gamma\left(\frac{q}{q-1}-3N\right)}{(q-1)^{3N}\,\Gamma\left(\frac{q}{q-1}\right)}\left[1+(q-1)\frac{p_T\cosh y-\mu(N+1)}{T}\right]^{\frac{q}{1-q}+3N}}$$

✓ **Zeroth term approximation** $(N_0 = 0)$

$$\boxed{\frac{d^2N}{dp_T dy} = \frac{gVp_T^2\cosh y}{(2\pi)^2}\left[1+(q-1)\frac{p_T\cosh y-\mu}{T}\right]^{\frac{q}{1-q}}}$$

✓ **Phenomenological Tsallis distribution:**

$$\boxed{\frac{d^2N}{dp_T dy} = \frac{gVp_T^2\cosh y}{(2\pi)^2}\left[1+(q-1)\frac{p_T\cosh y-\mu}{T}\right]^{\frac{q}{1-q}}}$$

- The ultrarelativistic transverse momentum distribution of Tsallis-2 statistics in the zeroth term approximation $(N_0 = 0)$ exactly coincides with the **phenomenological Tsallis distribution** introduced in [J. Cleymans, D. Worku, Eur. Phys. J. A 48 (2012) 160] and largely used in high-enrgy physics .

- Thus the well-known **phenomenological Tsallis distribution** corresponds to the Tsallis unnormalized (Tsallis-2) statistics for which the statistical averages are badly normalized.

Ultrarelativistic Transverse Momentum Distribution: Tsallis-1 Statistics

Ultrarelativistic Maxwell-Boltzmann Ideal Gas in the Grand Canonical Ensemble $\left(m=0,\ \ q<1\right)$

✓ **Mean occupation numbers:**

$$\langle n_{\vec{p}\sigma}\rangle = \sum_{\{n_{\vec{p}\sigma}\}} n_{\vec{p}\sigma} \frac{1}{\prod_{\vec{p}\sigma} n_{\vec{p}\sigma}!}\left[1+\frac{q-1}{q}\frac{\Lambda-\sum_{\vec{p}\sigma} n_{\vec{p}\sigma}(\varepsilon_{\vec{p}}-\mu)}{T}\right]^{\frac{1}{q-1}} = \sum_{N=0}^{N_0}\frac{\tilde{\omega}^N}{N!}h_0(0)\left[1+\frac{q-1}{q}\frac{\Lambda-\varepsilon_{\vec{p}}+\mu(N+1)}{T}\right]^{\frac{1}{q-1}+3N}$$

✓ **Norm function:**

A.S.P., Eur. Phys. J. A 53 (2017) 53;
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$$1 = \sum_{\{n_{\vec{p}\sigma}\}} \frac{1}{\prod_{\vec{p}\sigma} n_{\vec{p}\sigma}!}\left[1+\frac{q-1}{q}\frac{\Lambda-\sum_{\vec{p}\sigma} n_{\vec{p}\sigma}(\varepsilon_{\vec{p}}-\mu)}{T}\right]^{\frac{1}{q-1}} = \sum_{N=0}^{N_0}\frac{\tilde{\omega}^N}{N!}h_0(0)\left[1+\frac{q-1}{q}\frac{\Lambda+\mu N}{T}\right]^{\frac{1}{q-1}+3N}$$

where

$$h_\eta(\xi) = \frac{\left(\frac{q}{1-q}\right)^{3(N+\eta)}\Gamma\left(\frac{1}{1-q}-\xi-3(N+\eta)\right)}{\Gamma\left(\frac{1}{1-q}-\xi\right)}, \qquad \tilde{\omega} = \frac{gVT^3}{\pi^2}$$

- Expectation values of Tsallis-1 statistics are well normalized
- N_0 is a cut-off parameter
- Terms with $N > N_0$ increase and became divergent for *q<1*

Ultrarelativistic Transverse Momentum Distribution: Tsallis-1 Statistics (continuation)

A.S.P., Eur. Phys. J. A 53 (2017) 53;
Eur. Phys. J. A 52 (2016) 355

✓ **Ultrarelativistic transverse momentum distribution (Exact Solution):**

$$(m=0,\quad q<1)$$

$$\frac{d^2N}{dp_T dy}=\frac{gV}{(2\pi)^2}p_T^2\cosh y\sum_{N=0}^{N_0}\frac{\tilde{\omega}^N}{N!}\frac{\Gamma\left(\frac{1}{1-q}-3N\right)}{\left(\frac{1-q}{q}\right)^{3N}\Gamma\left(\frac{1}{1-q}\right)}\left[1+\frac{q-1}{q}\frac{\Lambda-p_T\cosh y+\mu(N+1)}{T}\right]^{\frac{1}{q-1}+3N}$$

✓ **Zeroth term approximation** $(N_0=0)$

$$\frac{d^2N}{dp_T dy}=\frac{gVp_T^2\cosh y}{(2\pi)^2}\left[1-\frac{q-1}{q}\frac{p_T\cosh y-\mu}{T}\right]^{\frac{1}{q-1}}$$

$$\Updownarrow \quad q\to 1/q$$

✓ **Phenomenological Tsallis distribution:**

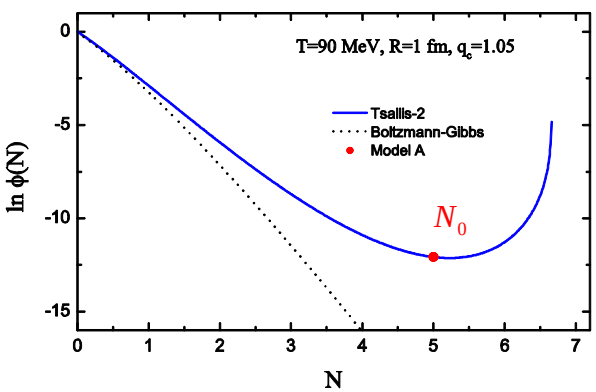
$$\frac{d^2N}{dp_T dy}=\frac{gVp_T^2\cosh y}{(2\pi)^2}\left[1+(q-1)\frac{p_T\cosh y-\mu}{T}\right]^{\frac{q}{1-q}}$$

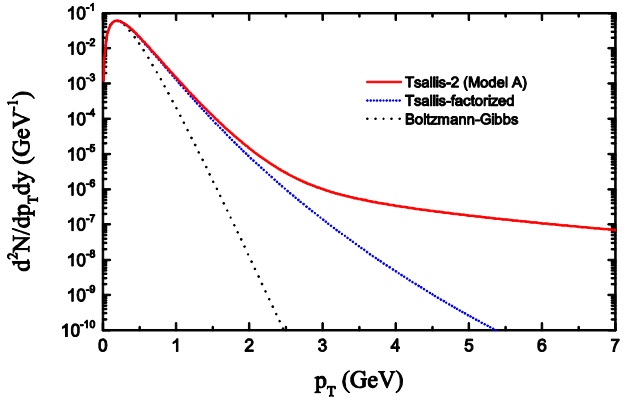
- The ultrarelativistic transverse momentum distribution of the Tsallis-1 statistics in the zeroth term approximation $(N_0=0)$ exactly recovers the ultrarelativistic transverse momentum distribution of the Tsallis-2 statistics in the zeroth term approximation under the transformation $(q\to 1/q)$
- Thus the ultrarelativistic phenomenological Tsallis distribution is obtained from the zeroth term approximation of the Tsallis-1 statistics by transformation $(q\to 1/q)$

The cut-off parameter of the Tsallis-2 statistics: Ultrarelativistic case

A.S.P., Eur. Phys. J. Web Conf. 138 (2017) 03008

✓ **Model A (from the minimum):** $\min(\ln\phi(N))\,|_{N=N_0}$



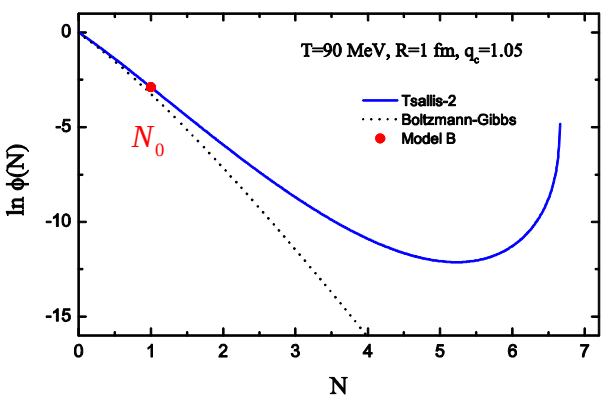


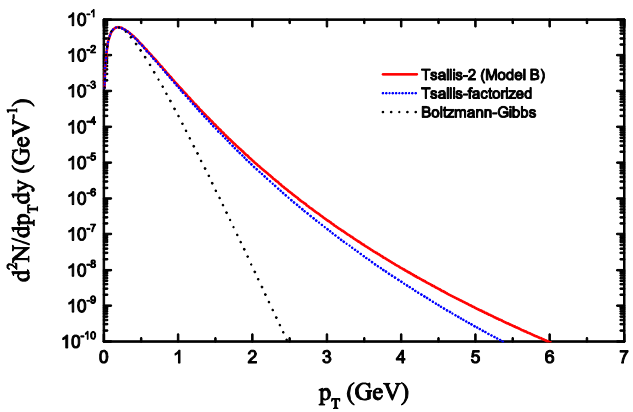
- Tsallis-2 statistics:

$$Z=\sum_{N=0}^{N_0}\phi(N),$$

$$\phi(N)=\frac{\tilde{\omega}^N}{N!}\frac{\Gamma\left(\frac{1}{q-1}-3N\right)}{(q-1)^{3N}\,\Gamma\left(\frac{1}{q-1}\right)}\times\left[1-(q-1)\frac{\mu N}{T}\right]^{\frac{1}{1-q}+3N}$$

✓ **Model B (from the inflection point):** $$\left.\frac{\partial^2\ln\phi(N)}{\partial N^2}\right|_{N=N_0}=0$$





- Boltzmann-Gibbs statistics:

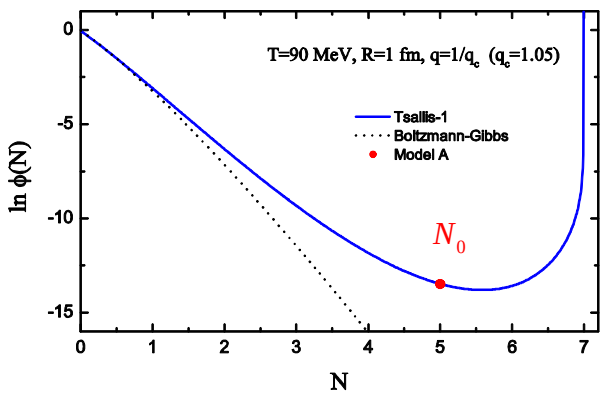
$$Z=\sum_{N=0}^{\infty}\phi(N),$$

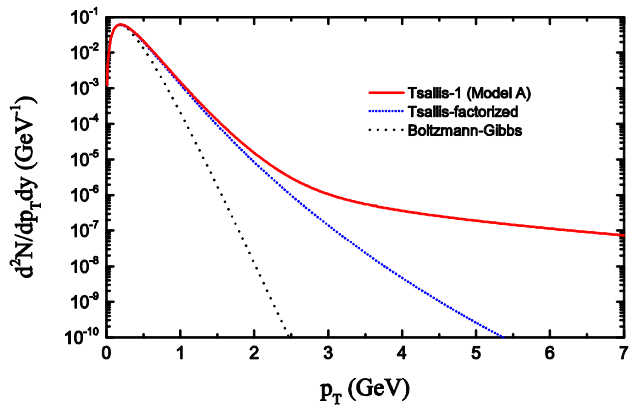
$$\phi(N)=\frac{\tilde{\omega}^N}{N!}e^{\frac{\mu N}{T}}$$

The cut-off parameter of the Tsallis-1 statistics : Ultrarelativistic case

A.S.P., Eur. Phys. J. Web Conf. 138 (2017) 03008

✓ **Model A (from the minimum):** $\min(\ln\phi(N))\,|_{N=N_0}$





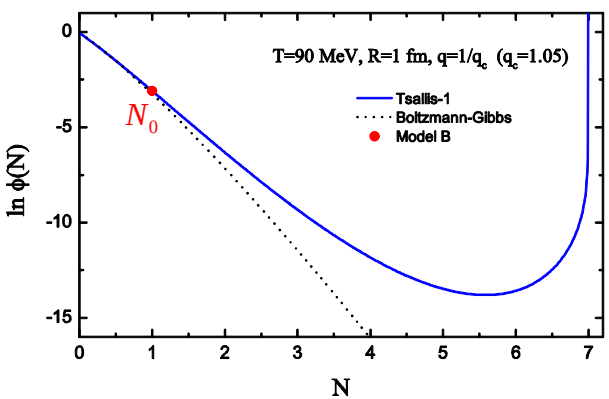
- Tsallis-1 statistics:

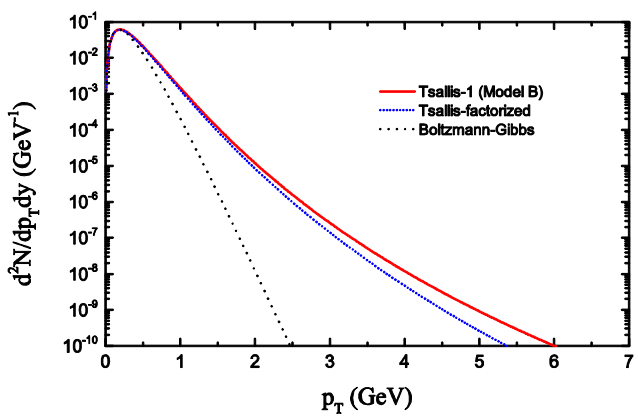
$$\sum_{N=0}^{N_0}\phi(N)=1$$

$$\phi(N)=\frac{\tilde{\omega}^N}{N!}\frac{\left(\frac{q}{1-q}\right)^{3N}\Gamma\left(\frac{1}{1-q}-3N\right)}{\Gamma\left(\frac{1}{1-q}\right)}\times\left[1+\frac{q-1}{q}\frac{\Lambda+\mu N}{T}\right]^{\frac{1}{q-1}+3N}$$

✓ **Model B (from the inflection point):**

$$\left.\frac{\partial^2\ln\phi(N)}{\partial N^2}\right|_{N=N_0}=0$$





- Boltzmann-Gibbs statistics:

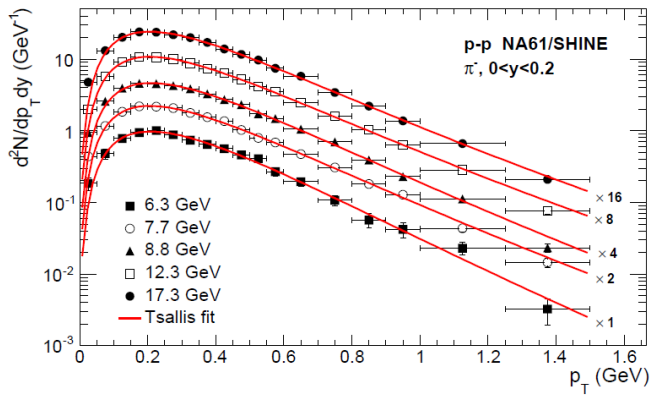
$$\sum_{N=0}^{N_0}\phi(N)=1$$

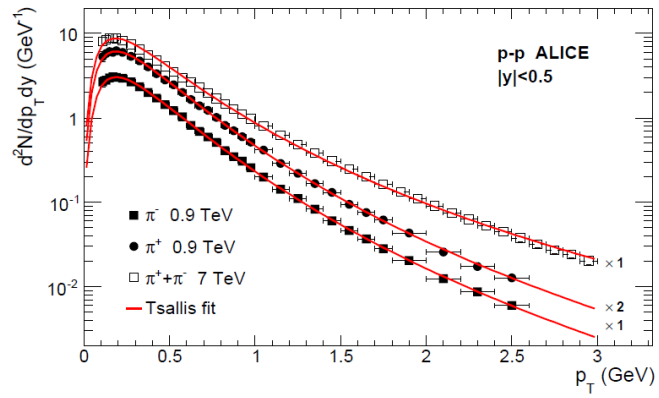
$$\phi(N)=\frac{\tilde{\omega}^N}{N!}e^{\frac{\Omega+\mu N}{T}}$$

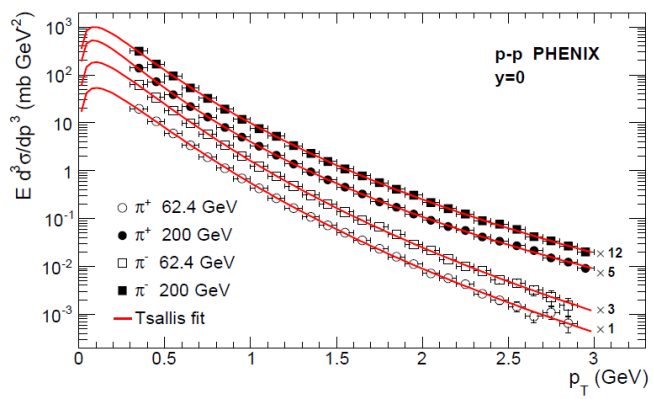
Model B of Tsallis-1 statistics and phenomenological Tsallis distribution: Ultrarelativistic case

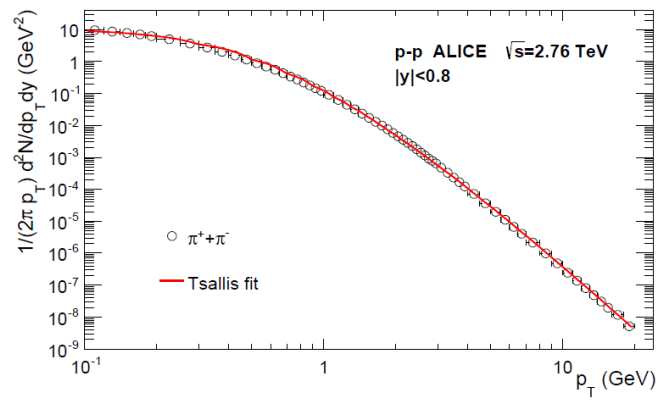
A.S.P., Eur. Phys. J. A 52 (2016) 355

Charged pions in *pp* collisions:









- Transverse momentum distributions of charged pions produced in *pp* collisions at SPS, RHIC and LHC energies
- The yields were integrated in the experimental rapidity interval $y_0 \le y \le y_1$
- The solid curves are the fits of the experimental data to the **ultrarelativistic (*m*=0) transverse momentum distributions** of
  1.) Tsallis-1 statistics
  2.) Tsallis-2 statistics
  3.) Phenomenological Tsallis distribution
- The curves are the same for all statistics but only the parameters are different.

Experimental Data:

NA61/SHINE, EPJC 74 (2014) 2794; PHENIX, PRC 83 (2011) 064903
ALICE, EPJC 71 (2011) 1655; ALICE, EPJC 75 (2015) 226;
ALICE, PLB 736 (2014) 196

Temperature for Model B of Tsallis-1 statistics and phenomenological Tsallis distribution: Ultrarelativistic case

Charged pions in *pp* collisions:

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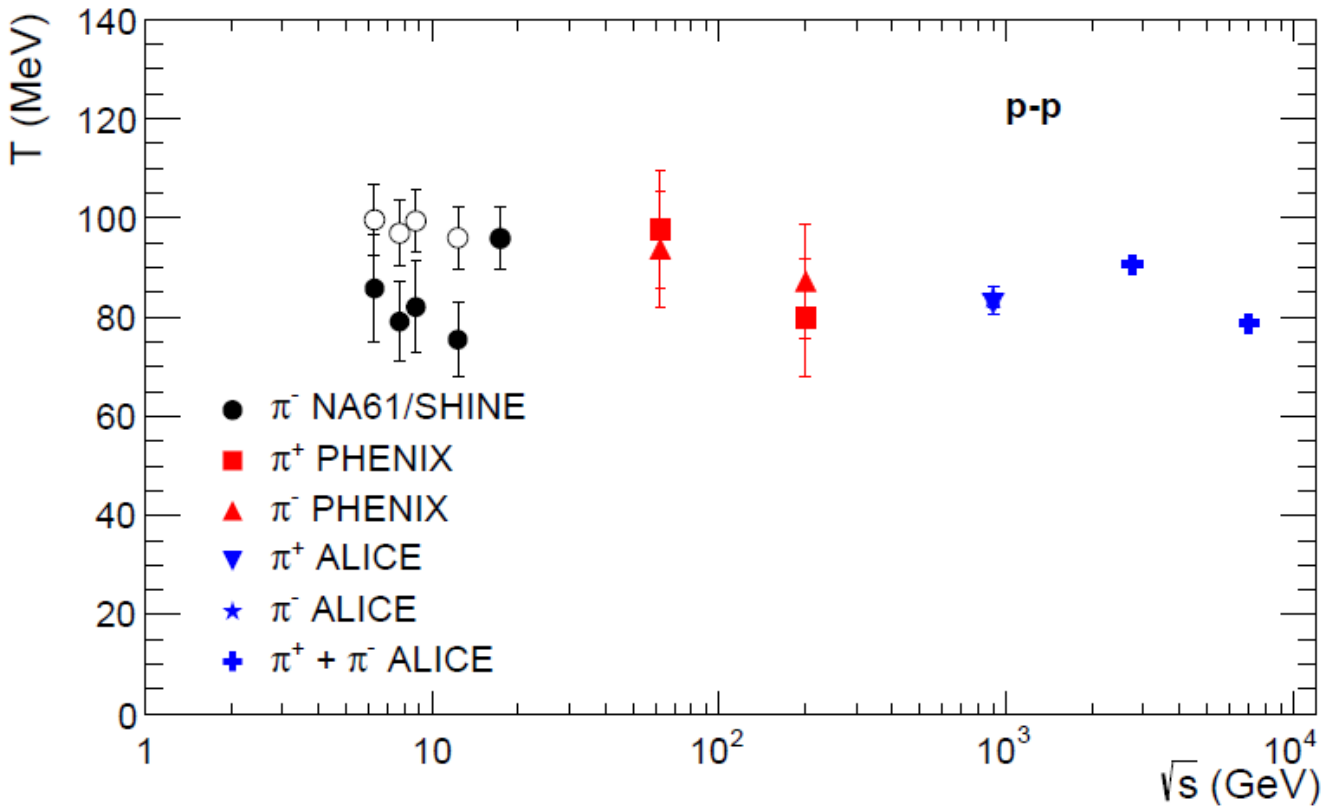

- ✓ Solid points are the results of the fit by Model B of Tsallis statistics
- ✓ Open symbols are the results of the fit by the phenomenological Tsallis distribution

Radius for Model B of Tsallis-1 statistics and phenomenological Tsallis distribution: Ultrarelativistic case

Charged pions in *pp* collisions:

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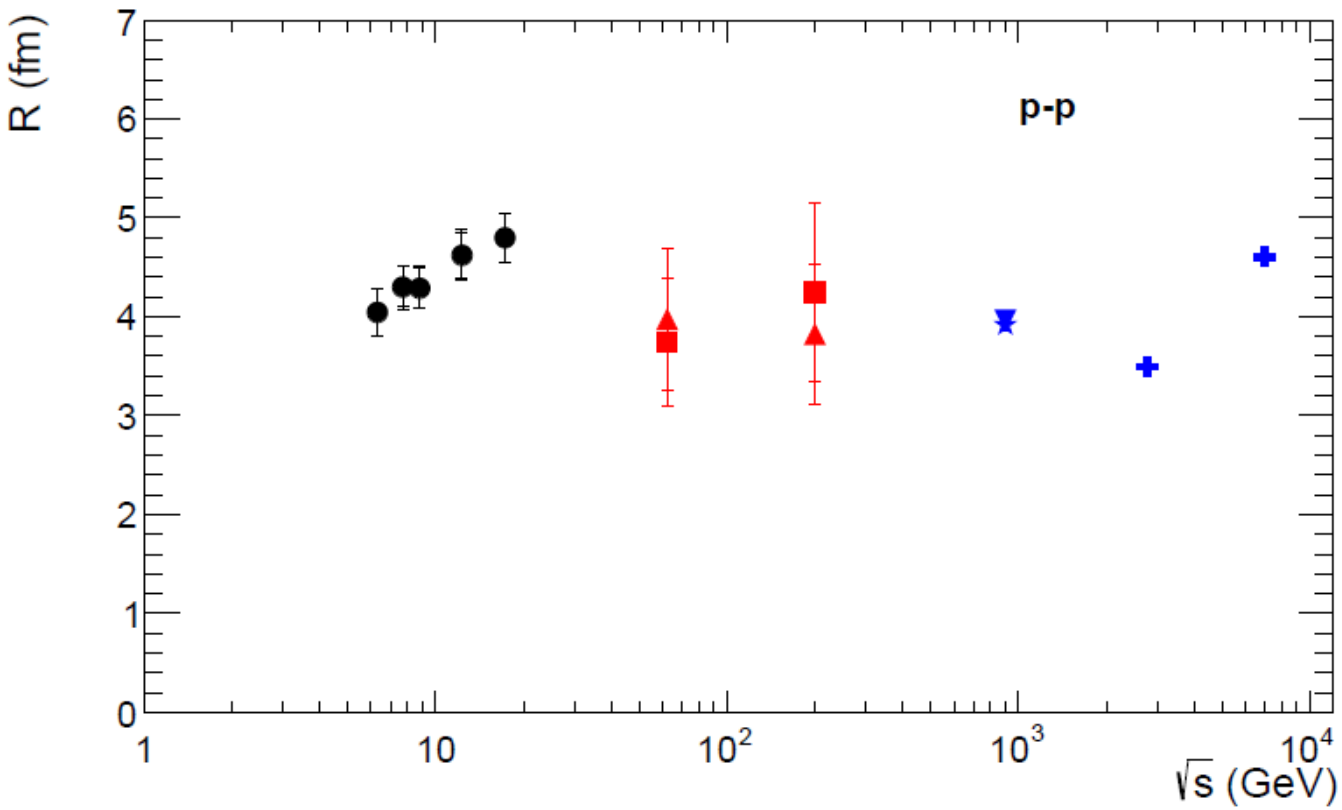
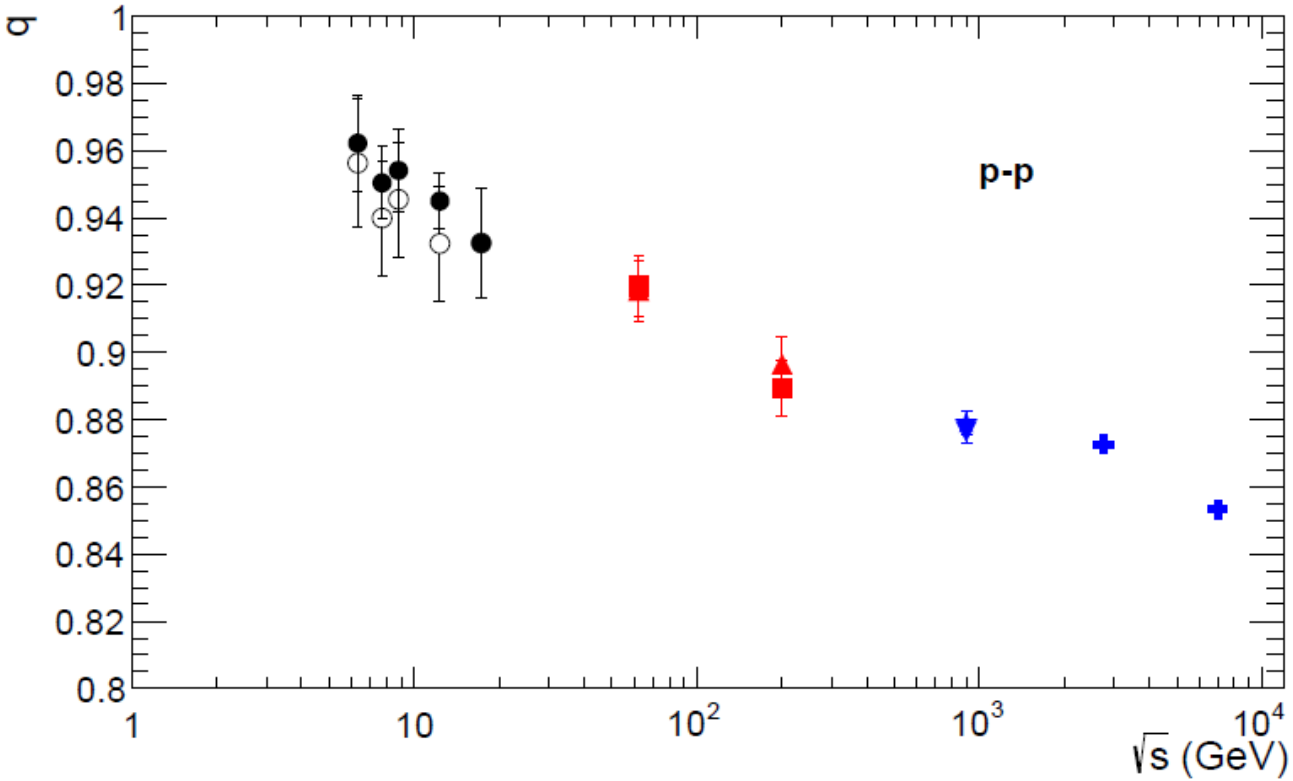


- ✓ Solid points are the results of the fit by Model B of Tsallis statistics
- ✓ Open symbols are the results of the fit by the phenomenological Tsallis distribution

Entropic parameter for Model B of Tsallis-1 statistics and phenomenological Tsallis distribution: Ultrarelativistic case

Charged pions in *pp* collisions:

A.S.P., Eur. Phys. J. A 52 (2016) 355



- ✓ Solid points are the results of the fit by Model B of Tsallis statistics
- ✓ Open symbols are the results of the fit by the phenomenological Tsallis distribution

Transverse Momentum Distribution: Tsallis-1 Statistics (Massive Particles)

✓ **Transverse momentum distribution (exact results):**

A.S.P., T. Bhattacharyya, arXiv:1903.06118 [nucl-th]

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{1}{n!\Gamma\left(\frac{1}{1-q}\right)} \int_0^{\infty} t^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]} \frac{\left(-\beta'\Omega_G(\beta')\right)^n}{e^{\beta'(m_T \cosh y-\mu)}+\eta} dt \quad \text{for} \quad q<1$$

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{q}{q-1}\right)}{n!} \frac{i}{2\pi} \oint_C (-t)^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]} \frac{\left(-\beta'\Omega_G(\beta')\right)^n}{e^{\beta'(m_T \cosh y-\mu)}+\eta} dt \quad \text{for} \quad q>1,$$

where $\varepsilon_{\mathbf{p}} = m_T \cosh y, \quad m_T = \sqrt{p_T^2 + m^2}$

$\eta = 1$ (Fermi-Dirac)

$\eta = -1$ (Bose-Einstein)

$\eta = 0$ (Maxwell-Boltzmann)

✓ **Norm function Λ:**

$$\sum_{n=0}^{\infty} \frac{1}{n!\Gamma\left(\frac{1}{1-q}\right)} \int_0^{\infty} t^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]} \left(-\beta'\Omega_G(\beta')\right)^n dt = 1 \quad \text{for} \quad q<1$$

$$\sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{q}{q-1}\right)}{n!} \frac{i}{2\pi} \oint_C dt (-t)^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]} \left(-\beta'\Omega_G(\beta')\right)^n = 1 \quad \text{for} \quad q>1,$$

Λ

where

$$-\beta'\Omega_G(\beta') = \sum_{\mathbf{p},\sigma} \ln\left[1+\eta e^{-\beta'(\varepsilon_{\mathbf{p}}-\mu)}\right]^{\frac{1}{\eta}}, \quad \beta' = \frac{t(1-q)}{qT}, \quad \varepsilon_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2},$$

Maxwell-Boltzmann Transverse Momentum Distribution: Tsallis-1 Statistics (Massive Particles)

✓ **Transverse momentum distribution (exact results):**

A.S.P., T. Bhattacharyya, arXiv:1903.06118 [nucl-th]

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{\omega^n}{n!} \frac{1}{\Gamma\left(\frac{1}{1-q}\right)} \int_0^{\infty} t^{\frac{q}{1-q}-n} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda - m_T \cosh y + \mu(n+1)}{T}\right]} \left(K_2\left(\frac{t(1-q)m}{qT}\right)\right)^n dt \quad \text{for} \quad q<1$$

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{(-\omega)^n}{n!} \Gamma\left(\frac{q}{q-1}\right) \frac{i}{2\pi} \oint_C (-t)^{\frac{q}{1-q}-n} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda - m_T \cosh y + \mu(n+1)}{T}\right]} \left(K_2\left(\frac{t(1-q)m}{qT}\right)\right)^n dt \text{ for } q>1$$

where $\varepsilon_{\mathbf{p}} = m_T \cosh y, \quad m_T = \sqrt{p_T^2 + m^2}$

✓ **Zeroth term approximation** $(n=0)$**:**

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \left[1 + \frac{1-q}{q}\frac{m_T \cosh y - \mu}{T}\right]^{\frac{1}{q-1}}$$

$q \to 1/q$

✓ **Phenomenological Tsallis distribution:**

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \left[1 + (q-1)\frac{m_T \cosh y - \mu}{T}\right]^{\frac{q}{1-q}}$$

Phenomenological Tsallis distribution corresponds to the transverse momentum distribution in the zeroth term approximation of the Tsallis-1 statistics under the transformation $q \to 1/q$

Transverse Momentum Distribution: Tsallis-2 Statistics (Massive Particles)

✓ **Transverse momentum distribution (exact results):**

A.S.P., T. Bhattacharyya, arXiv:1903.06118 [nucl-th]

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{1}{n! Z^q \Gamma\left(\frac{q}{q-1}\right)} \int_0^{\infty} t^{\frac{1}{q-1}} e^{-t} \frac{\left(-\beta' \Omega_G(\beta')\right)^n}{e^{\beta'(m_T \cosh y - \mu)} + \eta} dt \quad \text{for} \quad q > 1$$

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{1-q}\right)}{n! Z^q} \frac{i}{2\pi} \oint_C (-t)^{\frac{1}{q-1}} e^{-t} \frac{\left(-\beta' \Omega_G(\beta')\right)^n}{e^{\beta'(m_T \cosh y - \mu)} + \eta} dt \quad \text{for} \quad q < 1,$$

where $\varepsilon_{\mathbf{p}} = m_T \cosh y, \quad m_T = \sqrt{p_T^2 + m^2}$

$\eta = 1$ (Fermi-Dirac)

$\eta = -1$ (Bose-Einstein)

$\eta = 0$ (Maxwell-Boltzmann)

✓ **Norm function Z:**

$$Z = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma\left(\frac{1}{q-1}\right)} \int_0^{\infty} t^{\frac{1}{q-1}-1} e^{-t} \left(-\beta' \Omega_G(\beta')\right)^n dt \quad \text{for} \quad q > 1$$

$$Z = \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2-q}{1-q}\right)}{n!} \frac{i}{2\pi} \oint_C (-t)^{\frac{1}{q-1}-1} e^{-t} \left(-\beta' \Omega_G(\beta')\right)^n dt \quad \text{for} \quad q < 1,$$

$\longrightarrow$ Z

where

$$-\beta' \Omega_G(\beta') = \sum_{\mathbf{p},\sigma} \ln\left[1 + \eta e^{-\beta'(\varepsilon_{\mathbf{p}} - \mu)}\right]^{\frac{1}{\eta}}, \quad \beta' = \frac{t(q-1)}{T}, \quad \varepsilon_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2},$$

Maxwell-Boltzmann Transverse Momentum Distribution: Tsallis-2 Statistics (Massive Particles)

✓ **Transverse momentum distribution (exact results):**

A.S.P., T. Bhattacharyya, arXiv:1903.06118 [nucl-th]

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{\omega^n}{n!} \frac{1}{Z^q} \frac{1}{\Gamma\left(\frac{q}{q-1}\right)} \int_0^{\infty} t^{\frac{1}{q-1}-n} e^{-t\left[1-(1-q)\frac{m_T \cosh y - \mu(n+1)}{T}\right]} \left(K_2\left(\frac{t(q-1)m}{T}\right)\right)^n dt \quad \text{for} \quad q>1$$

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{(-\omega)^n}{n!} \frac{1}{Z^q} \Gamma\left(\frac{1}{1-q}\right) \frac{i}{2\pi} \oint_C (-t)^{\frac{1}{q-1}-n} e^{-t\left[1-(1-q)\frac{m_T \cosh y - \mu(n+1)}{T}\right]} \left(K_2\left(\frac{t(q-1)m}{T}\right)\right)^n dt \quad \text{for} \quad q<1$$

where $\varepsilon_{\mathbf{p}} = m_T \cosh y, \quad m_T = \sqrt{p_T^2 + m^2}$

✓ **Zeroth term approximation** $(n=0)$**:**

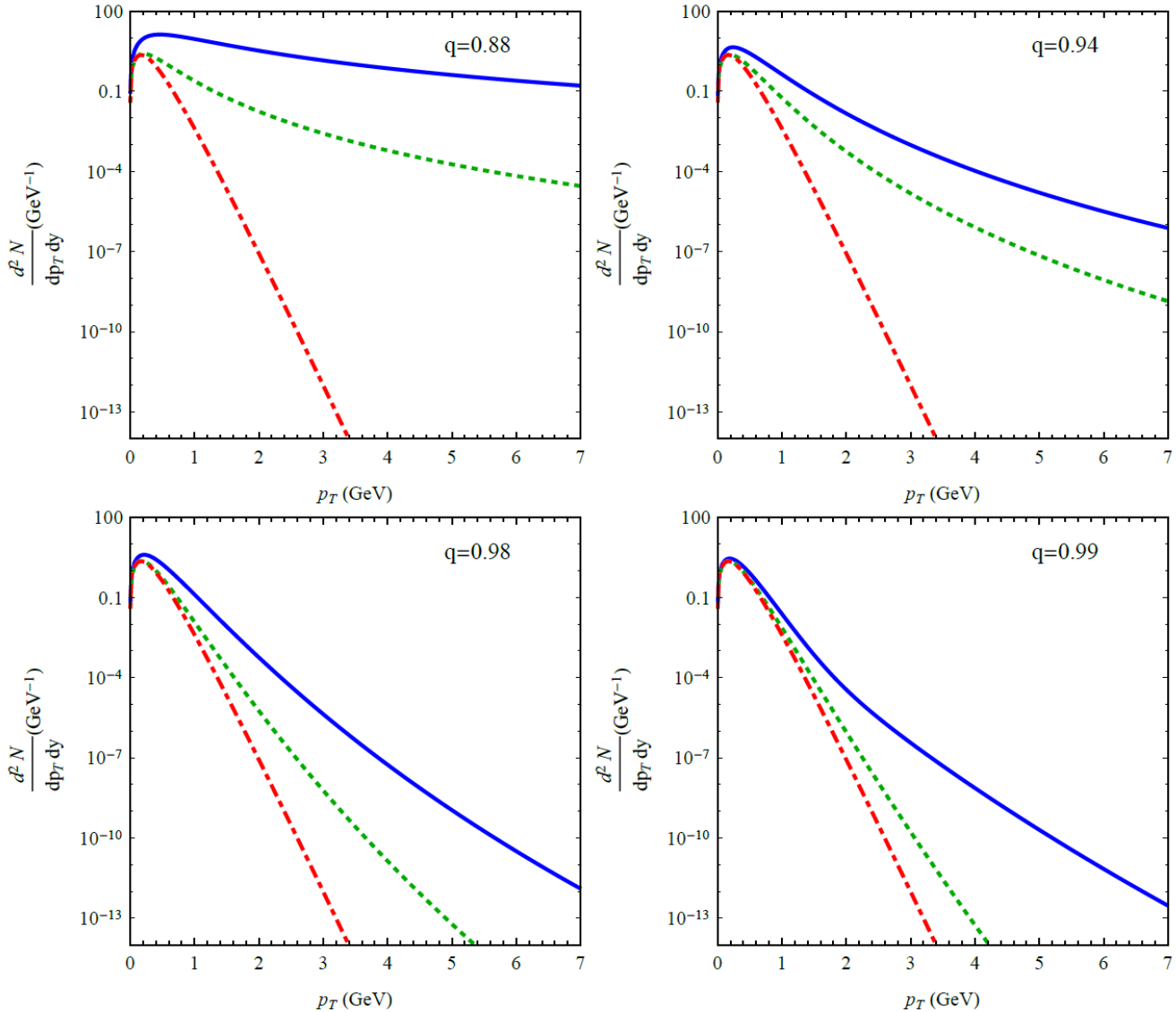
$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \left[1+(q-1)\frac{m_T \cosh y - \mu}{T}\right]^{\frac{q}{1-q}}$$

✓ **Phenomenological Tsallis distribution:**

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \left[1+(q-1)\frac{m_T \cosh y - \mu}{T}\right]^{\frac{q}{1-q}}$$

- **Phenomenological Tsallis distribution** is equivalent to the transverse momentum distribution in the zeroth term approximation of the Tsallis unnormalized (Tsallis-2) statistics.
- Zeroth term approximation of the Tsallis-1 statistics corresponds to zeroth term approximation of the Tsallis-2 statistics under the transformation $q \to 1/q$

Maxwell-Boltzmann Transverse Momentum Distribution: Model A of Tsallis-1 Statistics (Massive Particles)



A.S.P., T. Bhattacharyya, arXiv:1903.06118 [nucl-th]

FIG. 2. (Color online) The spectra of the Maxwell-Boltzmann massive particles in the Tsallis-1 statistics at mid-rapidity ($y = 0$) for different values of the entropic parameter q. Temperature $T = 82$ MeV, chemical potential $\mu = 0$, radius $R = 4$ fm and mass $m = 139.57$ MeV (pion mass). The solid, doted and dot-dashed lines correspond to the exact Tsallis-1 statistics, zeroth-term approximation and the Boltzmann-Gibbs statistics ($q = 1$), respectively.

Maxwell-Boltzmann Transverse Momentum Distribution: Model A of Tsallis-2 Statistics (Massive Particles)

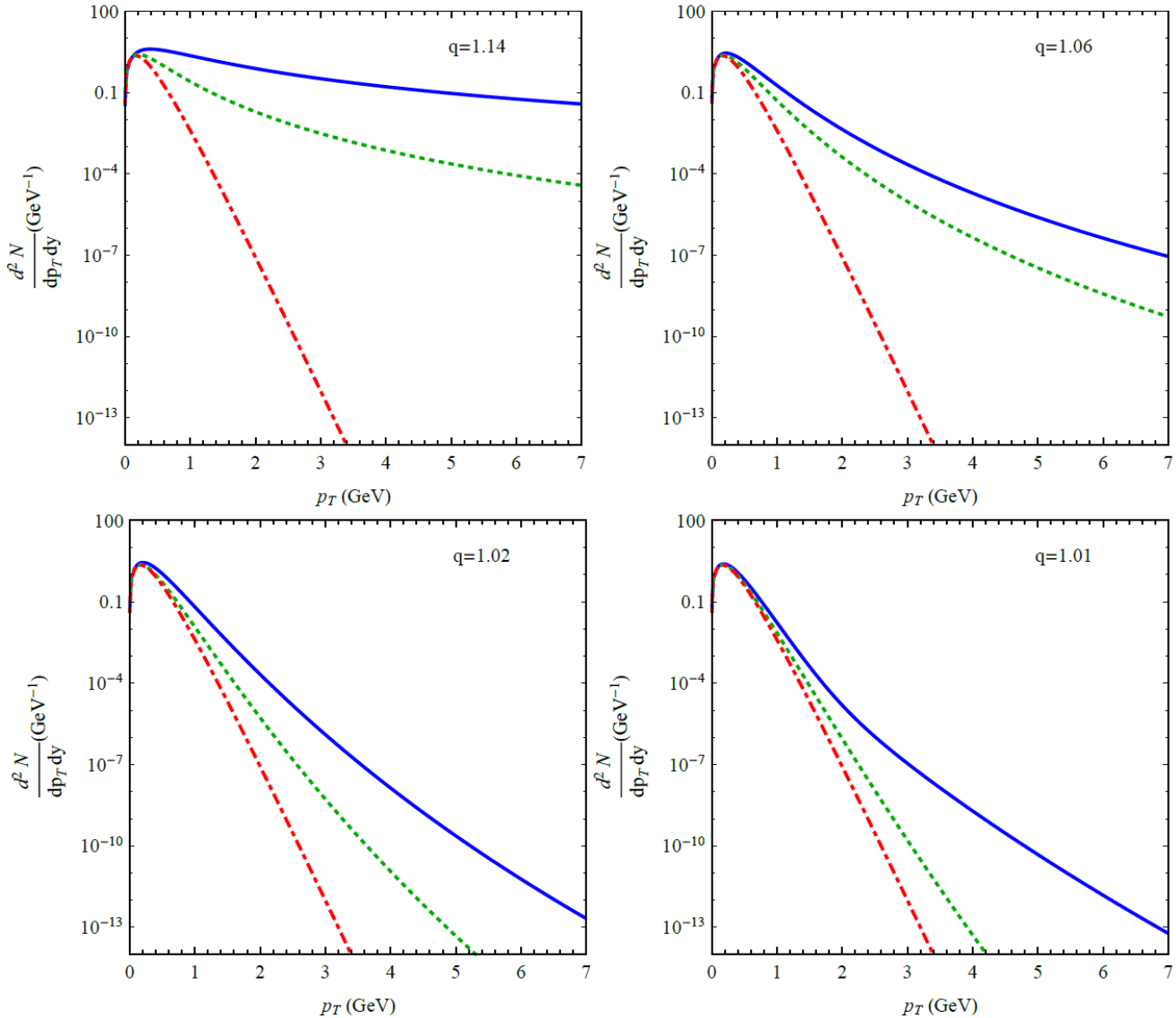


FIG. 3. (Color online) The spectra of the Maxwell-Boltzmann massive particles in the Tsallis-2 statistics at mid-rapidity ($y = 0$) for different values of the entropic parameter q. Temperature $T = 82$ MeV, chemical potential $\mu = 0$, radius $R = 4$ fm and mass $m = 139.57$ MeV (pion mass). The solid, dotted and dot-dashed lines correspond to the exact Tsallis-2 statistics, zeroth-term approximation and the Boltzmann-Gibbs statistics ($q = 1$), respectively.

A.S.P., T. Bhattacharyya, arXiv:1903.06118 [nucl-th]

Conclusions

1. We have obtained that the Tsallis statistics (Tsallis-1 statistics at $q<1$ and Tsallis-2 statistics at $q>1$) is divergent.
2. It is convergent only in the case of $q=1$ which corresponds to the standard Boltzmann-Gibbs statistics.
3. However, we have found that a few terms in a series expansion of quantities in the Tsallis statistics at $q \neq 1$ are convergent and they describe very well the experimental data on the transverse momentum distributions (TMD) of hadrons in the *pp* collisions at high energies (the standard Boltzmann-Gibbs statistics fails to describe these experimental data).
4. The analytical exact expressions for the TMD (ultrarelativistic and massive) of the Tsallis-1 and Tsallis-2 statistics were obtained.
5. We have demonstrated that the phenomenological Tsallis distribution is equivalent to the TMD of the Tsallis unnormalized (Tsallis-2) statistics in the zeroth term approximation (the statistical averages of the Tsallis-2 statistics are not consistent with norm equation of probabilities).
6. We have demonstrated that the phenomenological Tsallis distribution recovers the TMD of the Tsallis-1 statistics in the zeroth term approximation under the transformation of the parameter q to $1/q$.
7. We have found that the TMD of the Model B of the Tsallis statistics (the cut-off from the inflection point) differs from the phenomenological Tsallis distribution only at low energies of NICA and NA61/SHINE.
8. We have shown that the TMD of the Model A of the Tsallis statistics (the cut-off from the minimum point) differs essentially from the phenomenological Tsallis distribution.

Thank you for your attention!