Conformal Triangles and Zig-Zag Diagrams.

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План

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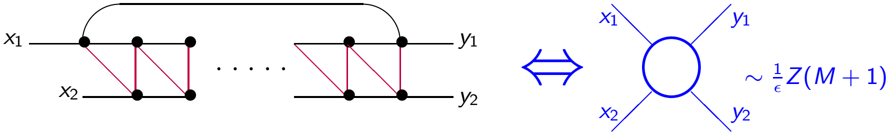
The 4-dimensional ϕ^4 field theory (and its multicomponent generalizations) serves the Brout-Englert-Higgs mechanism and thus is an essential part of the Standard Model of particle physics. It was shown by explicit evaluation (in MS scheme) of the Gell-Mann-Low β-function in $\phi^4_{D=4}$ theory that special Feynman diagrams – so-called zig-zag diagrams (the residue $Z(M+1)$ in the 4-point function)

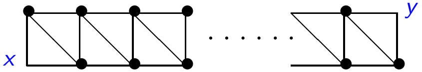


$$x_1 \overset{\beta}{\text{——}} x_2 = \frac{1}{(x_1-x_2)^{2\beta}}\ , \quad x_i, y_i \in \mathbb{R}^D, \qquad \bullet = \int d^D x, \quad D = 4 - 2\epsilon\ ,$$

give 44%, 46% and 47% of numerical contributions, respectively, to the $3, 4$ and 5 loop orders of β [D.J. Broadhurst and D. Kreimer (1995)].

It can be shown that the $(M+1)$-loop contribution $Z(M+1)$ to the β-function is given by M-loop two-point zig-zag diagram.

This two-point function is represented by the M-loop zig-zag diagram (by M-fold integral)



and has the general form $(D = 4)$

$$G_2(x, y) = \frac{\pi^{2M}}{(x-y)^2} \, Z(M+1) \,, \qquad (1)$$

where π^{2M} is the normalization factor, $x, y \in \mathbb{R}^4$ and $Z(M+1)$ is **the same constant** that contributes to the $(M+1)$-loop order β-function in the $\phi^4_{D=4}$ theory. The first nontrivial terms $Z(3) = 6\zeta_3$ and $Z(4) = 20\zeta_5$ were analytically evaluated in [K.G. Chetyrkin, A.L. Kataev, F.V. Tkachov, **1980**][1] and [K.G. Chetyrkin, F.V. Tkachov, **1981**], respectively. The constant $Z(5) = \frac{441}{8}\zeta_7$ of the zig-zag graph with 4 loops was calculated by D.Kazakov in **1983** . Here $\zeta_k := \sum_{n \geq 1} \frac{1}{n^k}$.

[1] The "two-loop fish diagram" was firstly evaluated in [E.De Rafael, J.L.Rosner, 1974].

The 5 loop zig-zag diagram contribution $Z(6) = 168\zeta_9$ to the β-function (in 6-loop order) was found by D.Broadhurst in **1985** . Then D.Broadhurst and D.Kreimer in **1995** evaluated $Z(M+1)$ numerically up to $(M+1) = 10$ loops, and based on these data they formulated a remarkable conjecture that the constant $Z(M+1)$ is given by the following expression

$$Z(M+1) = 4C_M \sum_{p=1}^{\infty} \frac{(-1)^{(p-1)(M+1)}}{p^{2(M+1)-3}} = \\ = \begin{cases} 4\, C_M\, \zeta_{2M-1} \ \text{for} \ M = 2N+1\,, \\ 4\, C_M\, (1 - 2^{2(1-M)})\, \zeta_{2M-1} \ \text{for} \ M = 2N\,, \end{cases} \qquad \zeta_k = \sum_{n>1} \frac{1}{n^k} \tag{2}$$

where M is the number of loops in the zig-zag diagrams and $C_M = \frac{(2M)!}{(M+1)!\, M!}$ is the Catalan number. Finally, the very nontrivial proof of the Broadhurst-Kreimer conjecture was found by F. Brown and O. Schnetz in **2013,2015**; based on [J.M.Drummond (2012)].

In this report, by using methods of D-dimensional CFT, the integral presentations for 4-point and 2-point zig-zag Feynman graphs are deduced. It gives a possibility to compute exactly a special class of zig-zag 2- and 4-point Feynman diagrams (for any M) in ϕ^4_D theory. In particular we find new rather simple proof of the Broadhurst-Kreimer conjecture.

Let $\{\hat{q}^\mu_a,\ \hat{p}^\nu_b\}$ $(a, b = 1, ..., n)$ be generators of a set of the D-dimensional Heisenberg algebras $\mathcal{H}_a$ (a=1,...,n)

$$[\hat{q}^\mu_a,\ \hat{q}^\nu_b] = 0 = [\hat{p}^\mu_a,\ \hat{p}^\nu_b]\ , \qquad [\hat{q}^\mu_a,\ \hat{p}^\nu_b] = i\,\delta^{\mu\nu}\,\delta_{ab} \qquad (\mu, \nu = 1, ..., D)\ .$$

We introduce states $|x_a\rangle$ which diagonalize coordinates $\hat{q}^\mu_a$:

$$\hat{q}^\mu_a|x_a\rangle = x^\mu_a|x_a\rangle\ ,$$

and form a basis in the representation spaces V_a of subalgebras $\mathcal{H}_a$. We also introduce the dual states $\langle x_a|$ such that the orthogonality and completeness conditions are valid

$$\langle x_a|x'_a\rangle = \delta^D(x_a - x'_a), \qquad \int d^D x_a\ |x_a\rangle\langle x_a| = I_a\ ,$$

where I_a is the unit operator in V_a and there are no summations over indices a. So, we have the algebra $\mathcal{H}^{(n)} = \oplus^n_{a=1}\mathcal{H}_a$ which acts in the space $V_1 \otimes \cdots \otimes V_n$ with bases vectors $|x_1\rangle \otimes \cdots |x_n\rangle$.

We use operators $(\hat{q}_a)^{2\alpha} = (\sum_\mu \hat{q}_a^\mu \hat{q}_a^\mu)^\alpha$ and $(\hat{p}_a)^{2\beta} = (\sum_\mu \hat{p}_a^\mu \hat{p}_a^\mu)^\beta$ with non-integer powers α and β, which are understood as integral operators defined via their integral kernels $\langle x|\,(\hat{q})^{-2\alpha}\,|y\rangle = (x)^{-2\alpha}\,\delta^D(x-y)$ and

$$\langle x|\frac{1}{(\hat{p})^{2\gamma}}|y\rangle = \int \frac{d^D k}{(2\pi)^D}\,\frac{e^{ik(x-y)}}{(k)^{2\gamma}} = \frac{a(\gamma)}{(x-y)^{2\gamma'}}\;,$$

$$a(\gamma) := \frac{2^{-2\gamma}}{\pi^{D/2}}\,\frac{\Gamma(\gamma')}{\Gamma(\gamma)}\;, \qquad \gamma' := D/2-\gamma\;.$$

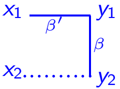
Consider the algebra $\mathcal{H}^{(2)} = \mathcal{H}_1 + \mathcal{H}_2$, which acts in $V_1 \otimes V_2$ with basis $|x_1, x_2\rangle := |x_1\rangle \otimes |x_2\rangle$ and introduce graph building operator:

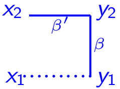
$$\hat{Q}_{12}^{(\beta)} := \frac{1}{a(\beta)}\,\mathcal{P}_{12}\,(\hat{p}_1)^{-2\beta}\,(\hat{q}_{12})^{-2\beta}\;,$$

where $(\hat{q}_{12})^2 = (\hat{q}_1^\mu - \hat{q}_2^\mu)(\hat{q}_1^\mu - \hat{q}_2^\mu)$ and $\mathcal{P}_{12}$ is the permutation operator

$$\mathcal{P}_{12}\,\hat{q}_1 = \hat{q}_2\,\mathcal{P}_{12}\;,\quad \mathcal{P}_{12}\,\hat{p}_1 = \hat{p}_2\,\mathcal{P}_{12}\;,\quad \mathcal{P}_{12}|x_1,x_2\rangle = |x_2,x_1\rangle\;,\quad (\mathcal{P}_{12})^2 = I\;.$$

We depict the kernel $\langle x_1, x_2 | \hat{Q}_{12}^{(\beta)} | y_1, y_2 \rangle$ of the operator $\hat{Q}_{12}^{(\beta)}$ as following

$\mathcal{P}_{12} \cdot$ 

$=$ 

$$= \frac{1}{a(\beta)} \langle x_1, x_2 | \, \mathcal{P}_{12} \, (\hat{p}_1)^{-2\beta} \, (\hat{q}_{12})^{-2\beta} \, | y_1, y_2 \rangle =$$

$$= \frac{1}{(x_2 - y_1)^{2\beta'} (y_1 - y_2)^{2\beta}} \, \delta^D (x_1 - y_2) \; ,$$

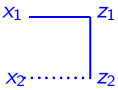
where

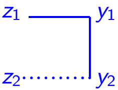
$$x_1 \cdots\cdots\cdots x_2 \;=\; \delta^D (x_1 - x_2) \; , \qquad x_1 \overset{\beta}{\text{———}} x_2 \;=\; (x_1 - x_2)^{-2\beta} \; .$$

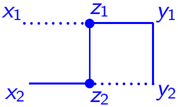
Now we note that $Q_{12}^{(\beta)}$ is the graph building operator for the planar zig-zag Feynman graphs:

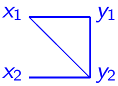
for 2 loops

$$\langle x_1, x_2 | \hat{Q}_{12} \underbrace{\;\; \hat{Q}_{12} | y_1, y_2 \rangle}_{\displaystyle \int dz_1 dz_2 | z_1, z_2 \rangle \langle z_1, z_2 |} =$$

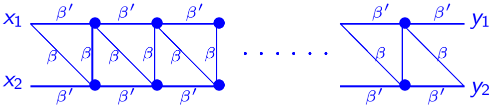
$= \int dz_1 dz_2 \;\; \mathcal{P}_{12} \cdot$ 

$\cdot \, \mathcal{P}_{12} \cdot$ 

$=$ 

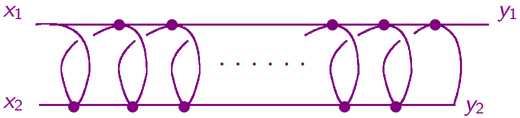
$=$ 

for even loops $(2N-2)$



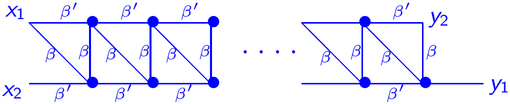
$$= \langle x_1, x_2 | (\hat{Q}_{12}^{(\beta)})^{2N} | y_1, y_2 \rangle (y_1 - y_2)^{2\beta} =$$

$=$



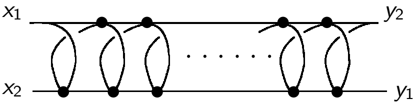
;

for odd loops $(2N-1)$



$$= \langle x_1, x_2 | (\hat{Q}_{12}^{(\beta)})^{2N+1} | y_1, y_2 \rangle (y_1 - y_2)^{2\beta} =$$

$=$

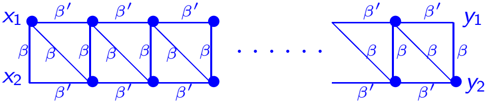


.

The vertices • denote the integration over $\mathbb{R}^D$.

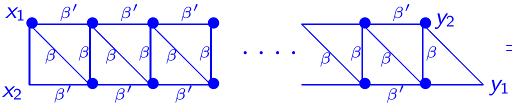
We stress that these Feynman integrals represent the contribution to the 4-point correlation functions in bi-scalar D-dimensional "fishnet" theory [V.Kazakov a.o. (2016,2018)]. For clarity, we present the zig-zag diagrams in the form of the spiral graphs having the cylindrical topology. We also stress that integral kernels, shown in the pictures, in the case $D=4$ and $\beta=1$, contribute to Green's functions of the standard $\phi^4_{D=4}$ field theory.

The next important statement is that $Q_{12}^{(\beta)}$ is also the graph building operator for the integrals of the planar zig-zag two-point Feynman graphs: for even number of loops $2N$



$$= \int d^D x_1 d^D y_2 \frac{\langle x_1, x_2 | (\hat{Q}_{12}^{(\beta)})^{2N} | y_1, y_2 \rangle}{(x_1 - x_2)^{2\beta}} \;;$$

for odd number of loops $(2N+1)$



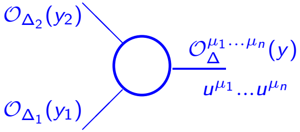
$$= \int d^D x_1 d^D y_2 \frac{\langle x_1, x_2 | (\hat{Q}_{12}^{(\beta)})^{2N+1} | y_1, y_2 \rangle}{(x_1 - x_2)^{2\beta}} \,.$$

Below we use these representations to evaluate exactly the corresponding class of 2-point and 4-point Feynman diagrams.

To evaluate all these diagrams we need to find eigenvalues and complete set of eigenvectors of $\hat{Q}_{12}^{(\beta)}$.

Remark. The elements $H_\beta := \mathcal{P}_{12}\, \hat{Q}_{12}^{(\beta)} \equiv (\hat{p}_1)^{-2\beta}(\hat{q}_{12})^{-2\beta}$ form a commutative set of operators $[H_\alpha,\ H_\beta] = 0$ $(\forall \alpha, \beta)$.

To find eigenvectors for the graph building operator $Q_{12}^{(\beta)}$ we consider the standard 3-point correlation function (in a CFT) of three fields $\mathcal{O}_{\Delta_1}$, $\mathcal{O}_{\Delta_2}$ and $\mathcal{O}_{\Delta}^{\mu_1\cdots\mu_n}$, where $\mathcal{O}_{\Delta_{1,2}}$ are scalar fields with conf. dimensions Δ_1, Δ_2, while $\mathcal{O}_{\Delta}^{\mu_1\cdots\mu_n}$ – (symmetric, traceless and transverse) tensor field with conf. dimension Δ. The conformally invariant expression of this correlation function (up to a normalization) is unique and well known [V.K. Dobrev, G. Mack, V.B. Petkova, S.G. Petrova, I.T. Todorov (1976,1977); E.S.Fradkin , M.Y.Palchik (1978);...]



$$= u^{\mu_1}\ldots u^{\mu_n}\, \langle \mathcal{O}_{\Delta_1}(y_1)\, \mathcal{O}_{\Delta_2}(y_2)\, \mathcal{O}_{\Delta}^{\mu_1\cdots\mu_n}(y)\rangle =$$

$$= \frac{\left(\frac{(u,y-y_1)}{(y-y_1)^2} - \frac{(u,y-y_2)}{(y-y_2)^2}\right)^n}{(y_1-y_2)^{2\eta}(y-y_1)^{2\delta}(y-y_2)^{2\rho}} \,,$$

where $u \in \mathbb{C}^D$ such that $(u,u) = u^\mu u^\mu = 0$ and

$$\eta = \frac{1}{2}(\Delta_1+\Delta_2-\Delta+n)\,, \;\; \delta = \frac{1}{2}(\Delta_1+\Delta-\Delta_2-n)\,, \;\; \rho = \frac{1}{2}(\Delta_2+\Delta-\Delta_1-n)\,.$$

We need the special form of the 3-point function (conformal triangle) when parameters $\Delta, \Delta_1, \Delta_2$ are related to two numbers $\alpha \in \mathbb{C}$, $\beta \in \mathbb{R}$:

$$\Delta_1 = \frac{D}{2}\,, \quad \Delta_2 = \frac{D}{2} - \beta\,, \quad \Delta = D - 2\alpha - \beta + n\,,$$

so we have for conformal triangle:

$$\langle y_1, y_2 | \Psi^{(n,u)}_{\alpha,\beta}(y)\rangle \;:=\; \frac{\left(\frac{(u,y-y_1)}{(y-y_1)^2} - \frac{(u,y-y_2)}{(y-y_2)^2}\right)^n}{(y_1-y_2)^{2\alpha}(y-y_1)^{2\alpha'}(y-y_2)^{2(\alpha+\beta)'}}\,.$$

Proposition 1. *The wave function* $|\Psi^{(n,u)}_{\alpha,\beta}(y)\rangle = u^{\mu_1}\cdots u^{\mu_n}|\Psi^{\mu_1\dots\mu_n}_{\alpha,\beta}(y)\rangle$ $(\forall\,\alpha,\beta \in \mathbb{C})$ *is the eigenvector for the graph building operator*

$$\hat{Q}^{(\beta)}_{12}\;|\Psi^{(n,u)}_{\alpha,\beta}(y)\rangle = \tau(\alpha,\beta,n)\;|\Psi^{(n,u)}_{\alpha,\beta}(y)\rangle\,,$$

with the eigenvalue

$$\tau(\alpha,\beta,n) = (-1)^n\,\pi^{D/2}\,\frac{\Gamma(\beta)\Gamma(\alpha)\Gamma\big((\alpha+\beta)'+n\big)}{\Gamma(\beta')\Gamma(\alpha'+n)\Gamma(\alpha+\beta)}\,.$$

The analogous statement, for $D=4$ and $\beta=1$, was made in [N.Gromov, V.Kazakov, and G.Korchemsky (2018)].

Note that with respect to the standard Hermitian scalar product the operator $\hat{Q}^{(\beta)}_{12} = \frac{1}{a(\beta)}\,\mathcal{P}_{12}\,(\hat{p}_1)^{-2\beta}\,(\hat{q}_{12})^{-2\beta}$ for $\beta \in \mathbb{R}$ is Hermitian up to the equivalence transformation:

$$(\hat{Q}^{(\beta)}_{12})^{\dagger} = \frac{1}{a(\beta)}\,(\hat{q}_{12})^{-2\beta}\,(\hat{p}_1)^{-2\beta}\,\mathcal{P}_{12} = U\,\hat{Q}^{(\beta)}_{12}\,U^{-1}\ ,$$

$$U := \mathcal{P}_{12}\,(\hat{q}_{12})^{-2\beta} = (\hat{q}_{12})^{-2\beta}\,\mathcal{P}_{12}\,.$$

Thus, we modify the scalar product in $V_1 \otimes V_2$

$$\langle\overline{\Psi}|\Phi\rangle := \langle\Psi|\,U\,|\Phi\rangle = \int d^4x_1 d^4x_2 \frac{\Psi^*(x_2\,,x_1)\,\Phi(x_1\,,x_2)}{(x_1 - x_2)^{2\beta}}\ ,$$

where $\beta \equiv D - \Delta_1 - \Delta_2$ and with respect to this scalar product the operator $\hat{Q}^{(\beta)}_{12}$ is Hermitian. We introduced the special conjugation

$$\langle\overline{\Psi}| := \langle\Psi|\,U = \langle\Psi|\,(\hat{q}_{12})^{-2\beta}\,\mathcal{P}_{12}\ ,$$

and operator U plays the role of the metric in $V_1 \otimes V_2$.

Complex parameter α should be partially fixed.

Indeed, we define operator

$$\hat{\mathrm{D}} = \frac{i}{2}\sum_{a=1}^{2}(\hat{q}_a\hat{p}_a + \hat{p}_a\hat{q}_a) + \frac{1}{2}(y^\mu\,\partial_{y^\mu} + \partial_{y^\mu}\,y^\mu) - \beta,$$

such that $[\hat{Q}^{(\beta)}_{12},\ \hat{\mathrm{D}}] = 0$ and it is diagonalized simultaneously with $\hat{Q}^{(\beta)}_{12}$:

$$\hat{\mathrm{D}}\,|\Psi^{(n,u)}_{\alpha,\beta}(y)\rangle = \Big(2\alpha + \beta - \frac{1}{2}D - n\Big)\,|\Psi^{(n,u)}_{\alpha,\beta}(y)\rangle\ .$$

For $\beta \in \mathbb{R}$, we obtain $\hat{\mathrm{D}}^\dagger = -U\,\hat{\mathrm{D}}\,U^{-1}$. Thus, operator $\hat{\mathrm{D}}$ is anti-Hermitian with respect to the new scalar product $\langle\Psi|\,U\,|\Phi\rangle$, and it gives the condition for eigenvalues of $\hat{\mathrm{D}}$:

$$2(\alpha^* + \alpha) = 2n + D - 2\beta \quad\Rightarrow\quad \alpha = \frac{1}{2}\,(n + D/2 - \beta) - i\nu\ , \quad \nu \in \mathbb{R}\ .$$

So, we see that the eigenvalue problem for $\hat{Q}^{(\beta)}_{12}$ is characterized by two real numbers $\beta, \nu \in \mathbb{R}$ and we have $\Delta = \frac{D}{2} + 2i\nu$.

It is remarkable fact that, under these conditions, the eigenvalue is real

$$\tau(\alpha,\beta,n) = (-1)^n \frac{\pi^{D/2}\Gamma(\beta)\,\Gamma(\frac{D}{4}+\frac{n}{2}-\frac{\beta}{2}+i\nu)\,\Gamma(\frac{D}{4}+\frac{n}{2}-\frac{\beta}{2}-i\nu)}{\Gamma(\beta')\,\Gamma(\frac{D}{4}+\frac{n}{2}+\frac{\beta}{2}+i\nu)\,\Gamma(\frac{D}{4}+\frac{n}{2}+\frac{\beta}{2}-i\nu)}\,.$$

In view of this, we introduce concise notation

$$|\Psi^{(n,u)}_{\nu,\beta,y}\rangle := |\Psi^{(n,u)}_{\alpha,\beta}(y)\rangle = u^{\mu_1}\cdots u^{\mu_n}|\Psi^{\mu_1\dots\mu_n}_{\alpha,\beta}(y)\rangle,$$

$$\Psi^{(n,u)}_{\nu,\beta,y}(x_1\,,x_2) := \langle x_1\,,x_2|\Psi^{(n,u)}_{\nu,\beta,y}\rangle\,.$$

Since the eigenvalue τ is real (it is invariant under the transformation $\nu\to-\nu$), two eigenvectors $|\Psi^{(n,u)}_{\nu,\beta,x}\rangle$ and $|\Psi^{(m,v)}_{\lambda,\beta,y}\rangle$, having different eigenvalues τ (e.g. $n\neq m$ and $\lambda\neq\pm\nu$), should be orthogonal to each other with respect to the new scalar product. Indeed, we have the following orthogonality condition for two conformal triangles (see, e.g., [V.K. Dobrev, G. Mack, I.T.Todorov, M.C.Mintchev, V.B.Petkova (1976-1978); N. Gromov, V. Kazakov, and G. Korchemsky (2019)])

$$\overline{\langle\Psi^{(m,v)}_{\lambda,\beta,y}}|\Psi^{(n,u)}_{\nu,\beta,x}\rangle = \int d^Dx_1\,d^Dx_2\,\langle\Psi^{(m,v)}_{\lambda,\beta,y}|U|x_1x_2\rangle\langle x_1x_2|\Psi^{(n,u)}_{\nu,\beta,x}\rangle =$$

$$
= \int d^D x_1 \, d^D x_2 \frac{(\Psi^{(m,v)}_{\lambda,\beta,y}(x_2\,,x_1))^* \; \Psi^{(n,u)}_{\nu,\beta,x}(x_1\,,x_2)}{(x_1 - x_2)^{2(D-\Delta_1-\Delta_2)}} =
$$

$$
= \delta_{nm} C_1(n\,,\nu)\,\delta_{nm}\,\delta(\nu-\lambda)\,\delta^D(x-y)\,(u,v)^n +
$$

$$
+ C_2(n\,,\nu)\,\delta_{nm}\,\delta(\nu+\lambda)\,\frac{\left((u,v) - 2\frac{(u,x-y)(v,x-y)}{(x-y)^2}\right)^n}{(x-y)^{2(D/2+2i\nu)}}, \quad (3)
$$

where $(u,v) = u^\mu v^\mu$, $\beta = D - \Delta_1 - \Delta_2 = \Delta_1 - \Delta_2$ and

$$
C_1(n\,,\nu) = \frac{(-1)^n \, 2^{1-n} \, \pi^{3D/2+1} \, n! \, \Gamma\left(2i\nu\right) \Gamma\left(-2i\nu\right)}{\Gamma\left(\frac{D}{2}+n\right) \left(\left(\frac{D}{2}+n-1\right)^2 + 4\nu^2\right) \Gamma\left(\frac{D}{2}+2i\nu-1\right) \, \Gamma\left(\frac{D}{2}-2i\nu-1\right)} \quad (4)
$$

We note that the coefficient C_1 is independent on β and plays the important role as the inverse of the Plancherel measure used in the completeness condition (see below). In contrast to this, the coefficient C_2 in (3) depends on β, but the explicit form for C_2 will not be important for us.

$$C_2(n,\nu) = 2\pi^{D+1}\,\frac{n!}{2^n}\,\frac{\Gamma\left(\frac{D}{4}-\frac{\Delta_1-\Delta_2}{2}+\frac{n}{2}-i\nu\right)}{\Gamma\left(\frac{D}{4}-\frac{\Delta_1-\Delta_2}{2}+\frac{n}{2}+i\nu\right)}\,\frac{\Gamma\left(\frac{D}{4}+\frac{\Delta_1-\Delta_2}{2}+\frac{n}{2}-i\nu\right)}{\Gamma\left(\frac{D}{4}+\frac{\Delta_1-\Delta_2}{2}+\frac{n}{2}+i\nu\right)}\,\cdot$$
$$\frac{\Gamma(2i\nu)\,\Gamma\left(\frac{D}{2}+2i\nu-1+n\right)}{\Gamma\left(\frac{D}{2}+n-2i\nu\right)\Gamma\left(\frac{D}{2}+2i\nu-1\right)\Gamma\left(\frac{D}{2}+n\right)} \tag{5}$$

Completeness (or resolution of unity I) for the basis of the eigenfunctions $|\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}\rangle$ is written as [V.K. Dobrev, G. Mack, I.T.Todorov, M.C.Mintchev, V.B.Petkova (1976-1978); N. Gromov, V. Kazakov, and G. Korchemsky (2019)]

$$I = \sum_{n=0}^{\infty}\int_0^{\infty}\frac{d\nu}{C_1(n,\nu)}\int d^D x\,|\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}\rangle\langle\overline{\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}}| =$$
$$= \sum_{n=0}^{\infty}\int_0^{\infty}\frac{d\nu}{C_1(n,\nu)}\int d^D x\,|\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}\rangle\langle\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}|\,U\,.$$

Substitution of this resolution of unity into expressions for zig-zag 4-point Feynman graphs gives (here M is a number of loops)

$$G_4^{(M)}(x_1,x_2;y_1,y_2) = \langle x_1,x_2|\big(\hat{Q}_{12}^{(\beta)}\big)^M|y_1,y_2\rangle(y_1-y_2)^{2\beta} =$$

$$= \sum_{n=0}^{\infty}\int_0^{\infty}\frac{d\nu}{C_1(n,\nu)}\int d^Dx\,\langle x_1,x_2|\big(\hat{Q}_{12}^{(\beta)}\big)^M|\Psi_{\nu,\beta,x}^{\mu_1\cdots\mu_n}\rangle\langle\Psi_{\nu,\beta,x}^{\mu_1\cdots\mu_n}|\,U|y_1,y_2\rangle(y_1-y_2)^{2\beta} =$$

$$= \sum_{n=0}^{\infty}\int_0^{\infty}d\nu\,\frac{\big(\tau(\alpha,\beta,n)\big)^M}{C_1(n,\nu)}\int d^Dx\,\langle x_1,x_2|\Psi_{\nu,\beta,x}^{\mu_1\cdots\mu_n}\rangle\langle\Psi_{\nu,\beta,x}^{\mu_1\cdots\mu_n}|y_2,y_1\rangle\,, \qquad (6)$$

where the integral over x in the right hand side of (6) is evaluated in terms of conformal blocks [F.A.Dolan, H.Osborn (2001,2004); H.Osborn, A.Petkou (1994)] (in four-dimensional case, this integral was considered in detail by [N. Gromov, V. Kazakov, and G. Korchemsky (2019)]).

Further we make use the relation between 4-point zig-zag functions $G_4^{(M)}(x_1,x_2;y_1,y_2)$ given in (6) and 2-point zig-zag functions $G_2^{(M)}(x_2,y_1)$

$$G_2^{(M)}(x_2,y_1) = \int d^Dx_1\,d^Dy_2\,\frac{\langle x_1,x_2|\big(\hat{Q}_{12}^{(\beta)}\big)^M|y_1,y_2\rangle}{(x_1-x_2)^{2\beta}}$$

Finally, we write explicit expressions for 2-point M-loop zig-zag diagrams as following

$$G_2^{(M)}(x_2,y_1) = \int d(x_1,y_2,x) \sum_{n=0}^{\infty} \int_0^{\infty} \frac{d\nu}{C_1(n,\nu)} \frac{\langle x_1,x_2|\big(\hat{Q}_{12}^{(\beta)}\big)^M |\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}\rangle\langle\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}|\,U|y_1,y_2\rangle}{(x_1-x_2)^{2\beta}} =$$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} d\nu \frac{(\tau(\alpha,\beta,n))^M}{C_1(n,\nu)} \int d(x_1,y_2,x) \frac{\langle x_1,x_2|\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}\rangle\langle\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}|y_2,y_1\rangle}{(x_1-x_2)^{2\beta}(y_1-y_2)^{2\beta}} =$$

$$= \frac{1}{(x_2-y_1)^{2\beta}} \frac{\Gamma(D/2-1)}{\Gamma(D-2)} \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(n+D-2)}{2^n\Gamma(n+D/2-1)} \int_0^{\infty} d\nu\, \frac{\tau^{M+3}(\alpha,\beta,n)}{C_1(n,\nu)}\,, \quad (7)$$

where $d(x_1,y_2,x) = d^Dx_1 d^Dy_2\, d^Dx$ and we apply the integral

$$\int d^Dx_1 d^Dy_2\, d^Dx \frac{\langle x_1,x_2|\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}\rangle\langle\Psi^{\mu_1\cdots\mu_n}_{\nu,\beta,x}|y_2,y_1\rangle}{(x_1-x_2)^{2\beta}(y_1-y_2)^{2\beta}} =$$

$$= \frac{(-1)^n\Gamma(n+D-2)\Gamma(D/2-1)}{2^n\Gamma(n+D/2-1)\Gamma(D-2)} \frac{\tau^3(\alpha,\beta,n)}{(x_2-y_1)^{2\beta}}\,. \quad (8)$$

The integral over ν in the right hand side of (7) for $\beta=1$ and even $D>2$ can be evaluated explicitly and gives the linear combination of ζ-values with rational coefficients.

To prove BK conjecture we need to consider the special case $\beta = 1$ and $D = 4$. In this case we have $\alpha = \frac{n+1}{2} - i\nu$ and the eigenvalue is simplified

$$\tau(\nu, n) := \tau(\alpha, \beta, n)|_{D=4,\beta=1} = \frac{(-1)^n (2\pi)^2}{(1+n)^2 + 4\nu^2} \, .$$

The coefficient C_1 in the definition of the Plancherel mesure for $D = 4$ and $\beta = 1$ is reduced to

$$C_1(n, \nu) = \frac{\pi^5}{2^{n+3}(1+n)\,\nu^2}\,\tau(\nu, n) \, .$$

Finally we substitute $\tau(\nu, n)$, $C_1(n, \nu)$ into (7), integrate over ν and obtain

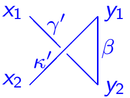
$$G_2(x_2, y_1)|_{D=4,\beta=1} \;=\; \frac{4\pi^{2M}}{(x_2 - y_1)^2}\, C_M \sum_{n=0}^{\infty} (-1)^{n(M+1)} \frac{1}{(n+1)^{2M-1}} \;, \quad (9)$$

where $C_M = \frac{1}{(M+1)}\binom{2M}{M}$ is a Catalan number. The relation (9) is equivalent the Broadhurst and Kreimer formula for the M loop zig-zag diagram (it corresponds to the $(M+1)$ loop contribution to the β-function in $\phi^4_{D=4}$ theory).

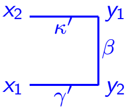
The generalization of the graph building operator is $(\beta = \kappa + \gamma)$:

$$Q_{12}^{(\kappa,\gamma)} := \frac{1}{a(\kappa)}\, \mathcal{P}_{12}\, \hat{p}_1^{-2\kappa}\, \hat{p}_2^{-2\gamma}\, \hat{q}_{12}^{-2\beta}\ .$$

We depict the integral kernel of the D-dimensional operator $Q_{12}^{(\kappa,\gamma)}$ as follows ($\beta = \kappa + \gamma$, $\beta' := D/2 - \beta$)



$=$

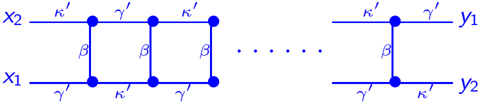


$$= \langle x_1, x_2 | Q_{12}^{(\kappa,\gamma)} | y_1, y_2 \rangle =$$

$$= \tfrac{1}{a(\kappa)} \cdot \langle x_1, x_2 |\, \mathcal{P}_{12}\, \hat{p}_1^{-2\kappa}\, \hat{p}_2^{-2\gamma}\, \hat{q}_{12}^{-2\beta}\, | y_1, y_2 \rangle =$$

$$= \frac{a(\gamma)}{(x_2 - y_1)^{2\kappa'}\, (x_1 - y_2)^{2\gamma'}\, (y_1 - y_2)^{2(\kappa+\gamma)}}\ .$$

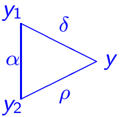
Thus, the operator $Q_{12}^{(\kappa,\gamma)}$ is the graph building operator for the ladder diagrams ($\beta = \kappa + \gamma$)



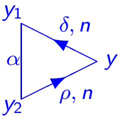
$$= \langle x_1, x_2 | (\hat{Q}_{12}^{(\kappa,\gamma)})^{2N} | y_1, y_2 \rangle (y_1 - y_2)^{2\beta}$$

Proposition 2. The eigenfunction for the operator $Q_{12}^{(\kappa,\gamma)}$ is given by 3-point correlation function (conformal triangle)

$$\langle y_1, y_2 | \Psi^{(n,u)}_{\delta,\rho}(y)\rangle \;=$$



$$\cdot \left(\frac{(u, y-y_1)}{(y-y_1)^2} - \frac{(u, y-y_2)}{(y-y_2)^2} \right)^n \;\equiv$$



$$Q_{12}^{(\kappa,\gamma)} \, |\Psi^{(n,u)}_{\delta,\rho}(y)\rangle = \bar{\tau}(\kappa,\gamma;\delta,\alpha;n) \, |\Psi^{(n,u)}_{\delta,\rho}(y)\rangle \; .$$

where $\alpha + \rho = \kappa'$, $\alpha + \delta = \gamma'$ and $\bar{\tau}(\kappa,\gamma;\delta,\alpha;n)$ is an eigenvalue

$$\bar{\tau}(\kappa,\gamma;\delta,\alpha;n) = \tau(\delta',\kappa,n) \cdot \tau(\alpha,\gamma,n) \, ,$$

$$\tau(\alpha,\beta,n) = (-1)^n \, \frac{\pi^{D/2}\Gamma(\beta)\Gamma(\alpha)\Gamma(\alpha'-\beta+n)}{\Gamma(\beta')\Gamma(\alpha'+n)\Gamma(\alpha+\beta)}$$

Remark 1. The conjugation transformation of the graph building operator $Q_{12}^{(\kappa,\gamma)}$ gives the operator ($\beta = \kappa + \gamma$)

$$Q_{12}^{(\kappa,\gamma,\xi)} = \hat{q}_{12}^{-2\xi} Q_{12}^{(\kappa,\gamma)} \hat{q}_{12}^{2\xi} = \frac{1}{a(\kappa)} \mathcal{P}_{12}\ \hat{q}_{12}^{-2\xi}\ \hat{p}_1^{-2\kappa}\ \hat{p}_2^{-2\gamma}\ \hat{q}_{12}^{2(\xi-\beta)} \equiv R_{12}^{(\kappa,\gamma,\xi)}(\beta)\ ,$$

which has the same spectrum. This conjugated operator is related to the R-operator [D. Chicherin, S. Derkachov, A. P. Isaev (2013)]

$$\mathcal{P}_{12}\ R_{12}^{(\kappa,\gamma,\xi)} = \mathcal{P}_{12}\ Q_{12}^{(\kappa,\gamma,\xi)} = \frac{1}{a(\kappa)} \hat{q}_{12}^{-2\xi}\ \hat{p}_1^{-2\kappa}\ \hat{p}_2^{-2\gamma}\ \hat{q}_{12}^{2(\xi-\beta)} \equiv \check{R}_{12}^{(\gamma,\xi)}(\beta)$$

and it is a conformal invariant solution of the Yang-Baxter equation

$$\check{R}_{12}^{(\gamma,\xi)}(\beta-\eta)\check{R}_{23}^{(\gamma,\xi)}(\beta)\check{R}_{12}^{(\gamma,\xi)}(\eta) = \check{R}_{23}^{(\gamma,\xi)}(\eta)\check{R}_{12}^{(\gamma,\xi)}(\beta)\check{R}_{23}^{(\gamma,\xi)}(\beta-\eta)\ .$$

The operator $R_{12}^{(\kappa,\gamma,\xi)}(\beta)$ intertwines two spaces $V_{\Delta_1} \otimes V_{\Delta_2} \to V_{\Delta_2} \otimes V_{\Delta_1}$, where V_Δ is the space of scalar conf. fields with dimensions Δ. Let we have $V_{\Delta_1} \otimes V_{\Delta_2} = \sum_{\Delta,n} V_\Delta^{(n)}$. Thus, eigenfunctions of $\hat{q}_{12}^{-2\xi} Q_{12}^{(\kappa,\gamma)} \hat{q}_{12}^{2\xi} = R_{12}^{(\kappa,\gamma,\xi)}(\beta)$ should describe the fusion of two scalar conformal fields with dimensions Δ_1 and Δ_2 into the composite tensor conformal field with dimension Δ.

Remark 2. The special case (for $D = 1$ and $\kappa = \gamma$) of this R-operator underlies Lipatov's integrable model of the high-energy asymptotics of multicolor QCD.

$$a(\kappa) R_{12}^{(\kappa,\xi)} = \hat{q}_{12}^{-2(\kappa+\xi)} \, \hat{p}_1^{-2\kappa} \, \hat{p}_2^{-2\kappa} \, \hat{q}_{12}^{-2(\kappa-\xi)} \quad \stackrel{\kappa \to 0}{\to} \quad 1 - \kappa h_{12}^{(\xi)} + \dots ,$$

$$h_{12}^{(\xi)} = 2 \ln q_{12}^2 + \hat{q}_{12}^{-2\xi} \ln(\hat{p}_1^2 \, \hat{p}_2^2) \, \hat{q}_{12}^{2\xi} ,$$

where $h_{12}^{(\xi)}$ is a local density of the Lipatov's Hamiltonian.

Conclusion.

In this report, we demonstrated how the recent progress in the investigations of the multidimensional conformal field theories (CFT) can be applied, e.g., in the analytical evaluations of massless Feynman diagrams. We believe that the approach described here gives the universal method of the evaluation of contributions into the special class of correlation functions and critical exponents in various CFT. We also wonder if it is possible to apply our D-dimensional generalizations to evaluation similar 4-points functions (with fermions) that arise in the generalized "fishnet" model, in double scaling limit of γ-deformed $N = 4$ SYM theory.

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