

# Super Yangians and Quantum Loop Superalgebras

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# Outline

## 1 Motivation

- Introduction
- Super Yangian and Quantum Loop Superalgebra

## 2 Our Results/Contribution

- Main Results
- Quantum affine Weyl groupoid and classification of Hopf superalgebra structures
- Main Results for representations

The object of our study is the Super Yangians, which related to the rational integrable models of Quantum Field Theory and

connections with other mathematical objects.

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We describe associative superalgebra and Hopf superalgebra structures on super Yangians and quantum loop superalgebra. We use an isomorphism (of associative superalgebras) between the completions of the Yangian of a special linear Lie superalgebra and the quantization of a loop superalgebra of a special linear Lie superalgebra. This last result is natural generalization of result by Sachin Gautam and Valerio Toledano Laredo on isomorphism between completions of Yangian of general linear algebra and quantum loop algebra of  $L\mathfrak{gl}_n$  ([4], see also the paper [6] for further development of this theory). S. Gautam, V. Toledano Laredo Meromorphic tensor equivalence for Yangians and quantum loop algebras. – Publ. Math. Inst. Hautes Études Sci., 125(2017), 267–337.

We also consider separately the problem on categories representation equivalence. We use idea by S. Gautam and V. Toledano-Laredo that the matrix coefficient of image of Cartan field under representation  $\rho(\psi(z)) = \frac{qz - q^{-1}a}{z - a}$  is the monodromy of the additive difference equation  $f(u+1) = Af(u)$ , where  $A(u) = \frac{u + \hbar - a}{u - a}$  is image Cartan field of Yangian (in the case of even roots) under representation. The main our results related to the problem of description Hopf superalgebra structures on super Yangian and quantum loop superalgebra and also on quantized enveloping superalgebras of some basic Lie superalgebras. The main tool of such description is a quantum Weyl groupoid. We define the structures of quantum Weyl groupoid and quantum affine Weyl groupoid in terms of action of elements of these groupoids and describe Hopf superalgebra structures on above mentioned quantum superalgebras.

The report presents a work that is a natural continuation of [3], [10]. At the end it is briefly presented some results related to the super Yangian of affine Kac-Moody algebras

# Short description. Basic problems.

Frame subtitles are optional. Use upper- or lowercase letters.

- Lie Superalgebras
- 2. Weyl groupoid .
- 3. Super Yangians and Quantum Loop Superalgebras.
- 4. Isomorphism between completions of Super Yangian and Quantum Loop Superalgebra.
- 5. Drinfeld comultiplication.
- 6. Representation Theory of Super Yangian and Quantum Loop Superalgebra.
- 7. Main result. Description of Hopf superalgebra structures.



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# Lie Superalgebras and Quantum Superalgebras.

- Bisuperalgebra structures.
- Problem of quantization
- Problem of description representations

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## Definition

The Lie superalgebra  $\mathfrak{g} = A(m, n)$  is generated by the generators:  $h_i, x_i^\pm, i \in I$ . The generators  $x_{m+1}^\pm$  are odd, while the remaining generators are even, that is, the parity function  $p$  takes on the following values:

$p(h_i) = 0, i \in I, p(x_j^\pm) = 0, j \neq m+1, p(x_{m+1}^\pm) = 1$ . These generators satisfy the following defining relations:

$$[h_i, h_j] = 0, \quad [h_i, x_j^\pm] = \pm a_{ij} x_j^\pm,$$

$$[x_i^+, x_j^-] = \delta_{ij} h_i,$$

$[[x_m^\pm, x_{m+1}^\pm], [x_{m+1}^\pm, x_{m+2}^\pm]] = 0$  if  $\alpha_{m+1}$  (This definition is suitable for arbitrary Dynkin diagram if  $\alpha_{m+1}$  be an arbitrary odd simple root.)

$$ad^{1-\tilde{a}_{ij}}(x_i^\pm)x_j^\pm = 0.$$

$$[x_i^\pm, [x_i^\pm, x_j^\pm]] = 0 \text{ if } |i-j| = 1, [x_i^\pm, x_j^\pm] = 0 \text{ elsewhere } (i \neq j).$$

# Lie bisuperalgebras and quantization.

In short, a Lie bialgebra is a Lie algebra  $\mathfrak{g}$  such that the dual vector space  $\mathfrak{g}^*$  is also a Lie algebra. Moreover, the bracket on  $\mathfrak{g}$  induced by the operation of the bracket on  $\mathfrak{g}^*$  it is one cocycle with coefficients in the module  $\mathfrak{g} \otimes \mathfrak{g}$ , where module structure is defined by  $a \cdot (b \otimes c) = [a \otimes 1 + 1 \otimes a, b \otimes c] = [a, b] \otimes c + b \otimes [a, c]$ . The main object of my report are Yangian and quantum loop algebra which are quantization of the following bialgebras, correspondingly: current Lie algebra  $\mathfrak{g}[t]$  and loop Lie algebra  $\mathfrak{g}[t, t^{-1}]$  with cobrackets which can be defined using some standard  $r$ -matrices or by equivalent way using language of Manin triples. Let  $A_{\hbar}$  be a QUE superalgebra:  $A_{\hbar}/\hbar A_{\hbar} \cong U(\mathfrak{g})$ . Then the Lie superalgebra  $\mathfrak{g}$  has a natural structure of a Lie superbialgebra defined by  $\delta(x) = \hbar^{-1}(\Delta(\tilde{x}) - \Delta^{op}(\tilde{x})) \mod \hbar$ , where  $x \in \mathfrak{g}$ ,  $\tilde{x} \in A_{\hbar}$  is a preimage of  $x$ ,  $\Delta$  is a comultiplication in  $A$  and  $\Delta^{op} := \tau_{U(\mathfrak{g}), U(\mathfrak{g})} \circ \Delta$ , where  $\tau_{U(\mathfrak{g}), U(\mathfrak{g})}$  be a (super)permutation of tensor multipliers.

More precisely, we will deal with the superanalogues of the Lie



bialgebras introduced above, that is,  $\mathfrak{g}$  will be some Lie superalgebra. More precisely, we restrict ourselves to considering the particular case when  $\mathfrak{g} = \mathfrak{sl}(m+1, n+1) = A(m, n)$  is a special linear superalgebra.

- Lie bisuperalgebra  $\mathfrak{A} = \mathfrak{g}[u]$ , where  $\mathfrak{g} = \mathfrak{sl}(m+1, n+1)$
- Cobracket is defined by  $\delta(a) = [a(u) \otimes 1 + 1 \otimes a(v), \frac{t}{u-v}]$ , where  $t$  is Casimir operator. .

### Two main examples

- Lie bisuperalgebra  $\mathfrak{B} = \mathfrak{g}[u, u^{-1}]$ .
- Cobracket

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# Basic definitions. Super Yangian

## Definition

Let,  $\mathfrak{g} = A(m, n)$ ,  $Y_{\hbar}(\mathfrak{g})$  be an associative (super)algebra generated by generators  $\{x_{i,r}^{\pm}, h_{i,r}\}_{i \in I, r \in \mathbb{Z}_+}$ , which satisfied the following defining relations:

Y1) For all  $i, j \in I$  and  $r, s \in \mathbb{Z}_+$   $[h_{i,r}, h_{j,l}] = 0$ .

Y2) For all  $i, j \in I$  and  $s \in \mathbb{Z}_+$   $[h_{i,0}, x_{j,s}^{\pm}] = \pm d_{ij} a_{ij} x_{j,s}^{\pm}$ .

Y3) For all  $i, j \in I$  and  $r, s \in \mathbb{Z}_+$

$$[h_{i,r+1}, x_{j,s}^{\pm}] - [h_{i,r}, x_{j,s+1}^{\pm}] = \pm \frac{d_{ij} a_{ij} \hbar}{2} (h_{i,r} x_{j,s}^{\pm} + x_{j,s}^{\pm} h_{i,r}).$$

Y4) For every  $i, j \in I$  and  $r, s \in \mathbb{Z}_+$

$$[x_{i,r+1}^{\pm}, x_{j,s}^{\pm}] - [x_{i,r}^{\pm}, x_{j,s+1}^{\pm}] = \pm \frac{d_{ij} a_{ij} \hbar}{2} (x_{i,r}^{\pm} x_{j,s}^{\pm} + x_{j,s}^{\pm} x_{i,r}^{\pm}).$$

Y5) For every  $i, j \in I$  and  $r, s \in \mathbb{Z}_+$   $[x_{i,r}^+, x_{j,l}^-] = \delta_{ij} \hbar h_{i,r+s}$ .

Y6) Let  $i \neq j \in I$  and let  $M = 1 - d_{ij} a_{ij}$ . Then for all

$k_1, \dots, k_M \in \mathbb{Z}$  and  $l \in \mathbb{Z}_+$

$$\sum_{\pi \in \mathfrak{S}_M} [x_{i,r_{\pi(1)}}^{\pm}, [x_{i,r_{\pi(2)}}^{\pm}, \dots, [x_{i,r_{\pi(M)}}^{\pm}, x_{j,l}^{\pm}] \dots]] = 0.$$

## Definition

Y7)  $[[x_{m,k}^{\pm}, x_{m+1,0}^{\pm}], [x_{m+1,0}^{\pm}, x_{m+2,t}^{\pm}]] = 0, k, t \in \mathbb{Z}.$

Let's note that  $Y_{\hbar}(\mathfrak{g})$  be a  $\mathbb{Z}_+$ -graded superalgebra with grading defining by the following conditions on generators:

$\deg(h_{i,r}) = \deg(x_{i,r}^{\pm}) = r$  и  $\deg(\hbar) = 1$ . Moreover,  $p(h_{i,r}) = 0$  for  $i \in I, r \in \mathbb{Z}_+$ , and  $p(x_{i,r}^{\pm}) = 0$  for  $i \in I \setminus \{m+1\}$ , and

$p(x_{m+1,r}^{\pm}) = 1$  for  $r \in \mathbb{Z}_+$ . As above, this definition is suitable for the an arbitrary Dynkin diagram, but in this case  $x_{m+1}^{\pm} = x_{\alpha_{m+1}}^{\pm}$  and  $\alpha_{m+1}$  be an arbitrary simple odd root.

The same way we can define super Yangian for arbitrary simple root system.

# Quantum Loop Superalgebra

## Definition

Let  $U_{\hbar}(\text{Lg})$  be an associative superalgebra over  $C[[\hbar]]$  generated by  $\{E_{i,k}, F_{i,k}, H_{i,k}\}_{i \in I, k \in \mathbb{Z}}$ , such that:

Q1) For all  $i, j \in I$  and  $r, s \in \mathbb{Z}$   $[H_{i,r}, H_{j,s}] = 0$ .

Q2) For all  $i, j \in I$  and  $k \in \mathbb{Z}$   $[H_{i,0}, E_{j,k}] = a_{i,j} E_{j,k}$ ,  
 $[H_{i,0}, F_{j,k}] = -a_{i,j} F_{j,k}$ .

Q3) For all  $i, j \in I$  and  $r, s \in \mathbb{Z} \setminus \{0\}$

$$[H_{i,r}, E_{j,k}] = \frac{[ra_{i,j}]_{q_i}}{r} E_{j,r+k}, \quad [H_{i,r}, F_{j,k}] = -\frac{[ra_{i,j}]_{q_i}}{r} F_{j,r+k}.$$

Q4) For all  $i, j \in I$  and  $k, l \in \mathbb{Z}$

$$E_{i,k+1} E_{j,l} - q_i^{a_{ij}} E_{j,l} E_{i,k+1} = q_i^{a_{ij}} E_{i,k} E_{j,l+1} - E_{j,l+1} E_{i,k},$$

$$F_{i,k+1} F_{j,l} - q_i^{-a_{ij}} F_{j,l} F_{i,k+1} = q_i^{-a_{ij}} F_{i,k} F_{j,l+1} - F_{j,l+1} F_{i,k}.$$

Q5) For all  $i, j \in I$  and  $k, l \in \mathbb{Z}$   $[E_{i,j}, F_{k,l}] = \delta_{i,j} \frac{\psi_{i,k+l} - \varphi_{i,k+l}}{q_i - q_i^{-1}}$ .

Q6) Let  $i \neq j \in I$  and let  $M = 1 - a_{ij}$ . For every  $k_1, \dots, k_M \in \mathbb{Z}$  and  $l \in \mathbb{Z}$

$$[E_{i,k_1}, \dots, E_{i,k_M}, F_{j,l}] = 0$$

## Definition

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$$\sum_{\pi \in \mathfrak{S}_M} \sum_{s=0}^M (-1)^s \begin{bmatrix} M \\ s \end{bmatrix}_{q_i} E_{i,k_{\pi(1)}} \cdot \dots \cdot E_{i,k_{\pi(s)}} \cdot E_{j,l} \cdot E_{i,k_{\pi(s+1)}} \cdot \dots \cdot E_{i,k_{\pi(M)}} = 0,$$

$$\sum_{\pi \in \mathfrak{S}_M} \sum_{s=0}^M (-1)^s \begin{bmatrix} M \\ s \end{bmatrix}_{q_i} F_{i,k_{\pi(1)}} \cdot \dots \cdot F_{i,k_{\pi(s)}} \cdot F_{j,l} \cdot F_{i,k_{\pi(s+1)}} \cdot \dots \cdot F_{i,k_{\pi(M)}} = 0,$$

$$\text{Q7) } [[E_{m,k}, E_{m+1,0}]_q, [E_{m+1,0}, E_{m+2,r}]_q]_q = 0,$$

$$[[F_{m,k}, F_{m+1,0}]_q, [F_{m+1,0}, F_{m+2,r}]_q]_q = 0,$$

Here the elements  $\psi_{i,r}, \varphi_{i,r}$  are defined by the following formulas:

$$\psi_i(z) = \sum_{r \geq 0} \psi_{i,r} z^{-r} = \exp\left(\frac{\hbar d_i}{2} H_{i,0}\right) \exp\left((q_i - q_i^{-1}) \sum_{s \geq 1} H_{i,s} z^{-s}\right),$$

$$\varphi_i(z) = \sum_{r \geq 0} \varphi_{i,r} z^{-r} =$$

$$\exp\left(-\frac{\hbar d_i}{2} H_{i,0}\right) \exp\left(-(q_i - q_i^{-1}) \sum_{s \geq 1} H_{i,-s} z^s\right),$$

where  $\psi_{i,-k} = \varphi_{i,k} = 0$  for  $k \geq 1$ . Here,  $p(H_{i,r}) = 0$  for

$i \in I, r \in \mathbb{Z}_+$ , and  $p(E_{i,r}) = p(F_{i,r}) = 0$  for  $i \in I \setminus \{m+1\}, r \in \mathbb{Z}_+$

and  $p(E_{m+1,r}) = p(F_{m+1,r}) = 0$  for  $r \in \mathbb{Z}$ .

Let us note, that this definition is suitable for the an arbitrary Dynkin diagram, but in this case  $E_{m+1} = E_{\alpha_{m+1}}$ ,  $F_{m+1} = F_{\alpha_{m+1}}$ ,  $\alpha_{m+1}$  be an arbitrary simple odd root.

### Example

Current Lie superalgebra  $\mathfrak{g}[t]$  quantization  $\rightarrow Y_{\hbar}(\mathfrak{g})$ , loop Lie superalgebra  $\mathfrak{g}[t, t^{-1}]$  quantization  $U_q(L\mathfrak{g})$ . Let's consider homomorphism defined on generators by the following formula  $\exp^* : \mathfrak{g}[t, t^{-1}] \rightarrow \hat{\mathfrak{g}}[s]$ ,  $X \otimes t^n \mapsto X \otimes e^{ns}$ ,  $n \in \mathbb{Z}$ . We would to quantize the mapping  $\exp^* : \mathfrak{g}[t, t^{-1}] \rightarrow \hat{\mathfrak{g}}[s]$  and obtain  $\Phi : \hat{U}_q(L\mathfrak{g}) \rightarrow \hat{Y}_{\hbar}(\mathfrak{g})$ .

### Example

S.Gautam and V. Toledano-Laredo generalize above mentioned result on the case Yangians and Quantum Loop Algebras.

# Isomorphism



Let  $\{E_{i,r}, F_{i,r}, H_{i,r}\}_{i \in I, r \in \mathbb{Z}}$  be current generators of Quantum Loop Superalgebra  $U_{\hbar}(L\mathfrak{g})$ , and  $\{e_{i,k}, f_{i,k}, h_{i,k}\}_{i \in I, k \in \mathbb{Z}_+}$  be generators of Yangian  $Y_{\hbar}(\mathfrak{g})$ . Let's define the map ([10])

$$\Phi : U_{\hbar}((L\mathfrak{g}) \rightarrow \widehat{Y_{\hbar}(\mathfrak{g})} \quad (1)$$

on generators by the following

$$\text{formulas: } \Phi(H_{i,r}) = \frac{\hbar}{q_i - q_i^{-1}} \sum_{k \geq 0} t_{i,k} \frac{r^k}{k!}, \quad \Phi(E_{i,r}) =$$

$$e^{r\sigma_i^+} \sum_{m \geq 0} g_{i,m}^+ e_{i,m}, \quad \Phi(F_{i,r}) = e^{r\sigma_i^-} \sum_{m \geq 0} g_{i,m}^- f_{i,m}.$$

Here we use the notations:  $q = e^{\hbar/2}$ ,  $q_i = q^{d_i}$ ,  $d_i$  be elements of symmetrizable matrix  $D = \text{diag}[d_1, \dots, d_{m+n+1}]$  of Cartan matrix  $A = (a_{i,j})$  of Lie Superalgebra  $\mathfrak{g} = A(m, n)$  ( $d_i = 1, i \in \{1, \dots, m\}$ ,  $d_i = -1, i \in \{m+1, \dots, m+n+1\}$ ).

We'll use the logarithmic generators  $\{t_{i,r}\}_{i \in I, r \in \mathbb{N}}$  of commutative subsuperalgebra  $Y_{\hbar}(\mathfrak{h}) \subset Y_{\hbar}(\mathfrak{g})$ :

$$\hbar \sum_{r > 0} t_{i,r} u^{-r-1} = \log(1 + \sum_{r > 0} h_{i,r} u^{-r-1}). \text{ The elements } \quad \leftarrow \rightarrow \quad \leftarrow \rightarrow \quad \leftarrow \rightarrow \quad \leftarrow \rightarrow$$

$\{g_{i,m}^{\pm}\}_{i \in I, m \in \mathbb{Z}_0}$  belongs to the completion  $\widehat{Y^0}$  of superalgebra  $Y^0 = Y_{\hbar}(\mathfrak{h})$  and defined as follows. Let's

$G(v) = \log \left( \frac{v}{e^{v/2} - e^{-v/2}} \right) \in Q[[v]]$  and define  $\gamma_i \in Y^0[v]$  by formula:  $\gamma_i(v) = \hbar \sum_{r \geq 0} \frac{t_{i,r}}{r!} \left( -\frac{d}{dv} \right)^{r+1} G(v)$ . Then,

$\sum_{m \geq 0} g_{i,m}^{\pm} v^m = \left( \frac{\hbar}{q_i - q_i^{-1}} \right)^{1/2} \exp \left( \frac{\gamma_i(v)}{2} \right)$ . Finally,  $\sigma_i^{\pm}$  are

homomorphisms of subsuperalgebras

$\sigma_i^{\pm} : Y_{\hbar}(\mathfrak{b}_{\pm}) (\subset Y_{\hbar}(\mathfrak{g})) \rightarrow Y_{\hbar}(\mathfrak{b}_{\pm})$ , which defined onto generators as follows  $\sigma_i^{\pm} : h_{i,k} \rightarrow h_{i,k}$ ,  $\sigma_i^{+} : x_{j,r}^{+} \rightarrow x_{j,r+\delta_{ij}}^{+}$ ,  $\sigma_i^{-} : x_{j,r}^{-} \rightarrow x_{j,r+\delta_{ij}}^{-}$ . These homomorphisms can be continued to homomorphism of

associative superalgebras  $\hat{Y}^\pm \rightarrow \hat{Y}^\pm$

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# Main result

## Theorem

- 1) *The above defined mapping  $\Phi : U_{\hbar}(L\mathfrak{g}) \rightarrow \widehat{Y_{\hbar}(\mathfrak{g})}$ , is homomorphism of associative superalgebras.*
- 2) *The mapping  $\Phi$  uniquely continued to homomorphism of topological completion :  $\hat{\Phi} : \widehat{U_{\hbar}(L\mathfrak{g})} \rightarrow \widehat{Y_{\hbar}(\mathfrak{g})}$  of these superalgebras. Moreover, mapping  $\hat{\Phi}$  be an isomorphism of topological superalgebras.*

# Drinfeld comultiplication

Now we consider so called Drinfeld comultiplication. Then Drinfeld comultiplication on super Yangian  $Y_h(\mathfrak{g})$ ,  $\mathfrak{g} = A(m, n)$ , is defined by the following formulas

$$\begin{aligned}\Delta_s(h_i(u)) &= h_i(u-s) \otimes h_i(u), \quad \Delta_s(x_i^+(u)) = \\ & x_i^+(u-s) \otimes 1 + \oint_{C_2} \frac{1}{u-v} h_i(v-s) \otimes x_i^+(v) dv, \quad \Delta_s(x_i^-(u)) = \\ & \oint_{C_1} \frac{1}{u-v} x_i^-(v-s) h_i(v) dv + 1 \otimes x_i^-(u).\end{aligned}$$

We also can define the Drinfeld comultiplication on  $U_q(L\mathfrak{g})$  by the formulas

$$\begin{aligned}\Delta_u(\Psi_i(z)) &= \Psi_i(u^{-1}z) \otimes \Psi_i(z), \quad \Delta_u(X_i^+(z)) = \\ & X_i^+(u^{-1}z) \otimes 1 + \oint_{C_2} \frac{zw^{-1}}{z-w} \Psi_i(u^{-1}w) \otimes X_i^+(w) dw, \quad \Delta_u(X_i^-(z)) = \\ & 1 \otimes X_i^-(u^{-1}z) + \oint_{C_1} \frac{zw^{-1}}{z-w} X_i^-(w) \otimes \Psi_i(u^{-1}w) dw. \\ X_i^\pm(z) &= \sum_{k \in \mathbb{Z}} X_{i,k}^\pm z^k, \quad \Psi_i(z) = \psi_i^+(z) - \psi_i^-(z), \quad X_{i,k}^+ = \\ E_{i,k}, X_{i,k}^- &= F_{i,k}.\end{aligned}$$

## Theorem

*Above defined mapping is isomorphism of Hopf superalgebra isomorphism*

$$\hat{\phi} : \widehat{U_{\hbar}(L\mathfrak{g})} \rightarrow \widehat{Y_{\hbar}(\mathfrak{g})}$$

*is Hopf superalgebra isomorphism of completions of super Yangian and quantum loop superalgebra with above defined Drinfeld comultiplications.*

# Second theorem

## Theorem

*.Mapping*

$\hat{\Phi} : \widehat{U_h(L\mathfrak{g})} \rightarrow \widehat{Y_h(\mathfrak{g})}$  is Hopf superalgebra isomorphism of completions of super Yangian and quantum loop superalgebra with above defined Drinfeld comultiplications and induced equivalence  $\Phi_{cat}$  of monoidal categories of representations of super Yangian and Quantum Loop Superalgebra.

## Corollary

*So we obtain exact faithful monoidal functor*

$\Phi_{cat} : \mathfrak{D}^\Pi(Y_h(\mathfrak{g})) \rightarrow \mathfrak{D}^\Omega(U_q(L\mathfrak{g}))$  which acts between some analogues of category  $\mathfrak{D}$  of representations.  
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- ① What happens to the structure of the Hopf superalgebra when the isomorphism defined above is applied?
- ② How to describe all possible structures of the Hopf superalgebra on the Yangian and the quantum loop superalgebra if we consider them only with a fixed structure of the associative superalgebra?

We define the Weyl groupoid as a some supercategory. Based on this abstract definition, we give an explicit realization of the Weyl quantum groupoid in terms of isomorphisms of quantum superalgebras generated by isomorphisms, which are induced by reflections (relatively both even and odd roots).

# Quantum Weyl groupoid for super Yangian.

Quantum Weyl groupoid for super Yangian can be defined as.

$$T_i = \exp \operatorname{adx}_{i,0}^+ \exp \operatorname{adx}_{i,0}^- \exp \operatorname{adx}_{i,0}^+.$$

## Theorem

$$T_i(x_{j,1}^+) = \begin{cases} -x_{i,1}^- + \frac{\hbar}{2}\{h_{i,0}, x_{i,0}^-\}, & \text{if } i = j, \\ [x_{i,0}^+, x_{j,1}^+], & \text{if } a_{ij} = \pm 1, \\ x_{j,1}^+, & \text{if } a_{ij} = 0 \end{cases},$$

$$T_i(\tilde{h}_{j,1}) = \begin{cases} -\tilde{h}_{i,1} - \frac{\hbar}{2}\{x_{i,0}^+, x_{i,0}^-\}, & \text{if } i = j, \\ \tilde{h}_{i,1} + \tilde{h}_{j,1} + \frac{\hbar}{2}\{x_{i,0}^+, x_{i,0}^-\}, & \text{if } a_{ij} = \pm 1, \\ \tilde{h}_{j,1}, & \text{if } a_{ij} = 0 \end{cases}.$$

$$T_i(x_{j,1}^-) = \begin{cases} -x_{i,1}^- + \frac{\hbar}{2}\{h_{i,0}, x_{i,0}^+\}, & \text{if } i = j, \\ -[x_{i,0}^-, x_{j,1}^-], & \text{if } a_{ij} = \pm 1, \\ x_{j,1}^-, & \text{if } a_{ij} = 0 \end{cases},$$

## Theorem

*The action of generators  $T_i$  is compatible with comultiplication and Drinfel'd comultiplication*

$$\Delta(T_i) = (T_i \otimes T_i) \circ \Delta, \quad \Delta_s^D(T_i) = (T_i \otimes T_i) \circ \Delta_s^D.$$

# Quantum Weyl groupoid for Quantum Loop Superalgebra

Quantum Weyl group action on quantum loop superalgebra. We introduce the element  $T_i^L$  which is natural analogue of Lusztig automorphisms.

$$T_i^L(x) = U_i x U_i^{-1},$$

$$U_i = \exp_{q_i^{-1}}(-q_i^{-1} F_{i,0} q_i^{H_{i,0}}) \exp_{q_i^{-1}}(E_{i,0}) \exp_{q_i^{-1}}(-q_i^{-1} F_{i,0} q_i^{H_{i,0}}) q_i^{H_{i,0}(H_{i,0}+1)}$$

$$\exp_p(x) = \sum_{n \geq 0} \frac{1}{[n]_p} p^{m(m-1)/2} x^n$$

### Theorem

*The action of generators  $T_i^L$  is compatible with comultiplication and Drinfel'd comultiplication*

$$\Delta_z(T_i^L) = (T_i^L \otimes T_i^L) \circ \Delta_z, \quad \Delta_z^D(T_i^L) = (T_i^L \otimes T_i^L) \circ \Delta_z^D$$



## Theorem

*Isomorphism  $\hat{\Phi}$  is compatible with action of quantum Weyl groups for super Yangian and Quantum Loop Superalgebra.*

# Outline

## 1 Motivation

- Introduction
- Super Yangian and Quantum Loop Superalgebra

## 2 Our Results/Contribution

- Main Results
- Quantum affine Weyl groupoid and classification of Hopf superalgebra structures
- Main Results for representations

# Category representations of super Yangian and Quantum Loop Superalgebra

Let  $\Pi, \Omega$  be some subsets of the complex plane are invariant with respect to the group of additive shifts on  $\hbar$ , respectively, multiplicative shifts by  $q$ .

We introduce the category  $\mathfrak{D}^\Pi(Y_\hbar(\mathfrak{g}))$  as a full subcategory of category  $\mathfrak{D}(Y_\hbar(\mathfrak{g}))$  consisting of the representations  $V$  such that, for every  $(\lambda, \{P_i^d\}, Q_{m+1}^d) \in \Pi_+^Y$  for which  $[V : L(\lambda, \{P_i^d\}, Q_{m+1}^d)] \neq 0$ , the roots of  $P_i^d, i \in I, Q_{m+1}^d$  lie in  $\Pi$ . Let similarly  $\Omega \in \mathbb{C}^*$  be a subset stable under multiplication by  $q^\pm$ . We define  $\mathfrak{D}^\Omega(U_q(L\mathfrak{g}))$  to be a the full subcategory of  $\mathfrak{D}(U_q(L\mathfrak{g}))$  consisting those  $U$  such that for every  $(\lambda, \{P_i^\delta\}, Q_{m+1}^\delta) \in \Pi_+^U$  for which  $[U : L(\lambda, \{P_i^\delta\}, Q_{m+1}^\delta)] \neq 0$ , the roots of  $P_i^\delta, i \in I, Q_{m+1}^\delta$  lie in  $\Omega$ .

## Theorem

- i)  $\mathfrak{D}^\Pi(Y_{\hbar}(\mathfrak{g}))$  and  $\mathfrak{D}^\Omega(U_q(L\mathfrak{g}))$  are Serre subcategories of  $\mathfrak{D}(Y_{\hbar}(\mathfrak{g}))$  and  $\mathfrak{D}(U_q(L\mathfrak{g}))$ , respectively. Other words, they are closed under taking direct sum, subobjects, quotients and extensions.*
- (ii)  $\mathfrak{D}^\Pi(Y_{\hbar}(\mathfrak{g}))$  and  $\mathfrak{D}^\Omega(U_q(L\mathfrak{g}))$  are closed under tensor product.*
- (iii) Categories  $\mathfrak{D}^\Pi(Y_{\hbar}(\mathfrak{g}))$  and  $\mathfrak{D}^\Omega(U_q(L\mathfrak{g}))$  are equivalent.*
- (iv) There is exist exact faithful monoidal functor*




$$\Phi : \mathfrak{D}^\Pi(Y_{\hbar}(\mathfrak{g})) \rightarrow \mathfrak{D}^\Omega(U_q(L\mathfrak{g})) \quad (2)$$

*which is category equivalence.*






# Summary

- The **first main message** of my talk is theorem about existence isomorphism between completions of super Yangian and Quantum Loop Superalgebra in category of Hopf superalgebras.
- The **second main message** of my talk is interpretation of above mentioned isomorphism and different Hopf superalgebra structures on super Yangian and Quantum Loop superalgebra in terms of Weyl groupoid actions.
- Perhaps a **third message**, is an application to representation theories of super Yangian and Quantum Loop Superalgebra. ..
- Outlook
  - What we have not done yet? Explicite description of automorphisms of super Yangian and Quantum Loop Superalgebra as Hopf superalgebras and associative superalgebras




# For Further Reading I

-  Stukopin V. Relation between categories of representations of the super-Yangian of a special linear Lie superalgebra and quantum loop superalgebra. – Theoretical and mathematical physics, v.204(2020), no 3, p. 1227 – 1243. .
-  S. Gautam, V. Toledano Laredo Meromorphic tensor equivalence for Yangians and quantum loop algebras. – Publ. Math. Inst. Hautes ´ Etudes Sci., 125(2017), 267–337..
-  Stukopin V. Relation between categories of representations of the super-Yangian of a special linear Lie superalgebra and quantum loop superalgebra. – Theoretical and mathematical physics, v.204(2020), no 3, p. 1227 – 1243.

## For Further Reading II

-  S. Gautam, V. Toledano Laredo Yangians and Quantum Loop Algebras.– Selecta Math., 19(2013), 271–336.
-  Huafeng Zhang Representations of Quantum Affine Superalgebras. – arXiv:math/1309. 5250 [math.QA].
-  S. Gautam, V. Toledano Laredo Meromorphic tensor equivalence for Yangians and quantum loop algebras. – Publ. Math. Inst. Hautes ´ Etudes Sci., 125(2017), 267–337.
-  S. Gautam, V. Toledano Laredo Yangians, quantum loop algebras and abelian difference equations. –J. Amer. Math. Soc., 29(2016), 775–824.
-  V. Stukopin Isomorphism between super Yangian and quantum loop superalgebra. – arXiv:math/1804. 06678 [math.QA].

## For Further Reading III

-  A. Mazurenko, V.A. Stukopin,  $\mathfrak{R}$ -matrix for quantum superalgebra  $\mathfrak{sl}(2|1)$  at roots of unity and its application to centralizer algebras, arXiv:1909.11613 [mathQA].
-  A. Mazurenko, V.A. Stukopin, Classification of Hopf superalgebras associated with quantum special linear superalgebra at roots of unity using Weyl groupoid, arXiv:2006.06610 math[QA].
-  A. Mazurenko, V.A. Stukopin, Relationship between super Yangians and quantum loop superalgebra via Weyl groupoid, in progress.