Lecture 4: Superconformal observables & instantons

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“Partition functions and automorphic forms”

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Plan

- Lecture 4:

- Superconformal indices

[Kinney, Maldacena, Minwalla, Raju] [Romelsberger] [Bhattacharya, Bhattacharyya, Minwalla, Raju]

- S^4, $S^4 \times S^1$ and instantons

[Pestun] [H.-C. Kim, S.-S. Kim, K. Lee]

- S^5, $S^5 \times S^1$ and instantons

[Kallen, Qiu, Zabzine] [H.-C.Kim, SK] [Lockhart, Vafa] [H.-C.Kim, J.Kim, SK] [Rastelli et.al.] [H.-C.Kim, SK, S.-S.Kim, K.Lee] ……

- Concluding remarks

SUSY partition functions on curved spaces

- S^n: $n = 4$ [Pestun], $n = 3$ [Kapustin,Willett,Yaakov], $n = 2$ [Doroud,Gomis,Le Floch,Lee] [Benini,Bobev]

  $n = 5$ [Kallen,Qiu,Zabzin] [H.-C.Kim,SK] [Lockhart,Vafa] [H.-C.Kim,J.Kim,SK] …

  $n = 6,7$ [Minahan, Zabzine]

- $S^{n-1} \times S^1$: $n = 4$ [Kinney,Maldacena,Minwalla,Raju] [Romelsberger]

  $n = 3$ [Bhattacharya,Minwalla] [SK]

  $n = 2$ (for gauge theories) [Benini, Eager, Hori, Tachikawa]

  $n = 5$ [H.-C.Kim, SK] [Lockhart, Vafa] [H.-C.Kim, J.Kim, SK]

- Others: e.g. $S^2 \times M_3$, $S^3 \times \Sigma_g$, $T^2 \times S^2$, disk $(\times S^1)$, $CP^2 \times S^1$, …

- Why?

- E.g. on S^n and $S^{n-1} \times R$, CFTs can be uniquely defined.
- Compact space, so that discrete questions can be posed and computed.
- SUSY QFTs on curve spaces are also better understood recently.
- strong-coupling calculation/studies possible for SUSY QFTs.

Superconformal index & $S^{n-1} \times S^1$

- In the 1st lecture, I introduced the "superconformal index" as a Witten index.
- Provided Hilbert space definition, and free field representation in 4d.

$$Z_{S^{d-1}\times\mathbb{R}}(t,y,\{m\}) = \int [dU] PE\left[\sum_{i\in\text{multiplets}} f_i(t,y,\{m\})\chi_{\mathbf{R}}(U)\right]$$

$$\int [dU] \to \int \prod_{i=1}^{r} da_i \cdot \prod_{\alpha>0}\left(2\sin\frac{\alpha(a)}{2}\right)^2$$

$$PE[f(t,y,\{m\})\chi_{\mathbf{R}}(U)] \equiv \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f(t^n,y^n,nm)\chi_{\mathbf{R}}(U^n)\right]$$

$$U \sim e^{ia}$$

- More abstractly in QFT, counting states w/ chemical potentials (like temperature) demands us to do path integral of Euclidean QFT, with time compactified to S^1.
- So, superconformal index is a SUSY partition function on (twisted) $S^{d-1} \times S^1$.

- $d = 4$:

- Integrand is free fields' indices, in "Coulomb phase" with nonzero a_i
- However, a_i is just part of the path integral variables, which we integrate at the last stage.
- While we keep them fixed, they formally play the role of Coulomb VEV.
- In combinatoric viewpoint, the integral imposes gauge singlet projection.

Index in the weakly-coupled QFT

- $d = 3$: Exist non-perturbative configurations that cannot be seen in strict free limit.

- Magnetic monopole operators: quantized fluxes on S^2.

$$\int_{S^2} \mathrm{tr}(F) \in 2\pi\mathbb{Z} \quad \text{with } \mathcal{L} \leftarrow -\frac{1}{4g_{YM}^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \cdots \ , \quad D_\mu \equiv \partial_\mu - iA_\mu$$

$$\int_{S^2} \mathrm{tr}(F) \in \frac{2\pi}{g_{YM}}\mathbb{Z} \quad \text{with } \mathcal{L} \leftarrow -\frac{1}{4}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \cdots \ , \quad D_\mu \equiv \partial_\mu - ig_{YM}A_\mu$$

- "Coulomb branch" like expressions: flux spread over S^2 [SK], …

- "Higgs branch" like expressions: vortices.

- $d = 2$: basically the elliptic genus that we explored so far (w/ slight change in fermion boundary conditions, from "R" to "NS")

- Almost take the form of integral over Coulomb VEV's, with integrand given by free fields's indices. (But the contour choice is complicated, due to nontrivial 0-mode structures.)

- $d = 5,6$: We shall consider rather interesting proposals for the indices, on $S^4 \times S^1$ and $S^5 \times S^1$, and their connections to the instanton partition functions.

$Z[S^4]$

- As a warm-up, first consider the 4d gauge theory's SUSY partition function on S^4.

- By a careful path integral calculus, one gets an "integral over Coulomb VEV"
- The saddle points consist of multi-instantons in Coulomb branch, localized either on north/south poles of S^4. [Pestun] (2007)

$$Z_{S^4}(g_{YM},\cdots) = \int [da] Z_{\mathbb{R}^4}(q=e^{2\pi i\tau}, a_i, \epsilon_{1,2}) Z_{\mathbb{R}^4}(q=e^{2\pi i\bar{\tau}}, a_i, \epsilon_{1,2})\Big|_{\epsilon_1=\epsilon_2=r^{-1}}$$

$$Z_{\mathbb{R}^4} = Z_{\rm cl} Z_{\rm pert} Z_{\rm inst} \qquad Z_{\rm cl} = \exp\left[-\pi i \tau r^2 {\rm tr}(a^2)\right] \qquad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$$

- Further generalized to squashed S^4 [Hama, Hosomichi]

5d superconformal index on $S^4 \times S^1$

- Now we consider the SUSY partition functions of 5d SCFTs on $S^4 \times S^1$.
- Strictly speaking, there's nowhere to start the QFT calculus, since we don't know its Lagrangian. So we don't know how to path-integrate.

- Here, one considers SCFT with Yang-Mills deformation, and try to obtain a formula by path-integrating Yang-Mills gauge fields & superpartners.
- Although 5d Yang-Mills is non-renormalizable, for SUSY path integrals one can perform an (almost) unambiguous calculation. [H.-C.Kim, S.-S.Kim, K. Lee] (2012)
- Apart from such conceptual issues, the calculus is similar to [Pestun] (2007)

- The Hilbert space definition of the index:

- 5d superconformal algebra is $F(4)$, which contains the bosonic algebra $SO(5,2) \times SU(2)_R$
- Pick a Q, S, which satisfy $\{Q, S\} = \Delta \equiv E - (3R + j_1 + j_2)$. ($Q = Q^{R=1/2}_{j_1=j_2=-1/2}$, $S = Q^\dagger$)

$$Z_{S^4\times S^1}(t,u,\{m_a\},q) = \mathrm{Tr}\left[(-1)^F t^{2(R+\frac{j_1+j_2}{2})} u^{j_1-j_2} e^{-m_a F_a} q^k\right]$$

5d index

- The result of SUSY path integral: (at infinite Yang-Mills coupling)

$$Z_{S^4\times S^1} = \oint [da] Z_{\mathbb{R}^4\times S^1}(ia, \epsilon_{1,2}, m_a, q) Z_{\mathbb{R}^4\times S^1}(-ia, \epsilon_{1,2}, -m_a, q^{-1})$$

$$(t, u) = e^{-\epsilon_\pm} = e^{-\frac{\epsilon_1 \pm \epsilon_2}{2}}$$

- Instantons at the south pole, anti-instantons at the north pole

- Although the claim is that the above expression holds exactly, we are only able to control the integrand (as before) in certain series expansions.
- Previously, we understood $Z_{R^4\times S^1}(\epsilon_{1,2}, a_i, m_a, q)$ as given by a series in q. This expansion cannot be sensibly used in this formula.

- Instead, the index can be expanded in t, since R-charge $R \sim$ BPS energy, so it is like low temperature expansion: generating function of BPS degeneracies.
- So we regard $Z_{R^4\times S^1}$ as a double series in t, q or t, q^{-1}, and collect in t order.

Enhanced global symmetries

- For instance, back to the 5d SU(2) theory at $N_f = 7$, where we expect $SO(14) \times U(1)_I \to E_8$ enhancement of symmetry: one obtains [Hwang, J.Kim, SK, Park] (2014)

$$1+\chi_{\mathbf{248}}^{E_8}t^2+\chi_2(u)\left[1+\chi_{\mathbf{248}}^{E_8}\right]t^3+\left[1+\chi_{\mathbf{27000}}^{E_8}+\chi_3(u)\left(1+\chi_{\mathbf{248}}^{E_8}\right)\right]t^4$$
$$+\left[\chi_2(u)\left(1+\chi_{\mathbf{248}}^{E_8}+\chi_{\mathbf{27000}}^{E_8}+\chi_{\mathbf{30380}}^{E_8}\right)+\chi_4(u)\left(1+\chi_{\mathbf{248}}^{E_8}\right)\right]t^5$$
$$+\left[2\chi_{\mathbf{248}}^{E_8}+\chi_{\mathbf{30380}}^{E_8}+\chi_{\mathbf{1763125}}^{E_8}+\chi_3(u)\left(2+2\chi_{\mathbf{133}}^{E_8}+\chi_{\mathbf{3875}}^{E_8}+2\chi_{\mathbf{27000}}^{E_8}+\chi_{\mathbf{30380}}^{E_8}\right)\right.$$
$$\left.+\chi_5(u)\left(1+\chi_{\mathbf{248}}^{E_8}\right)\right]t^6+\mathcal{O}\left(t^7\right) ,$$

- E_8 characters are obtained by summing over various instanton/anti-instanton sectors

$$\mathbf{248}=\mathbf{1}_0+\mathbf{14}_2+\mathbf{14}_{-2}+\mathbf{64}_{-1}+\overline{\mathbf{64}}_1+\mathbf{91}_0,$$
$$\mathbf{3875}=\mathbf{1}_4+\mathbf{1}_0+\mathbf{1}_{-4}+\mathbf{14}_2+\mathbf{14}_{-2}+\mathbf{64}_3+\mathbf{64}_{-1}+\overline{\mathbf{64}}_1+\overline{\mathbf{64}}_{-3}+\mathbf{91}$$
$$+\mathbf{104}_0+\mathbf{364}_2+\mathbf{364}_{-2}+\mathbf{832}_{-1}+\overline{\mathbf{832}}_1+\mathbf{1001}_0,$$
$$\mathbf{27000}=2\times\mathbf{1}_0+\mathbf{14}_2+\mathbf{14}_{-2}+2\times\mathbf{64}_{-1}+2\times\overline{\mathbf{64}}_1+2\times\mathbf{91}_0$$
$$+\mathbf{104}_4+\mathbf{104}_0+\mathbf{104}_{-4}+\mathbf{364}_2+\mathbf{364}_{-2}$$
$$+\mathbf{832}_3+\mathbf{832}_{-1}+\overline{\mathbf{832}}_1+\overline{\mathbf{832}}_{-3}+\mathbf{896}_2+\mathbf{896}_{-2}+\mathbf{1001}_0$$
$$+\mathbf{1716}_{-2}+\overline{\mathbf{1716}}_2+\mathbf{3003}_0+\mathbf{3080}_0+\mathbf{4928}_{-1}+\overline{\mathbf{4928}}_1,$$
$$\mathbf{30380}=\mathbf{1}_0+2\times\mathbf{14}_2+2\times\mathbf{14}_{-2}+\mathbf{64}_3+2\times\mathbf{64}_{-1}+2\times\overline{\mathbf{64}}_1+\overline{\mathbf{64}}_{-3}$$
$$+\mathbf{91}_4+3\times\mathbf{91}_0+\mathbf{91}_{-4}+\mathbf{104}_0+\mathbf{364}_2+\mathbf{364}_{-2}$$
$$+\mathbf{832}_3+2\times\mathbf{832}_{-1}+2\times\overline{\mathbf{832}}_1+\overline{\mathbf{832}}_{-3}+\mathbf{896}_2+\mathbf{896}_{-2}$$
$$+\mathbf{1001}_0+\mathbf{2002}_2+\mathbf{2002}_{-2}+\mathbf{3003}_0+\mathbf{4004}_0+\mathbf{4928}_{-1}+\overline{\mathbf{49}}$$

$$\mathbf{1763125}=2\times\mathbf{1}_0+2\times\mathbf{14}_2+2\times\mathbf{14}_{-2}+3\times\mathbf{64}_{-1}+3\times\overline{\mathbf{64}}_1+3\times\mathbf{91}_0$$
$$+\mathbf{104}_4+\mathbf{104}_0+\mathbf{104}_{-4}+\mathbf{364}_2+\mathbf{364}_{-2}+\mathbf{546}_6+\mathbf{546}_2+\mathbf{546}_{-2}+\mathbf{546}_{-6}$$
$$+2\times\mathbf{832}_3+2\times\mathbf{832}_{-1}+2\times\overline{\mathbf{832}}_1+2\times\overline{\mathbf{832}}_{-3}+2\times\mathbf{896}_2+2\times\mathbf{896}_{-2}$$
$$+2\times\mathbf{1001}_0+2\times\mathbf{1716}_{-2}+2\times\overline{\mathbf{1716}}_2+\mathbf{2002}_2+\mathbf{2002}_{-2}$$
$$+3\times\mathbf{3003}_0+2\times\mathbf{3080}_0+\mathbf{4004}_4+2\times\mathbf{4004}_0+\mathbf{4004}_{-4}$$
$$+3\times\mathbf{4928}_{-1}+3\times\overline{\mathbf{4928}}_1+\mathbf{5625}_4+\mathbf{5625}_0+\mathbf{5625}_{-4}$$
$$+\mathbf{5824}_3+\mathbf{5824}_{-1}+\mathbf{5824}_{-5}+\overline{\mathbf{5824}}_5+\overline{\mathbf{5824}}_1+\overline{\mathbf{5824}}_{-3}$$
$$+\mathbf{11648}_2+\mathbf{11648}_{-2}+\mathbf{17472}_3+\mathbf{17472}_{-1}+\overline{\mathbf{17472}}_1+\overline{\mathbf{17472}}_{-3}$$
$$+\mathbf{18200}_2+\mathbf{18200}_{-2}+\mathbf{21021}_0+\mathbf{21021}_{-4}+\overline{\mathbf{21021}}_4+\overline{\mathbf{21021}}_0$$
$$+\mathbf{24024'}_2+\mathbf{24024'}_{-2}+\mathbf{27456}_3+\overline{\mathbf{27456}}_{-3}+\mathbf{36608}_2+\mathbf{36608}_{-2}$$
$$+\mathbf{40768}_{-1}+\overline{\mathbf{40768}}_1+\mathbf{45760}_3+\mathbf{45760}_{-1}+\overline{\mathbf{45760}}_1+\overline{\mathbf{45760}}_{-3}$$
$$+\mathbf{58344}_0+\mathbf{58968}_0+\mathbf{64064'}_{-1}+\overline{\mathbf{64064'}}_1+\mathbf{115830}_{-2}+\overline{\mathbf{115830}}_2$$
$$+\mathbf{146432}_{-1}+\overline{\mathbf{146432}}_1+\mathbf{200200}_0.$$

6d index on $S^5 \times S^1$

- Here again, we have nowhere to start, not knowing Lagrangian descriptions.

- We only have various EFT descriptions. Here, we use and get insights from the 5d Yang-Mills descriptions for 6d QFT compactified on S^1.
- If the 5d SUSY path integral acquires contributions from instanton-like saddle points, we can hope that they reconstruct the 6th direction along S^1.
- In this case, the constructed S^1 will be Euclidean time.

- Indeed, SUSY path integral requires us to consider the Hopf fibration $S^1 \to S^5 \to CP^2$, and one gets the saddle point condition [Kallen, Zabzine] [Hosomichi, Seong, Terashima] [Kallen, Qiu, Zabzine] [H.-C. Kim, SK] (2012)

$$F_{\mu\nu} = \frac{1}{2}\sqrt{g_{S^5}}\epsilon_{\mu\nu\alpha\beta\gamma}\xi^\alpha F^{\beta\gamma}$$

ξ^μ : Killing vector along the fiber

- These configurations are often called "contact instantons"

Some details on $Z[S^5]$

- The 6d superconformal index: Hilbert space definition

- Choose $Q, S, \Delta \equiv \{Q, S\} = E - (4R + j_1 + j_2 + j_3)$ in $OSp(8^*|N)$, containing $SO(6,2) \times Sp(N)_R$
- index can carry chemical potentials for $j_1 - j_2$, $j_2 - j_3$, $E - R$:

$$Z_{S^5\times S^1}(\beta, a_{1,2,3}, \{m_a\}) = \mathrm{Tr}\left[(-1)^F e^{-\beta(E-R)} e^{-a_1 j_1 - a_2 j_2 - a_3 j_3} e^{-m_a F_a}\right] \quad , \quad a_1 + a_2 + a_3 = 0$$

- With $SU(3) \subset SO(6)$ angular momentum chemical potentials a_n on S^5, it further localizes the instantons on 3 $U(1)^2 \subset SU(3)$ fixed points on CP^2:

$$Z_{S^5}\left(\beta = \frac{g_{YM}^2}{r}, a_n, \{m_a\}\right) = \int \prod_{i=1}^{r}[dv_i] Z_{\mathbb{R}^4\times S^1}\left(q = e^{-\frac{4\pi^2}{\beta(1+a_3)}}, \frac{v_i}{1+a_3}, \frac{m_a}{1+a_3}, \epsilon_{1,2} = \frac{a_{1,2} - a_3}{1+a_3}\right)$$
$$\cdot Z_{\mathbb{R}^4\times S^1}(a_{1,2,3} \to a_{2,3,1}) \cdot Z_{\mathbb{R}^4\times S^1}(a_{1,2,3} \to a_{3,1,2})$$

- Depending on models, one can study either 5d SCFT or 6d SCFT with this Z_{S^5}.
  (E.g. [Chang, Fluder, Lin, Wang] (2017) for studies on 5d SCFTs)
- For instanton partition functions for 6d SCFTs, we have good reasons to believe that they are 6d partition functions on $R^4 \times T^2$ (lectures 2 & 3)… but, with opposite identifications of time/space directions on T^2.

Challenges

- Here we face a technical limitation, that we do not know the full functional form of the instanton partition function, $Z(q, a_i, \epsilon_{1,2}, \cdots)$.
- Only know expansion coefficients in $q = e^{-4\pi^2/\beta}$, when it is small.

- Especially, instead of the expansion in $q = e^{-4\pi^2/\beta}$, we often want an expansion in the fugacity $e^{-\beta}$ of the superconformal index.
- The setting in which q is the fugacity of the instanton partition function is not the same as the setting here (because of different notions of Euclidean time circle).
- We identified modular transformations of coefficients in $e^{-(v_i - v_{i+1})}$ expansion, but this appears useless here (since Coulomb VEV has to be integrated over).
- This is what makes it much more difficult to study it than 5d superconformal index, or the instanton partition function.

- Only consider cases in which exact resummation/re-expansion in $e^{-\beta}$ is known.

Enhanced SUSY limits

- Consider the 6d maximal SCFT (e.g. on M5-branes). This is an SCFT (twice as many SUSY than generic cases).

$$Z_{S^5\times S^1}(\beta, a_{1,2,3}, m) = \mathrm{Tr}\left[(-1)^F e^{-\beta(E-\frac{R_1+R_2}{2})} e^{-m(R_1-R_2)} e^{-a_1 j_1 - a_2 j_2 - a_3 j_3}\right] \quad , \quad a_1+a_2+a_3=0$$

- Index comes with chemical potentials, only commuting w/ a pair of real SUSY.
- If one tunes a chemical potential, it may commute with more SUSY, making the partition function simplify & make an exact re-summation of the integrand.
- One can show that the index respects 4 real SUSY at

$$m = \frac{1}{2} - a_n \quad (\text{for one of } n = 1,2,3)$$

- Simplest case: $m = \frac{1}{2}$, $a_1 = a_2 = a_3 = 0$ (respects 16 SUSY)

$$Z_{S^5} \to \frac{1}{N!}\int \prod_{i=1}^{N} dv_i \; e^{-\frac{2\pi^2 \sum_i v_i^2}{\beta}} \prod_{i\neq j} 2\sinh(\pi(v_i - v_j)) \cdot \eta\left(e^{-\frac{4\pi^2}{\beta}}\right)^{-1}$$

Results

- Can "S-dualize" Dedekind eta function, and Gaussian integrate over v_i's:

$$Z_{S^5} \to e^{\beta\left(\frac{N^3-N}{6}+\frac{N}{24}\right)} \prod_{n=1}^{N} \prod_{s=0}^{\infty} \frac{1}{1-e^{-(n+s)\beta}}$$

- Indeed, takes the form of an index: expanding in "fugacity" $e^{-\beta}$, coefficients are integers.
- This supports the emergence of temporal S^1, due to instanton sum

- There are two physics to be discussed, in this "unrefined limit"
- Zero point "energy": the leading part in the low temperature limit $\beta \to \infty$

$$Z \overset{\beta\to\infty}{\longrightarrow} e^{-\beta r \epsilon_0} (1 + \cdots)$$

$$\epsilon_0 \equiv -\frac{1}{r}\left(\frac{N^3-N}{6}+\frac{N}{24}\right)$$

- One curved space, there are nonzero vacuum 0-point energy: "Casimir energy"
- More precisely speaking, it is the 0-point value for $E - \frac{R_1+R_2}{2}$.
- Scales like N^3, like many other observables of the M5-brane system.

Spectrum & W-algebra

- Apart from the vacuum energy, the "spectral" part:

$$\prod_{n=1}^{N}\prod_{s=0}^{\infty}\frac{1}{1-e^{-(n+s)\beta}}$$

- This is known as the "vacuum character" of the W_N algebra, apart from the overall $U(1) \subset U(N)$ contribution $\prod_{n=1}^{\infty}\frac{1}{1-e^{-n\beta}}$

- In large N limit, it agrees with the index of gravitons in $AdS_7 \times S^4$. So AdS/CFT supports this formula, or depending on viewpoint, we tested AdS_7/CFT_6 using our field theory calculus.

- A physical explanation of this index was found by [Been, Rastelli, van Rees] (2014)

- There exists a subset of local BPS operators of CFT (~ states on $S^5 \times R$) that are closed under operator product expansion (OPE), and whose OPE yield W_N algebra.
- The local operators contributing to the index forms a vacuum character.
- Other W_N characters given by inserting defect operators to the index [Bullimore, H.-C. Kim]

Further problems

- Exact treatment?

- I explained situations in which the lack of exact knowledge of instanton partition function obstructs better physical explorations. (E.g. obstructing fugacity expansions)
- It is not just a matter of rigor. It obstructs pragmatic attempts of studies.

- Possibly new saddle point?

- The proposed expression for Z_{S^5} has certain technical uncertainties.
- It was not proved that the saddle points of path integral I explained are the most general solutions. [Zabzine et.al.]

- General 6d (1,0) SCFTs?

Concluding remarks

- In these lecture series, I tried explain the basics of instanton partition functions, and how they played crucial roles in recent advances in 5d & 6d SCFTs.

- I tried to avoid discussing open physical/conceptual issues (set aside an ultimate dream of realizing them using Lagrangians). But there are many issues concerning instantons that should be better understood.

- E.g. we only considered instantons in Coulomb phase, or on compact manifolds.
- However, instantons in the symmetric phase & non-compact space (e.g. R^4) has non-compact moduli from its "size".
- Its proper interpretations in higher dimensional QFTs are still unclear.

- Better physical understandings on them may lead to more interesting/subtle physical observables, other than Nekrasov's instanton partition functions.