The ring of weak Jacobi forms for D_8 root system

Dmitry Adler

2 February 2018

Weak Jacobi forms

Weak Jacobi form of weight k and index m (k, $m \in \mathbb{Z}$) for a root lattice L is a holomorphic function $\varphi : \mathcal{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}$, such that:

1) $\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{\mathfrak{z}}{c\tau+d}\right) = (c\tau+d)^k e^{\pi i m \frac{c(\mathfrak{z},\mathfrak{z})}{c\tau+d}} \varphi(\tau, \mathfrak{z})$;

2) $\varphi(\tau, \mathfrak{z} + l\tau + l') = e^{-2\pi i m(l,\mathfrak{z}) - \pi i m(l,l)\tau} \varphi(\tau, \mathfrak{z})$ for $l, l' \in L$;

3) $\varphi(\tau, w(\mathfrak{z})) = \varphi(\tau, \mathfrak{z})$ for w from Weyl group;

4) $\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{\mathfrak{z}}{c\tau+d}\right)$ has a Fourier development:

$$\sum_{\lambda \in L^*, n \geqslant 0} a(n, l) q^n \zeta^\lambda,$$

where $q = e^{2\pi i \tau}$, $\zeta^\lambda = e^{2\pi i (\mathfrak{z}, \lambda)}$.

The main results

Theorem. The ring of weak Jacobi forms for root lattice D_n with $n \leqslant 8$ is a free algebra over the ring of modular forms with $n+1$ generators. These generators can be written in an explicit form.

Remark. This theorem is also true in the case $n > 8$, but construction of some generators is much more difficult.

Remark. The lattice D_n is a set $\{(x_1, \ldots, x_n) \in \mathbb{Z}^n \,|\, x_1 + \ldots + x_n \equiv 0 \mod 2\}$.

Generators in case D_8

We can predict the following list of generators:

$$\varphi_{0,1}, \varphi_{-2,1}, \varphi_{-4,1}, \varphi_{-8,1},$$

$$\varphi_{-6,2}, \varphi_{-8,2}, \varphi_{-10,2}, \varphi_{-12,2}, \varphi_{-14,2}$$

Generators in case D_8

We can predict the following list of generators:

$$\varphi_{0,1}, \varphi_{-2,1}, \varphi_{-4,1}, \varphi_{-8,1},$$

$$\varphi_{-6,2}, \varphi_{-8,2}, \varphi_{-10,2}, \varphi_{-12,2}, \varphi_{-14,2}$$

Construction of $\varphi_{-4,1}$

For any lattice we can consider

$$\vartheta_L(\tau, Z) = \sum_{l \in L} q^{\frac{(l,l)}{2}} e^{2\pi i (l,Z)}.$$

Let us consider theta-functions for lattices $E_8 = \langle D_8, \frac{e_1+\ldots+e_8}{2} \rangle$ and $D_{16}^{+} = \langle D_{16}, \frac{e_1+\ldots+e_{16}}{2} \rangle$.

Construction of $\varphi_{-4,1}$

One can compute that

$$E_4(\tau) \cdot \vartheta_{E_8}(\tau, Z) - \vartheta_{D_{16}^+}\Big|_{D_8}(\tau, Z) =$$

$$= \left(128 - 16\sum_{j=1}^{8}\zeta_j - 16\sum_{j=1}^{8}\zeta_j^{-1} + \sum \zeta_1^{\pm\frac{1}{2}} \dots \zeta_8^{\pm\frac{1}{2}}\right) q + q^2(\dots),$$

where $\zeta_j = e^{2\pi i z_j}$.

Construction of $\varphi_{-4,1}$

The form

$$\varphi_{-4,1}(\tau, Z) = \frac{E_4(\tau) \cdot \vartheta_{E_8}(\tau, Z) - \vartheta_{D_{16}^+}\Big|_{D_8}(\tau, Z)}{\Delta(\tau)} =$$

$$= 128 - 16\sum_{j=1}^{8} \zeta_j - 16\sum_{j=1}^{8} \zeta_j^{-1} + \sum \zeta_1^{\pm\frac{1}{2}} \ldots \zeta_8^{\pm\frac{1}{2}} + q(\ldots).$$

is a weak Jacobi form of weight -4 and index 1 for D_8.

Construction of $\varphi_{-2,1}$

The differential operator

$$H_k = \frac{1}{2\pi i}\frac{\partial}{\partial \tau} + \frac{1}{8\pi^2}\left(\frac{\partial}{\partial \mathfrak{z}}, \frac{\partial}{\partial \mathfrak{z}}\right) + (2k-n)G_2(\tau)\times$$

is well-defined on the space of (weak) Jacobi form.

More precisely,

$$H_k : J_{k,m}^{(weak)} \to J_{k+2,m}^{(weak)}.$$

Construction of $\varphi_{-2,1}$

In our case $k = -4$, $n = 8$, and

$$\left(\frac{\partial}{\partial \mathfrak{z}}, \frac{\partial}{\partial \mathfrak{z}}\right) = \sum_{j=1}^{8} \frac{\partial^2}{\partial z_j^2}.$$

We get

$$H_{-4}(\varphi_{-4,1}) = \frac{1}{3}\left(256 - 8\sum_{j=1}^{8}\zeta_j - 8\sum_{j=1}^{8}\zeta_j^{-1} - \sum \zeta_1^{\pm\frac{1}{2}} \ldots \zeta_8^{\pm\frac{1}{2}}\right) +$$

$$+q(\ldots).$$

Generators for D_n with $n < 8$

Using generators for the case D_8 and assuming $z_8 = 0$, we obtain generators for the case D_7.

Assuming after that $z_7 = 0$, we obtain generators for the case D_6, and so on.

Thank you for your attention!