Elliptic polylogarithms, Eichler integrals and Feynman integrals

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IPhT Saclay &

Partition Functions And Automorphic Forms
Dubna

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- Hodge structure

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▶ Elliptic polylogarithm

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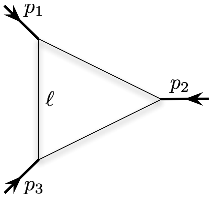
- J. Broedel, C. Duhr, F. Dulat, L. Tancredi, “Elliptic polylogarithms and iterated integrals on elliptic curves I: general formalism”, [arXiv:1712.07089]

A very partial list of references IV

- J. Broedel, C. Duhr, F. Dulat, L. Tancredi, "Elliptic polylogarithms and iterated integrals on elliptic curves II: an application to the sunrise integral", [arXiv:1712.07095]

Part I

Polylogarithms



We consider the 3-mass triangle $p_1 + p_2 + p_3 = 0$ and $p_i^2 \neq 0$

$$I_{\triangleright}(p_1^2, p_2^2, p_3^2) = \int_{\mathbb{R}^{1,3}} \frac{d^4\ell}{\ell^2(\ell + p_1)^2(\ell - p_3)^2}$$

Which can be represented as using $1/p^2 = \int_0^\infty dx\, \exp(-xp^2)$

$$I_{\triangleright} = \int_{\substack{x\geqslant 0\\ y\geqslant 0}} \frac{dxdy}{(p_1^2 x + p_2^2 y + p_3^2)(xy + x + y)}$$

and evaluated as

$$I_{\triangleright} = \frac{D(z)}{\left(p_1^4 + p_2^4 + p_3^4 - (p_1^2 p_2^2 + p_1^2 p_3^2 + p_2^2 p_3^2)\right)^{\frac{1}{2}}}$$

z and $\bar{z}$ roots of $(1-x)(p_3^2 - x p_1^2) + p_2^2 x = 0$

- Single-valued Bloch-Wigner dilogarithm for $z \in \mathbb{C} \backslash \{0, 1\}$

$$D(z) = \Im\mathrm{m}(Li_2(z)) + \arg(1-z)\log|z|$$

- The permutation of the 3 masses: $\{z, 1-\bar{z}, \frac{1}{\bar{z}}, 1-\frac{1}{z}, \frac{1}{1-z}, -\frac{\bar{z}}{1-\bar{z}}\}$ this set is left invariant by the $D(z)$

- The integral has branch cuts arising from the square root since $D(z)$ is analytic

The triangle graph motive [Bloch, Kreimer] I

$$I_{\triangleright} = \int_{\substack{x \geqslant 0 \\ y \geqslant 0}} \frac{dxdy}{(p_1^2 x + p_2^2 y + p_3^2)(xy + x + y)}$$

The integral is defined over the domain $\Delta = \{[x, y, z] \in \mathbb{P}^2, x, y, z \geqslant 0\}$ and the denominator is the quadric

$$C_{\triangleright} := (p_1^2 x + p_2^2 y + p_3^2 z)(xy + xz + yz)$$

Let $L = \{x = 0\} \cup \{y = 0\} \cup \{z = 0\}$ and $D = \{x + y + z = 0\} \cup C_{\triangleright}$

$$\frac{dxdy}{(p_1^2 x + p_2^2 y + p_3^2)(xy + x + y)} \in H := H^2(\mathbb{P}^2 - D, L \backslash (L \cup C_{\triangleright}) \cap L, \mathbb{Q})$$

We need to consider the relative cohomology because the domain Δ is not in $H_2(\mathbb{P}^2 - D)$ because $\partial\Delta \neq \emptyset$

The triangle graph motive [Bloch, Kreimer] II

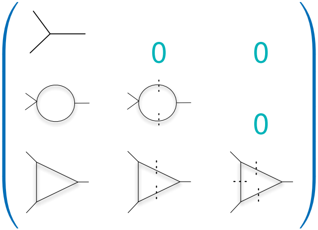
Since $\partial\Delta \cap C_{\triangleright} = \{[1,0,0],[0,1,0],[0,0,1]\}$ one needs to perform a blow-up these 3 points.

One can define a mixed Tate Hodge structure [Bloch, Kreimer] with weight $W_0 H \subset W_2 H \subset W_4 H$ and grading

$$gr_0^W H = \mathbb{Q}(0), \qquad gr_2^W H = \mathbb{Q}(-1)^5, \qquad gr_4^W H = \mathbb{Q}(-2)$$

The Hodge matrix and unitarity

$$\begin{pmatrix} 1 & 0 & 0 \\ -Li_1(z) & 2i\pi & 0 \\ -Li_2(z) & 2i\pi \log z & (2i\pi)^2 \end{pmatrix}$$



The triangle graph motive [Bloch, Kreimer] III

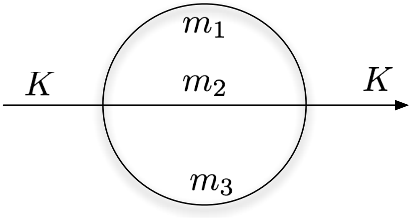
- The construction is valid for all one-loop amplitudes in four dimensions
- It was shown by [Bloch, Esnault, Kreimer; Bloch, Kreimer; Schnetz, Brown] that the 1-loop triangle integral involves only mixed Tate Hodge structure.

Part II

Elliptic polylogarithms and modular forms

The Feynman parametrisation is given by

$$\mathcal{J}^2_\ominus = \int_{\substack{x\geqslant 0\\ y\geqslant 0}} \frac{dxdy}{(m_1^2 x + m_2^2 y + m_3^2)(x+y+xy) - p^2 xy} = \int_\Delta \Omega_\ominus \,.$$



the two-loop sunset integral

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- The sunset integral is the integration of the 2-form ω

$$\Omega_{\ominus} = \frac{z dx \wedge dy + x dy \wedge dz + y dz \wedge dx}{A_{\ominus}(x,y,z)} \in H^2(\mathbb{P}^2 - \mathcal{E}_{p^2})$$

- The graph is based on the elliptic curve $\mathcal{E}_{p^2} : A_{\ominus}(x,y,z) = 0$

$$A_{\ominus}(x,y,z) := (m_1^2 x + m_2^2 y + m_3^2 z)(xz + xy + yz) - p^2 xyz \,.$$

the two-loop sunset integral

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- The domain of integration Δ is

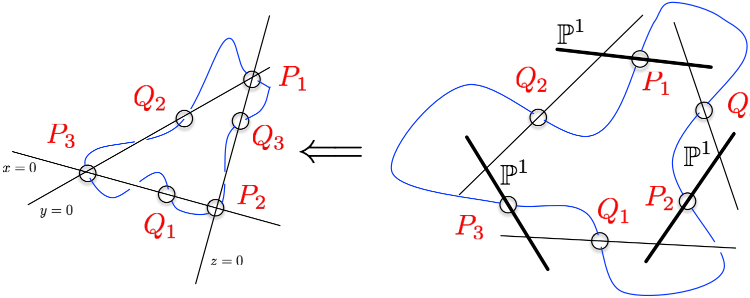
$$\Delta := \{[x:y:z] \in \mathbb{P}^2 | x \geqslant 0, y \geqslant 0, z \geqslant 0\}$$

the sunset graph mixed Hodge structure

- The elliptic curve intersects the domain of integration Δ

$$\Delta \cap \{A_\ominus(x,y,z)=0\} = \{[1:0:0],[0:1:0],[0:0:1]\}$$

- We need to blow-up work in $\mathbb{P}^2 - \mathcal{E}_{p^2}$



the sunset graph mixed Hodge structure

- The domain of integration $\Delta \notin H_2(\mathbb{P}^2 - \mathcal{E}_{p^2})$ because $\partial\Delta \neq \emptyset$: Need to pass to the relative cohomology
- If $P \to \mathbb{P}^2$ is the blow-up and $\hat{\mathcal{E}}_{p^2}$ is the strict transform of $\mathcal{E}_{p^2}$
- Hexagon $\mathfrak{h}^0$ union of strict transform of $\partial\mathcal{D} = \{xyz = 0\}$ and the 3 $\mathbb{P}^1$ divisors
- Then in P we have resolved the two problems $\mathfrak{h} = \mathfrak{h}^0 - (\mathfrak{h}^0 \cap \hat{\mathcal{E}}_{K^2})$

$$\Delta \cap \hat{\mathcal{E}}_{p^2} = \emptyset; \qquad \tilde{\Delta} \in H_2(P - \hat{\mathcal{E}}_{p^2}, \mathfrak{h})) \simeq H^2(P - \hat{\mathcal{E}}_{K^2}, \mathfrak{h})^\vee$$

- The sunset integral is a period of the mixed Hodge structure $H^2_{p^2}$: $\mathcal{I}_\ominus = \langle \Omega_\ominus, \tilde{\Delta} \rangle$

- We have a variation (with respect to p^2) of Hodge structures

$$H^2_{p^2} := H^2(P - \hat{\mathcal{E}}_{p^2}, \mathfrak{h} - (\mathfrak{h} \cap \hat{\mathcal{E}}_{p^2}))$$

The sunset motive

We have the follow (short) sequence

$$H^1(\mathbb{G}_m^2, \mathbb{Q}(2)) \xrightarrow{\alpha} H^1(\mathcal{E}^0_{K^2}, \mathbb{Q}(2)) \to H^2(\mathbb{G}_m^2, \mathcal{E}^0_{K^2}; \mathbb{Q}(2)) \to H^2(\mathbb{G}_m^2, \mathbb{Q}(2)) \to 0.$$

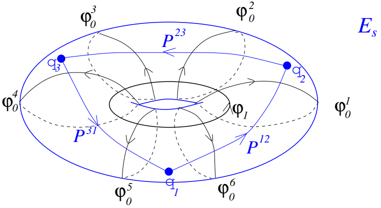
with $\mathcal{E}^0_{K^2} = \mathcal{E}_{K^2} - \{P_1, P_2, P_3, Q_1, Q_2, Q_3\}$ and $\mathbb{P}^2 - \mathfrak{h} = \mathbb{G}_m^2$

- Since $\mathrm{Image}(\alpha) = \mathrm{span}\langle d\log(X/Z), d\log(Y/Z)\rangle$
- Introducing the regulator $\mathcal{L}_2\left\{\frac{X}{Z}, \frac{Y}{Z}\right\} = F(P_3) + F(Q_2) - F(P_2) - F(Q_3)$

$$F(x) = -\int_{x_0}^{x} \log\left(\frac{X}{Z}(y)\right) d\log y$$

- with the 2-torsion relations $Q_i = -P_i$ for $i = 1, 2, 3$

The sunset elliptic dilogarithm



Representing the ratio of the coordinates on the sunset cubic curve as functions on $\mathcal{E}_\ominus \simeq \mathbb{C}^\times / q^{\mathbb{Z}}$

$$\frac{X}{Z}(x) = \frac{\theta_1(x/Q_1)\theta_1(x/P_3)}{\theta_1(x/P_1)\theta_1(x/Q_3)} \qquad \frac{Y}{Z}(x) = \frac{\theta_1(x/Q_2)\theta_1(x/P_3)}{\theta_1(x/P_2)\theta_1(x/Q_3)}$$

$\theta_1(x)$ is the Jacobi theta function

$$\theta_1(x) = q^{\frac{1}{8}} \frac{x^{1/2} - x^{-1/2}}{i} \prod_{n \geqslant 1} (1 - q^n)(1 - q^n x)(1 - q^n / x)\,.$$

The sunset elliptic dilogarithm

We find

$$\mathcal{J}_{\ominus}(s) \equiv \frac{i\varpi_r}{\pi}\left(\hat{E}_2\left(\frac{P_1}{P_2}\right) + \hat{E}_2\left(\frac{P_2}{P_3}\right) + \hat{E}_2\left(\frac{P_3}{P_1}\right)\right) \quad \text{mod periods}$$

where

$$\hat{E}_2(x) = \sum_{n\geqslant 0}\left(\mathrm{Li}_2\left(q^n x\right) - \mathrm{Li}_2\left(-q^n x\right)\right) - \sum_{n\geqslant 1}\left(\mathrm{Li}_2\left(q^n/x\right) - \mathrm{Li}_2\left(-q^n/x\right)\right) .$$

The sunset elliptic dilogarithm

- The elliptic dilogarithm $\hat{E}_2(x)$ is not invariant under q-translation and transforms according

$$\hat{E}_2(qx) = \hat{E}_2(x) - \frac{\pi^2}{2} + i\pi \log(x)$$
$$\hat{E}_2(x/q) = \hat{E}_2(x) + \frac{\pi^2}{2} - i\pi \log(x/q)\,.$$

This is because the Feynman integral we are studying is a multivalued function. Shifting the point P in $\mathbb{C}^\times/q^{\mathbb{Z}}$ changes the expression for $\mathcal{I}_\ominus$ by a period of the elliptic curve $\mathcal{E}_{K^2}$

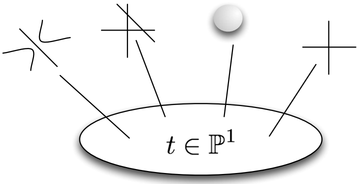
the elliptic curve of the sunset integral

- With all mass equal $m_i = m$ and $t = p^2/m^2$ the integral reduces to

$$\mathcal{J}_\ominus(t) = \frac{1}{m^2}\int_0^\infty\int_0^\infty \frac{dxdy}{(x+y+1)(x+y+xy)-txy}\,.$$

$$\mathcal{E}_t:\ (x+y+1)(x+y+xy)-txy=0\,.$$

- Special values
  - At $t=0$, $t=1$ and $t=+\infty$ the elliptic curve factorizes.
  - At $t=9$ we have the 3-particle the threshold $t\in\mathbb{C}\backslash[9,+\infty[$.



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$$\mathcal{E}_t:\ (x+y+1)(x+y+xy)-txy=0\,.$$

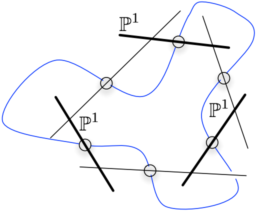
- Family of elliptic curve surface with 4 singular fibers leads to a K_3 pencil

$$\begin{array}{ccc} \mathcal{E}_t & \longrightarrow & \overline{\mathcal{E}}_t \\ f\big\downarrow & & \bar{f}\big\downarrow \\ X_1(6) & \longrightarrow & X_1(6)\cup\{cusps\} \end{array}$$

- This is a universal family of $X_1(6)$ modular curves with a point of order 6

The sunset integral and the motive

$$\begin{array}{ccc} \mathcal{E}_t & \longrightarrow & \overline{\mathcal{E}}_t \\ f \downarrow & & \downarrow \bar{f} \\ X_1(6) & \longrightarrow & X_1(6) \cup \{cusps\} \end{array}$$



- Elliptic local system V on $X_1(6)$ with fibre $\mathbb{Q}^2$
- For $s_1, s_2 \in \mathfrak{h} \cap \mathcal{E}_t$ then the divisor $s_1 - s_2$ on $\mathcal{E}_t$ is of torsion of order 6
- therefore $H_t^2 = \mathbb{Q}(0)^3 \oplus \hat{H}_t^2$

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow \hat{H}_t^2 \longrightarrow H^1(\mathcal{E}_t, \mathbb{Q}(-1)) \to 0$$

the sunset integral as an elliptic dilogarithm

- The Hauptmodul t is given by [Zagier; Stienstra]

$$t = 9 + 72\frac{\eta(2\tau)}{\eta(3\tau)}\left(\frac{\eta(6\tau)}{\eta(\tau)}\right)^5$$

- The period ϖ_r and ϖ_c, with $q := \exp(2i\pi\tau(t))$ are given by

$$\varpi_r \sim \frac{\eta(\tau)^6\eta(6\tau)}{\eta(2\tau)^3\eta(3\tau)^2}; \qquad \varpi_c = \tau\,\varpi_r$$

The regulator map I

- The regulator map is

$$\begin{array}{rcl} K_2(\mathbb{Q}(\mathcal{E}_t)) & \to & H^1(\mathcal{E}_t,\mathbb{R}) \\ \{\frac{x}{z},\frac{y}{z}\} & \mapsto & \{\gamma \mapsto \int_\gamma \eta(\frac{x}{z},\frac{y}{z})\}; \qquad \gamma \in H_1(\mathcal{E}_t,\mathbb{R}) \end{array}$$

- Since the elliptic curve $\mathcal{E}_t \cong \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) \cong \mathbb{C}^*/q^{\mathbb{Z}}$ the regulator can be expressed as a Kronecker-Eisenstein series [Bloch]
- Since for $s_1, s_2 \in \mathfrak{h} \cap H_t^2$ then the divisor $s_1 - s_2$ on $\mathcal{E}_t$ is of torsion of order 6

The regulator map II

$$\int_\gamma \eta(\frac{x}{z},\frac{y}{z}) = \Re\mathrm{e}\left(R_\tau(\zeta_6)+R_\tau(\zeta_6^2)\right)$$

$$R_\tau(\zeta_6) = \int d\tau \sum_{(m,n)\neq(0,0)} \frac{\psi(m,n)(\epsilon_1\tau+\epsilon_2)}{(m+n\tau)^3}$$

- Character $\psi : \varpi_c\mathbb{Z}\oplus\varpi_r\mathbb{Z}\to S^1$ and $(\zeta_6)^6 = 1$
- The sunset integral is then given by [Bloch, Vanhove]

$$\mathcal{I}^2_\ominus(t) = \text{periods} + \text{real-period}\left(R_\tau(\zeta_6)+R_\tau(\zeta_6^2)\right)$$

the sunset integral as an elliptic dilogarithm

This leads to the solutions of the PF equation [Bloch, Vanhove]

$$\mathcal{J}^2_{\ominus}(t) = i\pi\varpi_r(t)(1-2\tau) - \frac{6\varpi_r(t)}{\pi}\, E_{\ominus}(\tau)\,,$$

$$E_{\ominus}(\tau) = \frac{D(\zeta_6)}{\Im\mathrm{m}(\zeta_6)} - \frac{1}{2i}\sum_{n\geqslant 0}\left(\mathrm{Li}_2\left(q^n\zeta_6\right) + \mathrm{Li}_2\left(q^n\zeta_6^2\right) - \mathrm{Li}_2\left(q^n\zeta_6^4\right) - \mathrm{Li}_2\left(q^n\zeta_6^5\right)\right)$$

- We have $\mathrm{Li}_2(x)$ and not the Bloch-Wigner $D(x)$

the sunset integral as an elliptic dilogarithm

This leads to the solutions of the PF equation [Bloch, Vanhove]

$$\mathcal{J}^2_\ominus(t) = i\pi\varpi_r(t)(1-2\tau) - \frac{6\varpi_r(t)}{\pi} E_\ominus(\tau),$$

$$E_\ominus(\tau) = \frac{1}{2}\sum_{n\neq 0} \frac{\psi_2(n)}{n^2} \frac{1}{1-q^n}$$

- $\psi_2(n)$ is an odd mod 6 character

$$\psi_2(n) = \begin{cases} 1 & \textit{for } n \equiv 1 \mod 6 \\ -1 & \textit{for } n \equiv 5 \mod 6 \end{cases}$$

The sunset Eisenstein series

- The integral is given by

$$\mathcal{J}^2_{\ominus}(t) = \int_0^\infty \int_0^\infty \frac{dxdy}{(x+y+1)(x+y+xy) - txy}$$

- The 2-form has only log-pole on $\mathcal{E}_t$ and there is a residue 1-form

$$\mathcal{J}^2_{\ominus}(t) = periods + \varpi_r \left\langle \epsilon_1\tau + \epsilon_2, \int d\tau \sum_{(m,n)\neq(0,0)} \frac{\psi_2(n)(\epsilon_1\tau+\epsilon_2)}{(m+n\tau)^3} \right\rangle$$

- Character $\psi : Lattice(\mathcal{E}_t) \to S^1$. Pairing $\langle \epsilon_1, \epsilon_2 \rangle = -\langle \epsilon_2, \epsilon_1 \rangle = 2i\pi$
- The amplitude integral is *not* the regulator map which involves a real projection $r : K_2(\mathcal{E}_t) \to H^1(\mathcal{E}_t, \mathbb{R})$
- The amplitude is multivalued in t whereas the regulator is single-valued

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- The regulator is an Eichler integral

$$\mathcal{J}^2_{\ominus}(t) = periods + \varpi_r \int_\tau^{i\infty} \sum_{(m,n)\neq(0,0)} \frac{\psi_2(n)(\tau-x)}{(m+nx)^3} dx$$

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- The regulator is an Eichler integral

$$\mathcal{J}^2_{\ominus}(t) = periods + \varpi_1 \sum_{(m,n)\neq(0,0)} \frac{\psi_2(n)}{n^2(m+n\tau)}$$

three-loop banana graph: Picard-Fuchs operator

- The geometry of the 3-loop banana graph is a K_3 surface (Shioda-Inose family for $\Gamma_1(6)^{+3}$) with Picard number 19 and discriminant of Picard lattice is 6

$$(m_4^2 + \sum_{i=1}^{3} m_i^2 x_i)(1 + \sum_{i=1}^{3} x_i^{-1}) \prod_{i=1}^{3} x_i - t \prod_{i=1}^{3} x_i = 0$$

- The all equal mass case with $t = p^2/m^2$ satisfies the Picard-Fuchs equation [vanhove]

$$\Big(t^2(t-4)(t-16)\frac{d^3}{dt^3} + 6t(t^2-15t+32)\frac{d^2}{dt^2} + (7t^2-68t+64)\frac{d}{dt} + t - 4\Big) J^2_{\ominus}(t) = -4!$$

- One miracle is that this picard-fuchs operator is the symmetric square of the picard-fuchs operator for the sunset graph [Verrill]

three-loop banana graph: solution

- It is immediate to use the Wronskian method to solve the differential equation [Bloch, Kerr, Vanhove]

$$m^2\, I^2_{\ominus}(t) = 40\pi^2 \log(q)\, \varpi_1(\tau)$$
$$-48\varpi_1(\tau)\left(24\mathcal{L}i_3(\tau,\zeta_6) + 21\,\mathcal{L}i_3(\tau,\zeta_6^2) + 8\mathcal{L}i_3(\tau,\zeta_6^3) + 7\mathcal{L}i_3(\tau,1)\right)$$

with $\mathcal{L}i_3(\tau, z)$ [Zagier; Beilinson, Levin]

$$\mathcal{L}i_3(\tau,z) := \mathrm{Li}_3\,(z) + \sum_{n\geqslant 1}\left(\mathrm{Li}_3\,(q^n z) + \mathrm{Li}_3\left(q^n z^{-1}\right)\right)$$
$$-\left(-\frac{1}{12}\log(z)^3 + \frac{1}{24}\log(q)\log(z)^2 - \frac{1}{720}(\log(q))^3\right).$$

three-loop banana graph: solution

- Which can be written using as an Eisenstein series

$$m^2\, I^2_{\ominus}(t(\tau)) = \varpi_1(\tau)\, \left(-4(\log q)^3 + \frac{1}{2}\sum_{n\neq 0} \frac{\psi_3(n)}{n^3}\frac{1+q^n}{1-q^n}\right)$$

where $\psi_3(n)$ is an even mod 6 character

- Arising from the regulator for $\mathrm{Sym}^2 H^1(\mathcal{E}_\ominus(t))$ with $dz = \epsilon_1\tau + \epsilon_2$

$$\varpi_1\, \left\langle dz^2, \int \sum_{m,n} \frac{d\tau dz^2\, \psi_3(n)}{(m+n\tau)^4} \right\rangle$$

- Again this is given by the Eichler integral

$$\mathrm{period} + \varpi_1 \int_\tau^{i\infty} \sum_{m,n} \frac{(x-\tau)^2\, \psi_3(n)}{(m+nx)^4}\, dx$$

Part III

More masses and loops

Higher-order bananas I

The integral we have been discussing are given by

$$I_n^2 = \int_{x_i \geqslant 0} \frac{1}{(1+\sum_{i=1}^n x_i)(1+\sum_{i=1}^n x_i^{-1}) - t} \prod_{i=1}^n \frac{dx_i}{x_i}$$

The differential equation is

$$\left(\sum_{r=0}^{n-1} q_r(t) \left(t\frac{d}{dt}\right)^r\right) I_n^2(t) = -n!$$

- with [vanhove]

Higher-order bananas II

$$
\begin{aligned}
q_{n-1}(t) &= t^{\lfloor \frac{n}{2} \rfloor + \eta(n)} \prod_{i=0}^{\lfloor \frac{n}{2} \rfloor} (t - (n-2i)^2) \\
q_{n-2}(t) &= \frac{n-1}{2} \frac{dq_{n-1}(t)}{dt} \\
q_0(t) &= t - n \,.
\end{aligned}
$$

- with $\eta(n) = 0$ if $n \equiv 1 \mod 2$ and 1 if $n \equiv 0 \mod 2$.

When the masses are varying the PF operators changes order we have

- Order L for all equal non vanishing masses $m_1 = m_2 = \cdots = m_{L+1}$ and $S_{L+1} = -(L+1)!$

Higher-order bananas III

- Each time a mass becomes degenerate the order increases. Max order is $2L$ when all the masses are different.

In the $L = 3$ sunset case we have

$$L_{PF} = \sum_{r=0}^{n} p_r \left(p^2 \frac{d}{dp^2}\right)^r$$

with

- $L = 3$ and degree $p_r = 2$ for $m_1 = m_2 = m_3 = m_4$
- $L = 4$ and degree $p_r = 8$ for $m_1 = m_2 \neq m_3 = m_4$
- $L = 4$ and degree $p_r = 7$ for $m_1 = m_2 = m_3 \neq m_4$
- $L = 5$ and degree $p_r = 15$ for $m_1 = m_2 \neq m_3 \neq m_4$
- $L = 6$ and degree $p_r = 25$ for $m_1 \neq m_2 \neq m_3 \neq m_4$

The sunset Gromov-Witten invariants

The holomorphic period around $s(=1/t)=0$

$$\pi_0 = \int_{\varphi_0} \Omega_\ominus = \sum_{m\geqslant 0} s^m \sum_{b_1+b_2+b_2=m} m_1^{b_1} m_2^{b_2} m_3^{b_3} \left(\frac{m!}{b_1!b_2!b_3!}\right)^2$$

and the logarithmic Mahler measure defined by $\pi_0 = \frac{d}{ds} R_0$

$$R_0 = i\pi - \int_{|x|=|y|=1} \log(s^{-1} - (m_1^2 x + m_2^2 y + m_3^2)(x^{-1}+y^{-1}+1)) \frac{d\log x d\log y}{(2\pi i)^2}.$$

The sunset Feynman integral leads to Gromov-Witten numbers

$$\mathcal{J}_\ominus(s) = -\pi_0 \left(3R_0^3 + \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell>0 \\ (\ell_1,\ell_2,\ell_3)\in\mathbb{N}^3\setminus(0,0,0)}} \ell(1-\ell R_0) N_{\ell_1,\ell_2,\ell_3} \prod_{i=1}^{3} m_i^{\ell_i} e^{\ell R_0} \right).$$

The sunset Gromov-Witten invariants

The local Gromov-Witten numbers N_{ℓ_1,ℓ_2,ℓ_3} can be expressed in terms of the virtual integer number of degree ℓ rational curves by

$$N_{\ell_1,\ell_2,\ell_3} = \sum_{d|\ell_1,\ell_2,\ell_3} \frac{1}{d^3} n_{\frac{\ell_1}{d},\frac{\ell_2}{d},\frac{\ell_3}{d}} .$$

$\underline{\ell}$	(100)	$\overset{k>0}{(k00)}$	(110)	(210)	(111)	(310)	(220)	(211)	(221)
$N_{\underline{\ell}}$	2	$2/k^3$	-2	0	6	0	$-1/4$	-4	10
$n_{\underline{\ell}}$	2	0	-2	0	6	0	0	-4	10

$\underline{\ell}$	(410)	(320)	(311)	(510)	(420)	(411)	(330)	(321)	(222)
$N_{\underline{\ell}}$	0	0	0	0	0	0	$-2/27$	-1	$-189/4$
$n_{\underline{\ell}}$	0	0	0	0	0	0	0	-1	-48

The sunset Gromov-Witten invariants

For the all equal masses case $m_1 = m_2 = m_3 = 1$, the mirror map is

$$Q = e^{R_0} = -q \prod_{n\geqslant 1} (1 - q^n)^{n\delta(n)}; \qquad \delta(n) := (-1)^{n-1} \left(\frac{-3}{n}\right),$$

where $\left(\frac{-3}{n}\right) = 0, 1, -1$ for $n \equiv 0, 1, 2 \mod 3$.

The local Gromov-Witten numbers

$$\frac{N_\ell}{6} = 1, -\frac{7}{8}, \frac{28}{27}, -\frac{135}{64}, \frac{626}{125}, -\frac{751}{54}, \frac{14407}{343}, -\frac{69767}{512}, \frac{339013}{729}, -\frac{827191}{500}, \frac{8096474}{1331}, -\frac{367837}{16}, \frac{195328680}{2197}, -\frac{137447647}{392}, \frac{4746482528}{3375}, -\frac{23447146631}{4096}, \frac{115962310342}{4913}, -\frac{574107546859}{5832}, \frac{2844914597656}{6859}, -\frac{1410921149451}{800}, \frac{10003681368433}{1323}, \ldots$$

The sunset mirror symmetry

Why is this all happening?

- The sunset elliptic curve is embedded into a singular compactification X_0 of the local Hori-Vafa 3-fold

$$Y := \{1 - s(m_1^2 x + m_2^2 y + m_3^2)(1 + x^{-1} + y^{-1}) + uv = 0\} \subset (\mathbb{C}^*)^2 \times \mathbb{C}^2 \,,$$

- The GW numbers are computed for the local mirror symmetry of a semi-stably degenerating a family of elliptically-fibered Calabi-Yau 3-folds $\mathrm{X}_{z_0} \to \mathrm{X}_0$

The sunset mirror symmetry

- Iritani's quantum $\mathbb{Z}$-variation of Hodge structure on the even cohomology of the Batyrev mirror X° of X allows to compare the asymptotic Hodge theory of this B-model to that of the mirror (elliptically fibered) A-model Calabi-Yau X°

- We have an isomorphism of A- and B-model $\mathbb{Z}$-variation of Hodge structure

$$H^3(\mathrm{X}_{z_0}) \cong H^{even}(\mathrm{X}^\circ_{Q_0}),$$

  and taking (the invariant part of) limiting mixed Hodge structure on both sides yields the relation between regulator periods and local Gromov-Witten numbers

- The computation of the GW numbers uses the mirror map $(K^2, m_1, m_2, m_3) \mapsto Q(K^2, m_1, m_2, m_3) = e^{R_0}$

The sunset Yukawa coupling

- The Yukawa coupling of the non-compact CY X

$$Y_{ijk} = \int_X \tilde{\Omega} \wedge \nabla_{\delta_i \delta_j \delta_k} \tilde{\Omega}$$

- descends to the local Yukawa of the sunset elliptic curve

$$Y^{\mathrm{loc}}_{0ij} \propto Y_{\ominus} = \int \Omega_{\ominus} \wedge \nabla_{\frac{d}{ds}} \Omega_{\ominus} = \frac{1}{2i\pi} \frac{\partial^2 R_1}{\partial R_0^i \partial R_0^j}$$

- The same construction applies to the 3-loop banana graph (4-fold CY) and the 4-loop banana graph (5-fold CY). Polylogarithms are believed to not be enough from 4-loop

- At higher-loop loop the geometry is more intricate but we could expect more connection between Gromov-Witten prepotential and (massive) quantum field theory Feynman integrals.