Dark matter with a nontrivial motion as a gravitational effect of modified theories of gravity

S.A. Paston

Saint Petersburg State University

The mystery of dark matter:

The hypothesis of its existence provides a good fit to evidence in a wide range of scales –

from the galactic scale (rotation curves)
to the cosmological one (the formation of cosmic structures and the prediction of the total mass of the matter in the Universe).

In a framework of the ΛCDM model – a modern standard model in cosmology – the properties of the dark matter are close to the non-relativistic dust-like matter.

However, attempts to detect dark matter (to detect its interaction with regular matter) are still unsuccessful!

Maybe, dark matter *doesn't genuinely exist*,
but it is an effect of a description of the gravitational interaction?

Maybe, dark matter *doesn't genuinely exist*,
but it is an effect of a description of the gravitational interaction?

Possible modifications of gravity:

- MOND (Modified Newtonian Dynamics),
- $f(R)$ gravity,
- scalar-tensor gravity,
- mimetic gravity,
- embedding gravity;
- ...

In order to be able to explain all effects related to the dark matter, the modified gravity should contain a sufficient number of degrees of freedom!

Mimetic gravity

$$S = S^{\mathrm{EH}}[g(...)] + S_m[g(...)], \qquad S^{\mathrm{EH}} = -\frac{1}{2\varkappa}\int d^4x\sqrt{-g}\,R \quad (1)$$

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}(\partial_\alpha\varphi)(\partial_\beta\varphi) \qquad \left(\Rightarrow \quad g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) \equiv 1\right) \quad (2)$$

$$G_{\mu\nu} - \varkappa\, T_{\mu\nu} = \varkappa\rho(\partial_\mu\varphi)(\partial_\nu\varphi), \qquad D_\mu(\rho g^{\mu\nu}\partial_\nu\varphi) = 0,$$

$$\rho \equiv \frac{1}{\varkappa}\, g^{\mu\nu}(G_{\mu\nu} - \varkappa\, T_{\mu\nu}) \quad (3)$$

(A.H. Chamseddine, V. Mukhanov, JHEP, 2013:11 (2013), 135, arXiv:1308.5410)

Mimetic gravity

$$S = S^{\mathrm{EH}}[g(\ldots)] + S_m[g(\ldots)], \qquad S^{\mathrm{EH}} = -\frac{1}{2\varkappa}\int d^4x\sqrt{-g}\,R \quad (1)$$

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}(\partial_\alpha\varphi)(\partial_\beta\varphi) \qquad \left(\Rightarrow \quad g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) \equiv 1\right) \quad (2)$$

$$G_{\mu\nu} - \varkappa\, T_{\mu\nu} = \varkappa\rho(\partial_\mu\varphi)(\partial_\nu\varphi), \qquad D_\mu(\rho g^{\mu\nu}\partial_\nu\varphi) = 0,$$

$$\rho \equiv \frac{1}{\varkappa}\, g^{\mu\nu}(G_{\mu\nu} - \varkappa\, T_{\mu\nu}) \quad (3)$$

(A.H. Chamseddine, V. Mukhanov, JHEP, 2013:11 (2013), 135, arXiv:1308.5410)

Equivalent formulation in the form of GR with additional *fictitious* (dark?!) matter:

$$S = S^{\mathrm{EH}} + S_m + S^{+} \quad (4)$$

$$S^{+} = -\frac{1}{2}\int d^4x\,\sqrt{-g}\,\rho\Big(1 - g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi)\Big) \quad (5)$$

(A. Golovnev, Phys.Lett.B, 728 (2014), 39, arXiv:1310.2790)

Looking at the form of equations

$$G_{\mu\nu} = \varkappa\Big(T_{\mu\nu} + \rho(\partial_\mu\varphi)(\partial_\nu\varphi)\Big), \qquad D_\mu(\rho g^{\mu\nu}\partial_\nu\varphi) = 0 \tag{6}$$

we can see that ρ – is the density of the fictitious matter and its velocity is

$$u_\mu = \partial_\mu\varphi, \tag{7}$$

which means that the matter moves potentially.

Not enough degrees of freedom!

Complications of mimetic gravity

Modification of the action for fictitious matter:

$$S^{+} = -\frac{1}{2}\int d^4x\sqrt{-g}\left(\rho\Big(1-g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi)\Big)-V(\rho)+\frac{1}{2}\gamma(\rho)\left(\Box\rho\right)^2\right) \quad (8)$$

(A. Chamseddine, V. Mukhanov, A. Vikman, JCAP, 2014:06 (2014), 017, arXiv:1403.3961; L. Mirzagholi, A. Vikman, JCAP, 2015:06 (2015), 028, arXiv:1412.7136)

Modification of the mimetic change of variables:

$$S = S^{\mathrm{EH}}[g(...)] + S_m[g(...)] \quad (9)$$

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}\tilde{g}^{\gamma\delta}\left(\partial_\gamma\varphi + \alpha\partial_\gamma\beta\right)\left(\partial_\delta\varphi + \alpha\partial_\delta\beta\right) \quad (10)$$

$$u_\mu = \partial_\mu\varphi + \alpha\partial_\mu\beta \quad (11)$$

(S. P., Phys. Rev. D 96 (2017) 084059, arXiv:1708.03944)

Embedding gravity

$$S = S^{\mathrm{EH}}[g(...)] + S_m[g(...)], \qquad S^{\mathrm{EH}} = -\frac{1}{2\varkappa}\int d^4x\sqrt{-g}\,R, \quad (12)$$

$$g_{\mu\nu} = \eta_{ab}(\partial_\mu y^a)(\partial_\nu y^b), \quad (13)$$

where $a, b = 0, \ldots, 9$.

(T. Regge, C. Teitelboim, *"General relativity a la string: a progress report"*, Proceedings of the First Marcel Grossmann Meeting (Trieste, Italy, 1975), 1977, p. 77, arXiv:1612.05256)

The simple geometric sense – a metric becomes the induced metric of a 4-dimensional surface in the ambient space which is described by the embedding function $y^a(x^\mu)$.

The original idea of embedding theory is to rewrite the GR in a way similar to the String theory with the hope that the quantization procedure will improve.

The original idea of embedding theory is to rewrite the GR in a way similar to the String theory with the hope that the quantization procedure will improve.

But – there are *extra* solutions!

Equation of motion (Regge-Teitelboim equations)

$$D_\mu \Big(\big(G^{\mu\nu} - \varkappa\, T^{\mu\nu} \big)\, \partial_\nu y^a \Big) = 0 \tag{14}$$

Regge and Teitelboim: introduce *ad hoc* additional constraint

$$G^{\mu 0} - \varkappa\, T^{\mu 0} = 0 \tag{15}$$

The Regge-Teitelboim equations can be rewritten in the form of Einstein's equations with additional contribution $\tau^{\mu\nu}$ – EMT of the *fictitious* (dark?!) matter:

$$G^{\mu\nu} = \varkappa\left(T^{\mu\nu} + \tau^{\mu\nu}\right), \qquad D_\mu\left(\tau^{\mu\nu}\partial_\nu y^a\right) = 0 \tag{16}$$

(M. Pavsic, *Class. Quant. Grav.*, 2 (1985), 869, arXiv:1403.6316)

The corresponding action can be written as

$$S = S^{\mathrm{EH}} + S_m + S^+, \tag{17}$$

where

$$S^+ = \frac{1}{2}\int d^4x\,\sqrt{-g}\,\Big((\partial_\mu y^a)(\partial_\nu y^b)\eta_{ab} - g_{\mu\nu}\Big)\tau^{\mu\nu}, \tag{18}$$

(S. P., *Phys. Rev. D*, 96 (2017), 084059, arXiv:1708.03944)
if dark matter is described by variables y^a and $\tau^{\mu\nu}$;

or

$$S^+ = \int d^4x\,\sqrt{-g}\,\Big(j^\mu_a\partial_\mu y^a - \mathbf{tr}\sqrt{g_{\mu\nu}j^\nu_a\eta^{ab}j^\alpha_b}\Big), \tag{19}$$

(S. P., A. Sheykin, *Eur. Phys. J. C*, 78: 12 (2018), 989, arXiv:1806.10902)
if, besides y^a, some currents j^μ_a are chosen to be independent variables that describe dark matter.

It is these currents that turn out to be conserved due to one of the equations of motion:

$$D_\mu j_a^\mu = 0 \quad \Leftrightarrow \quad \partial_\mu \left(\sqrt{-g}\, j_a^\mu\right) = 0, \qquad j_a^\mu = \tau^{\mu\nu}\partial_\nu y_a \tag{20}$$

If we assume that all these currents are non-relativistic in the ambient space:

$$j_a^\mu = \delta_a^0 j^\mu + \delta j_a^\mu, \qquad \delta j_a^\mu \to 0, \tag{21}$$

then it will bring us to the *non-relativistic limit* of embedding theory, where the fictitious matter appears non-relativistic as well.

In this limit the action corresponding to the dark matter (19) transforms into the action

$$S^{+} = \int d^4x \sqrt{-g} \left(j^{\mu} \partial_{\mu} y^{(0)} - \sqrt{j^{\mu} g_{\mu\nu} j^{\nu}} \right) \tag{22}$$

It describes dust matter with potential motion.

However, this limit is singular, as one of the equations of motion has the form

$$\partial_{\mu} y^{a} = \beta^{-1}_{\mu\nu} j^{\nu a}, \qquad \beta_{\mu}{}^{\alpha} = \sqrt{g_{\mu\nu} j^{\nu}_{a} \eta^{ab} j^{\alpha}_{b}} \tag{23}$$

Embedding theory in a weak gravity limit

Weak gravity:

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}, \qquad \delta g_{\mu\nu} \ll 1 \tag{24}$$

One should pick a corresponding embedding function

$$y^a(x) = \bar{y}^a(x) + \delta y^a(x) \tag{25}$$

Embedding theory in a weak gravity limit

Weak gravity:

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}, \qquad \delta g_{\mu\nu} \ll 1 \tag{24}$$

One should pick a corresponding embedding function

$$y^a(x) = \bar{y}^a(x) + \delta y^a(x) \tag{25}$$

The embedding of the metric of Minkowski space should play the role of the background $\bar{y}^a(x)$. Note that the choice is not unique!

The simplest option — 4-dimensional plane — leads to non-linearizability of the Regge-Teitelbboim equations and to non-linearity of the relation between $\delta g_{\mu\nu}$ and δy^a.

The variation of the metric in the linear approximation:

$$
\begin{aligned}
g_{\mu\nu} &= (\partial_\mu y^a)(\partial_\nu y_a) \quad \Rightarrow \\
\Rightarrow \quad \delta g_{\mu\nu} &= (\partial_\mu \delta y^a)(\partial_\nu y_a) + (\mu \leftrightarrow \nu) = \\
&= \left(\partial_\mu(\delta y^a_\parallel + \delta y^a_\perp)\right)(\partial_\nu y_a) + (\mu \leftrightarrow \nu) = \\
&= D_\mu \xi_\nu + D_\nu \xi_\mu - 2\delta y_{\perp a} b^a_{\mu\nu}, \qquad (26)
\end{aligned}
$$

where $\xi_\mu = \delta y^a_\parallel \partial_\mu y_a$, and $b^a_{\mu\nu} = D_\mu \partial_\nu y^a$ – second fundamental form of the surface.

The linearity of the relation between $\delta g_{\mu\nu}$ and δy^a requires

$$
\operatorname{rank} b^a_{\mu\nu} = 6, \qquad (27)
$$

i.e. the property of *unfoldness* of the embedding.

The background should be unfolded embedding of the Minkowski metric. Examples of such embeddings may be found in
(S. P., T. Zaitseva, Universe, 7:12 (2021), 477, arXiv:2111.04188)

The unfolded background leading to the non-relativistic motion of the fictitious matter is:

$$\bar{y}^0 = x^0, \qquad \bar{y}^I = \bar{y}^I(x^i), \tag{28}$$

where $\bar{y}^I(x^i)$ – is unfolded embedding $\mathbb{R}^3$ into $\mathbb{R}^9$.

The unfolded background leading to the non-relativistic motion of the fictitious matter is:

$$\bar{y}^0 = x^0, \qquad \bar{y}^I = \bar{y}^I(x^i), \tag{28}$$

where $\bar{y}^I(x^i)$ – is unfolded embedding $\mathbb{R}^3$ into $\mathbb{R}^9$.

The embedding functions $y^a(x)$ for the corresponding 4-dimensional surface in the leading order w.r.t. $1/c$ have form

$$y^0 = x^0 + \frac{1}{c}\,\psi\left(\frac{x^0}{c}, x^i\right) + o\left(\frac{1}{c^2}\right),$$

$$y^I = \bar{y}^I\left(\frac{x^0}{c}, x^i\right) + \frac{1}{c^2}\bar{\alpha}^{lmI}\left(\frac{1}{2}(\partial_l\psi)(\partial_m\psi) - \varphi\delta_{lm}\right) + o\left(\frac{1}{c^2}\right), \tag{29}$$

where

$$\bar{\alpha}^{ik}_L = \bar{\alpha}^{ki}_L, \qquad \bar{\alpha}^{ik}_L\partial_m\bar{y}^L = 0, \qquad \bar{\alpha}^{ik}_L\bar{b}^L_{lm} = \frac{1}{2}\left(\delta^i_l\delta^k_m + \delta^i_m\delta^k_l\right) \tag{30}$$

Non-relativistic equations of motion of the fictitious matter in the leading order w.r.t. $1/c$ have form

$$\partial_t \bar{y}^I = \gamma^I, \qquad \partial_t \psi = \varphi + \frac{1}{2}\gamma^I\gamma^I, \qquad \partial_t \bar{\rho}_\tau = -\partial_i \left(\bar{\rho}_\tau v_\tau^i\right),$$

$$\bar{\rho}_\tau \left(\partial_t + v_\tau^i \partial_i\right) v_\tau^m =$$

$$= -\bar{\rho}_\tau \partial_m \varphi + \partial_l \left(\bar{\rho}_\tau \left[v_\tau^l v_\tau^m + \bar{\alpha}_L^{lm} \left(\bar{\alpha}_L^{ik} \left((\partial_i \gamma^I)(\partial_k \gamma^I) + \partial_i \partial_k \varphi \right) + 2 v_\tau^i \partial_i \gamma^L \right) \right] \right)$$

where $\gamma^I = (\partial_k \bar{y}^I)\partial_k \psi + \bar{\alpha}_I^{ik} \partial_i \partial_k \psi$, and φ – is a Newtonian gravitational potential corresponding to matter distribution with the density $\rho + \bar{\rho}_\tau$.

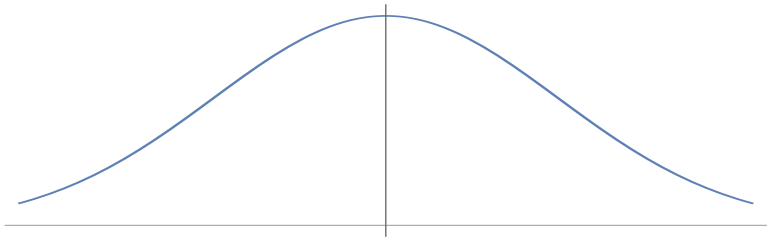
The parameters describing the fictitious matter are:
density $\bar{\rho}_\tau$, velocity v_τ^i, and also the quantity ψ and 3 additional functions parametrizing the embedding $\bar{y}^I(x^i)$ of the flat metric.

The self-interaction is in place!

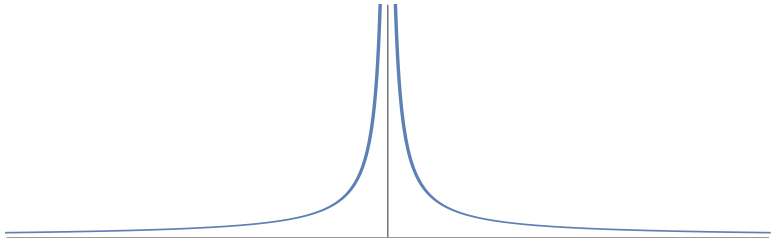
(S. P., Universe, 6:10 (2020), 163, arXiv:2009.06950)

Core-cusp problem

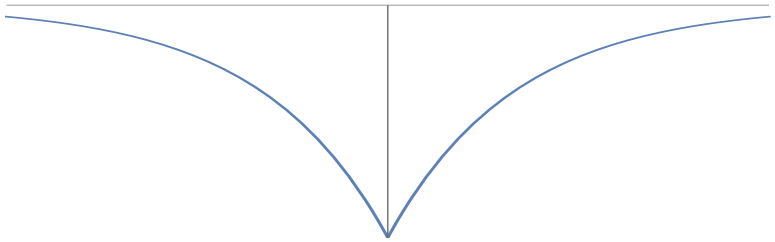
Observations generally show a smooth density distribution at the centers of galaxies – core:



while numerical simulations result in an increase
$\rho(r) \sim r^{\alpha}, \quad \alpha \approx -1$ in density at the center – cusp:



which corresponds to the gravitational potential



The relation between density profile and particles distribution function

We consider a spherically symmetric and static on average distribution of particles.

Density $\rho(x)$ is related with the gravitational potential $\varphi(x)$ by:

$$\partial_k \partial_k \varphi(x) = 4\pi G \rho(x) \tag{32}$$

Finite motion can be closed and open as well:

It is defined by normalized energy $\varepsilon = E/m$ and angular momentum $\ell_k = L_k/m$ with addition of a vector τ_k and initial phase γ.
A motion of a single particle is given by the function $\hat{x}_i\left(t, \varepsilon, \ell_k, \tau_l, \gamma\right)$.

The distribution function of particles f:

$$dN = f\left(\varepsilon, \ell_k, \tau_l, \gamma\right)\, d\varepsilon\, d^3\ell\, d\tau\, d\gamma \tag{33}$$

Then the density can be written as:

$$\rho(x_i) = m \int d\varepsilon\, d^3\ell\, d\tau\, d\gamma\, f\left(\varepsilon, \ell_k, \tau_l, \gamma\right) \delta\left(x_i - \hat{x}_i\left(t, \varepsilon, \ell_k, \tau_l, \gamma\right)\right) \tag{34}$$

Taking stationarity into account we have

$$\rho(r) = \frac{m}{4\pi r^2 T} \int d\varepsilon\, d^3\ell\, d\tau\, d\gamma\; f\left(\varepsilon, \ell_k, \tau_l, \gamma\right) \times$$

$$\times \int\limits_{S_r} d^2x \int\limits_0^T dt\, \delta\left(x_i - \hat{x}_i\left(t, \varepsilon, \ell_k, \tau_l, \gamma\right)\right) =$$

$$= \frac{m}{4\pi r^2 T} \int d\varepsilon\, d^3\ell\, d\tau\, d\gamma\; f\left(\varepsilon, \ell_k, \tau_l, \gamma\right) \frac{n}{|v_r|} \tag{35}$$

for the radial distribution.

In the spherically symmetric field, the energy

$$E = \frac{m}{2}\left(v_r^2 + v_\tau^2\right) + m\varphi(r) \tag{36}$$

is conserved as well as the angular momentum. Therefore,

$$|v_r| = \sqrt{2\varepsilon - 2\varphi(r) - \frac{\ell^2}{r^2}}, \quad \text{since} \quad L = mrv_\tau. \tag{37}$$

At large T

$$n\left(\varepsilon, \ell, r\right) \approx \frac{2T}{\hat{T}\left(\varepsilon, \ell_k\right)} \theta\left(2\varepsilon - 2\varphi(r) - \frac{\ell^2}{r^2}\right), \tag{38}$$

which leads to

$$\rho(r) = \frac{m}{2\pi r^2} \int d\varepsilon\, d^3\ell\, \frac{f(\varepsilon, \ell_k)\, \theta\left(2\varepsilon - 2\varphi(r) - \frac{\ell^2}{r^2}\right)}{\hat{T}\left(\varepsilon, \ell_k\right) \sqrt{2\varepsilon - 2\varphi(r) - \frac{\ell^2}{r^2}}}, \tag{39}$$

where

$$f(\varepsilon, \ell_k) = \int d\tau\, d\gamma\, f(\varepsilon, \ell_k, \tau_l, \gamma) \tag{40}$$

Integrating over all directions of the vector ℓ_k, we obtain:

$$\rho(r) = \frac{m}{2\pi r^2} \int d\varepsilon \int\limits_0^\infty d\ell \, \frac{\hat{f}(\varepsilon,\ell)\,\theta\left(2\varepsilon - 2\varphi(r) - \frac{\ell^2}{r^2}\right)}{\hat{T}\,(\varepsilon,\ell)\sqrt{2\varepsilon - 2\varphi(r) - \frac{\ell^2}{r^2}}} \tag{41}$$

Let's discuss the behavior of the $\rho(r)$ at $r \to 0$.

The only contribution comes from the area $\ell \leqslant r\sqrt{2\varepsilon - 2\varphi(r)}$, so the asymptotic of the $\rho(r)$ at $r \to 0$ is defined by the asymptotic at $\ell \to 0$ of the functions in the integral.

Function $\hat{T}(\varepsilon,\ell)$ has finite limit, while for $\hat{f}(\varepsilon,\ell)$ several cases are possible:

Case 1) $\hat{f}(\varepsilon,\ell) \approx \hat{f}(\varepsilon,0) + \hat{f}'(\varepsilon,0)\ell$ c $\hat{f}(\varepsilon,0) \neq 0$ for some ε

Case 2) $\hat{f}(\varepsilon,\ell) \approx \hat{f}'(\varepsilon,0)\ell$ for all ε

Case 3) $\hat{f}(\varepsilon,\ell)$ not analytical w.r.t. ℓ

Performing a change of variables $\ell = r\tilde{\ell}$ in the integral, we obtain:

Case 1)

$$\rho(r) = \frac{m}{2\pi r}\int d\varepsilon\, d\tilde{\ell}\,\frac{\hat{f}(\varepsilon,0)\,\theta\left(2\varepsilon - 2\varphi(r) - \tilde{\ell}^2\right)}{\hat{T}\,(\varepsilon,0)\sqrt{2\varepsilon - 2\varphi(r) - \tilde{\ell}^2}} = \frac{m}{4r}\int d\varepsilon\,\frac{\hat{f}(\varepsilon,0)}{\hat{T}\,(\varepsilon,0)} \quad (42)$$

which gives cusp-profile с $\alpha = -1$, т.е. $\rho(r) \sim 1/r$.

Case 2)

$$\rho(r) = \frac{m}{2\pi}\int d\varepsilon\, d\tilde{\ell}\,\frac{\hat{f}'(\varepsilon,0)\tilde{\ell}\,\theta\left(2\varepsilon - 2\varphi(r) - \tilde{\ell}^2\right)}{\hat{T}\,(\varepsilon,0)\sqrt{2\varepsilon - 2\varphi(r) - \tilde{\ell}^2}} =$$

$$= \frac{m}{2\pi}\int d\varepsilon\,\frac{\hat{f}'(\varepsilon,0)}{\hat{T}\,(\varepsilon,0)}\sqrt{2\varepsilon - 2\varphi(r)} \quad (43)$$

which gives core-profile.

Asymptotic of the distribution function at $\ell \to 0$

Situation A) – a formation of the static structure takes place in a preliminary given spherically symmetric potential.

Possible realization – the spherically symmetric potential is already created by the dark matter and we consider a formation of a static structure of the regular matter inside this potential.

Spherical symmetry $\Longrightarrow$ the angular momentum is conserved $\Longrightarrow$ the distribution function $f(\varepsilon, \ell_k)$ doesn't change with time and it is enough to find it at the initial moment.

Using the distribution function $\chi(x_i, v_k)$ of particles over its coordinates x_i and velocities v_k, we have

$$f(\varepsilon, \ell_k) = \int d^3x\, d^3v\, \chi(x_i, v_i)\delta(\ell_i - \epsilon_{ikl} x_k v_l)\delta\left(\varepsilon - \frac{v^2}{2} - \varphi(x_i)\right) \quad (44)$$

If we assume spherical symmetry and take $\ell_k = (\ell, 0, 0)$, we find

$$f(\varepsilon, \ell_k) = \frac{1}{\ell}\int dx_2\, dx_3\, dv_2\, dv_3\, \left[\chi(x_i, v_i)\delta\left(\varepsilon - \frac{v^2}{2} - \varphi(x_i)\right)\right]\Bigg|_{x_1=v_1=0} \times \\ \times\delta(\ell - x_2 v_3 + x_3 v_2) \quad (45)$$

Here coefficient at $1/\ell$ has a finite limit at $\ell \to 0$, which is true even without spherical symmetry.

As a result for

$$\hat{f}(\varepsilon, \ell) = \int\limits_{S_\ell} d^2\ell \, f\left(\varepsilon, \ell_k\right) \tag{46}$$

at $\ell \to 0$ we have $\hat{f}(\varepsilon, \ell) \approx C(\varepsilon)\ell$, i. e. *Case 2)* takes place, and hence, the core-profile will arise.

Situation B) – the formation of the static distribution in the given potential with the symmetry reduced to the axial symmetry.

A possible realization – the potential with axial symmetry already formed by the dark matter and we consider the formation of the static structure from ordinary matter inside this potential. Such symmetry agrees better with the observed galaxies.

Due to the lack of spherical symmetry, the previously mentioned description of particles' trajectories in terms of parameters $\varepsilon, \ell_k, \tau_l, \gamma$ will no longer be exact, but approximate, and not all of these parameters will be conserved with time.

Making the change of variables in (39) we obtain

$$\rho(r) = \frac{mr}{2\pi} \int d\varepsilon \, d^3\tilde{\ell} \, \frac{f(\varepsilon, r\tilde{\ell}_k) \, \Theta\left(2\varepsilon - 2\varphi(r) - \tilde{\ell}^2\right)}{\hat{T}\left(\varepsilon, r\tilde{\ell}_k\right) \sqrt{2\varepsilon - 2\varphi(r) - \tilde{\ell}^2}}, \tag{47}$$

hence

$$\rho(r) \sim r \, f(\varepsilon, r\tilde{\ell}_k) \tag{48}$$

With axial symmetry only ℓ_3 is conserved and the distribution $f(\varepsilon, \ell_k)$ of particles over $\ell_{||} \equiv \ell_{1,2}$ can change but only in a way that the value

$$\int d^2\ell_{||} \, f(\varepsilon, \ell_k) \tag{49}$$

is preserved.
As a result, the fastest growth of $f(\varepsilon, \ell_3, \ell_{||}) \sim 1/\ell_{||}^{\beta}$, with $\beta < 2$, and hence we can have *weak* cusp-profile $\rho(r) \sim r^{\alpha}$ with $\alpha > -1$.

Situation C) – the formation of the static distribution in a completely reduced symmetry or the gravitational potential forms simultaneously with the static structure and significant deviations from spherical symmetry are possible in this process.

Possible realization – we consider a formation of the static structure of the dark matter particles. Such a setup corresponds to numerical simulations.

Since no exact symmetry presents, the distribution $f(\varepsilon, \ell_k)$ over all components ℓ_k will change in time conserving only the value

$$\int d^3\ell\, f(\varepsilon, \ell_k) \tag{50}$$

As a result, the fastest growth of $f(\varepsilon, \ell_k) \sim 1/\ell^\beta$, with $\beta < 3$, and hence we have *strong* cusp-profile $\rho(r) \sim r^\alpha$ with $\alpha > -2$.

If we assume that $\hat{f}(\varepsilon, \ell)$ is analytical

$$\hat{f}(\varepsilon, \ell) \approx \hat{f}(\varepsilon, 0) + \hat{f}'(\varepsilon, 0)\ell, \tag{51}$$

then *case 1)* takes place and cusp-profile with $\alpha = -1$ arise, i.e. $\rho(r) \sim 1/r$.

(A. Kapustin, S. Paston, arXiv:2207.04288)