

Heat Kernel for Higher-Order Minimal Operators

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Outline:

Introductory part

- Motivation and our approach
- Definitions and general relations
- DeWitt's method and difficulties of its generalization

Our results

- “Generalized Exponential Functions” (GEF)
- Powers of a Laplace type operator
- σ -tensors and “generalized Fourier transform”
- Expansions for higher-order operators

Publications:

- A. O. Barvinsky, P. I. Pronin, and W. Wachowski, Phys. Rev. D100, 105004 (2019), arXiv:1908.02161 [hep-th].
- A. O. Barvinsky and W. Wachowski, arXiv:2112.03062 [hep-th].

Motivation

Effective action

$$\Gamma^{(1)}[\varphi] = \frac{i}{2} \log \det \hat{F}(\nabla) = \frac{1}{2} \int_0^\infty \frac{d\tau}{(4\pi i)^{d/2}} \tau^{-d/2-1} \int d^d x \sqrt{g} \operatorname{tr} \hat{K}(\tau|x, x).$$

It follows that, for example, for $d = 4$ the divergent part of the effective action is expressed in terms of the heat kernel coefficients \hat{a}_0 , \hat{a}_1 and \hat{a}_2 .

Key result (DeWitt)

For the minimal 2nd order operator

$$\hat{F}(\nabla) = -\hat{1}\square + \hat{P} + \frac{\hat{1}}{6} R, \quad (\square = g^{ab} \nabla_a \nabla_b)$$

$$[\hat{a}_0] = \hat{1}, \quad [\hat{a}_1] = -\hat{P},$$

$$[\hat{a}_2] = \frac{1}{180} (R_{abcd} R^{abcd} - R_{ab} R^{ab} + \square R) \hat{1} + \frac{1}{2} \hat{P}^2 + \frac{1}{12} \hat{\mathcal{R}}_{ab} \hat{\mathcal{R}}^{ab} - \frac{1}{6} \square \hat{P}.$$

Motivation II

Theories with higher derivatives or non-minimal operator

- R^2 -gravity
- Hořava–Lifshitz type models

$$\begin{aligned}\mathbb{B}^i{}_j(\nabla) = & -\frac{1}{2\sigma}\delta_j^i\Delta^3 - \frac{1}{2\sigma}\Delta^2\nabla_j\nabla^i - \frac{\xi}{2\sigma}\nabla^i\Delta\nabla^k\nabla_j\nabla_k \\ & - \frac{\xi}{2\sigma}\nabla^i\Delta\nabla_j\Delta + \frac{\lambda}{\sigma}\Delta^2\nabla^i\nabla_j + \frac{\lambda\xi}{\sigma}\nabla^i\Delta^2\nabla_j.\end{aligned}$$

Universal functional traces

$$\hat{F}(\nabla) = \hat{1}\square^M + \hat{P}(\nabla),$$

$$\text{Tr ln } \hat{F} = M \text{Tr ln } \hat{1}\square + \text{Tr ln} \left(\hat{1} + \frac{\hat{P}(\nabla)}{\square^M} \right), \quad \left[\nabla_{a_1} \cdots \nabla_{a_n} \frac{1}{\square^m} \delta(x, x') \right].$$

Two approaches in heat kernel studies

“Mathematicians”

Names: Hadamard — Minakshisundaram — Seeley — Gilkey — ...

Problems: Proof of general theorems. Connection with the theory of pseudodifferential operators, spectral geometry, index theorems, etc.

Feature: Only compact manifolds are considered.

“Physics”

Names: Fock — Schwinger — DeWitt — ...

Barvinsky–Vilkovisky (1985), Gusynin, Avramidi, Vassilevich, ...

Problems: Gradient expansion in effective field theory. An efficient algorithm for calculating the heat kernel coefficients is important.

Feature: Mainly interesting is the case of asymptotically flat space.

Heat kernel definition

Let a d -dimensional Riemannian manifold with metric g_{ab} is given (Euclidean case, $T^a{}_{bc} = 0$). Some bundle is given over it, which sections are the collection of fields $\varphi = \varphi^A$ (we will omit indices in the bundle, and label matrix-valued quantities with caps). Then the Riemann tensor and the curvature in the bundle are defined in the usual way:

$$[\nabla_a, \nabla_b]v^c = R^c{}_{dab}v^d, \quad [\nabla_a, \nabla_b]\varphi = \hat{\mathcal{R}}_{ab}\varphi.$$

Let, further, $\hat{F}(\nabla)$ is a strictly positive (pseudo-)differential operator acting on φ . Then the operator $e^{-\tau\hat{F}}$ satisfies the operator equation

$$\left(\partial_\tau + \hat{F}\right)e^{-\tau\hat{F}} = 0.$$

We define the heat kernel of the operator $\hat{F}(\nabla)$ as the kernel of the operator $e^{-\tau\hat{F}}$

$$\hat{K}(\tau|x, x'|) = e^{-\tau\hat{F}} \frac{1}{\sqrt{g}} \delta(x, x'),$$

$$\left(\partial_\tau + \hat{F}_x\right) \hat{K}(\tau|x, x'|) = 0, \quad \hat{K}(0|x, x') = \frac{1}{\sqrt{g}} \delta(x, x').$$

Resolvents and complex powers

$$\hat{F}_x \hat{G}_F(x, x') = \frac{\hat{1}}{\sqrt{g}} \delta(x, x').$$

Resolvents

$$\frac{1}{\hat{F} + \lambda} = \int_0^\infty d\tau e^{-\tau(\hat{F} + \lambda)}, \quad \hat{G}_{F+\lambda}(x, x') = \int_0^\infty d\tau e^{-\lambda\tau} \hat{K}(\tau|x, x'|).$$

Complex powers

$$\hat{F}^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} e^{-\tau \hat{F}}, \quad \hat{G}_{F^s}(x, x') = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \hat{K}(\tau|x, x'|),$$
$$e^{-\tau \hat{F}} = \frac{1}{2\pi i} \int_{w-i\infty}^{w+i\infty} \frac{\tau^{-s} \Gamma(s)}{\hat{F}^s} ds, \quad \hat{K}(\tau|x, x'|) = \frac{1}{2\pi i} \int_{w-i\infty}^{w+i\infty} \tau^{-s} \Gamma(s) \hat{G}_{F^s}(x, x') ds.$$

Why do we need heat kernels?

ζ -functional regularization

We need to know how to regularize expressions like $\hat{G}(x, x)$. Let us define

$$\hat{\zeta}_F(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \hat{K}(\tau|x, x).$$

It is known that $\hat{K}(\tau|x, x) \sim \tau^{-d/N}$. Therefore, this integral converges on the UV limit for $\text{Re } s > d/N$, and we continue analytically to the region where it diverges. Then we can propose the following regularization rule:

$$\hat{G}_{F^s}(x, x) \stackrel{\text{reg}}{=} \hat{\zeta}_F(s, x).$$

Compact manifolds

Let the manifold be compact. Then there is an orthonormal basis of eigenfunctions

$$|n\rangle = \varphi_n(x), \quad \hat{F}|n\rangle = \lambda_n |n\rangle, \quad \lambda_n > 0.$$

Functions of the operator

$$\begin{aligned} \hat{F} &= \sum_n \lambda_n |n\rangle \langle n|, & e^{-\tau \hat{F}} &= \sum_n e^{-\tau \lambda_n} |n\rangle \langle n|, \\ \frac{1}{\hat{F} + \lambda} &= \sum_n \frac{|n\rangle \langle n|}{\lambda_n + \lambda}, & \hat{F}^{-s} &= \sum_n \lambda_n^{-s} |n\rangle \langle n|. \end{aligned}$$

Heat trace and the operator ζ -function

$$\mathrm{Tr} e^{-\tau \hat{F}} = \sum_n e^{-\tau \lambda_n} = \int d^d x \sqrt{g} \, \mathrm{tr} \hat{K}(\tau|x, x),$$

$$\zeta_F(s) = \mathrm{Tr} \hat{F}^{-s} = \sum_n \lambda_n^{-s} = \int d^d x \sqrt{g} \, \mathrm{tr} \hat{\zeta}_F(s, x).$$

DeWitt's method

Laplace type operator

$$\hat{F}(\nabla) = \hat{1}\square + \hat{P}(x), \quad \square = -g^{ab}\nabla_a\nabla_b.$$

DeWitt's ansatz

$$\hat{K}(\tau|x, x') = \frac{\Delta^{1/2}(x, x')}{(4\pi\tau)^{d/2}} \exp\left(-\frac{\sigma(x, x')}{2\tau}\right) \sum_{m=0}^{\infty} \tau^m \hat{a}_m(F|x, x'),$$

$$\sigma_a = \nabla_a \sigma, \quad \sigma_a \sigma^a = 2\sigma; \quad \Delta(x, x') = \frac{\det(-\nabla_a \nabla_{b'} \sigma)}{\sqrt{g(x) g(x')}},$$

$$[\hat{K}(\tau)] = (4\pi\tau)^{-d/2} \sum \tau^m [\hat{a}_m].$$

Recurrence relations

$$(m + \sigma^a \nabla_a) \hat{a}_m = \Delta^{-1/2} \hat{F}(\nabla) \Delta^{1/2} a_{m-1}, \quad \sigma^a \nabla_a \hat{a}_0 = 0.$$

Inapplicability of the semiclassical approximation

WKB ansatz

$$\sqrt{\det \left[-\frac{1}{2\pi} \frac{\partial^2 S(\tau|\mathbf{x}, \mathbf{y})}{\partial x^a \partial y^b} \right]} \exp [-S(\tau|\mathbf{x}, \mathbf{y})],$$

where the action $S(\tau|\mathbf{x}, \mathbf{y})$ is the solution of the Hamilton–Jacobi equation

$$-\frac{\partial S}{\partial \tau} + F \left(-\frac{\partial S}{\partial \mathbf{x}} \right) = 0.$$

A power of Laplacian $F(\mathbf{p}) = (-\mathbf{p}^2)^\nu$

$$S(\tau|\mathbf{x}, \mathbf{y}) \sim \left(\frac{(\mathbf{x} - \mathbf{y})^2}{4\nu^2 \tau^{1/\nu}} \right)^{\frac{\nu}{2\nu-1}},$$

$$\left| \det \frac{\partial^2 S(\tau|\mathbf{x}, \mathbf{y})}{\partial x^a \partial y^b} \right|^{1/2} \sim \tau^{-d/2\nu} \left(\frac{(\mathbf{x} - \mathbf{y})^2}{4\tau^{1/\nu}} \right)^{-\frac{d}{2} \frac{\nu-1}{2\nu-1}}.$$

Where do $\mathcal{E}_{\nu,\alpha}(z)$ come from?

A power of Laplacian $(-\square)^\nu$ in flat space

$$(\partial_\tau + (-\square)^\nu) K_{\nu,d}(\tau|\mathbf{x}) = 0, \quad K_{\nu,d}(0|\mathbf{x}) = \delta(\mathbf{x}),$$
$$\nu = 1 \quad \Rightarrow \quad K_{1,d}(\tau|\mathbf{x}) = \frac{1}{(4\pi\tau)^{d/2}} \exp\left(-\frac{\mathbf{x}^2}{4\tau}\right).$$

What happens when $\nu \neq 1$?

Acting by analogy

$$K_{\nu,d}(\tau|\mathbf{x}) = \int \frac{d^d \mathbf{k}}{(2\pi)^d} \exp\left(-k^{2\nu}\tau + i\mathbf{k}\mathbf{x}\right),$$

depends on \mathbf{x} only as $|\mathbf{x}| = \sqrt{\mathbf{x}^2}$, $K_{\nu,d}(c^{2\nu}\tau|c\mathbf{x}) = c^{-d} K_{\nu,d}(\tau|\mathbf{x})$,

$$K_{\nu,d}(\tau|\mathbf{x}) = \frac{\tau^{-d/2\nu}}{(4\pi)^{d/2}} \mathcal{E}_{\nu,d/2}\left(-\frac{\mathbf{x}^2}{4\tau^{1/\nu}}\right).$$

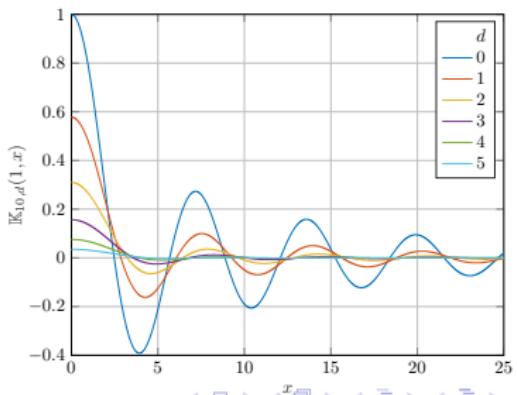
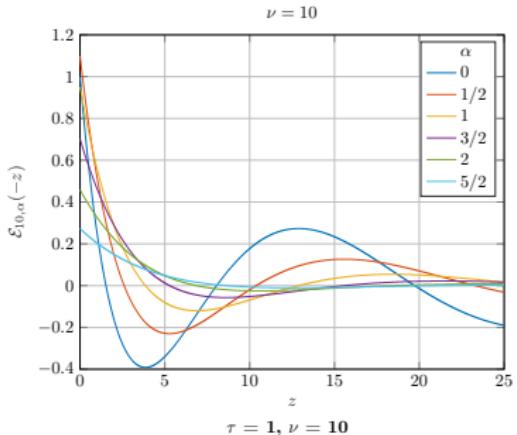
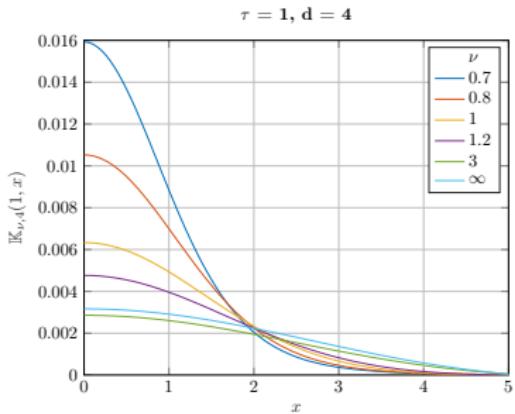
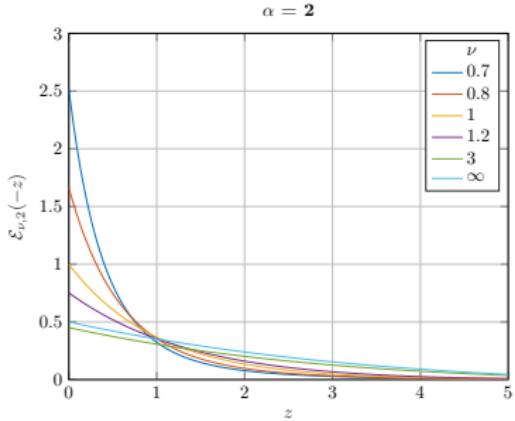
Taylor expansion of $\mathcal{E}_{\nu,\alpha}(z)$

$$\begin{aligned}(-\square)^m K_{\nu,d}(\tau, \mathbf{x}) \Big|_{\mathbf{x}=0} &= \frac{\tau^{-\frac{d/2+m}{\nu}}}{(4\pi)^{d/2}} \frac{\Gamma(d/2+m)}{\Gamma(d/2)} \mathcal{E}_{\nu,d/2}^{(m)}(0) \\&= \int \frac{d^d \mathbf{k}}{(2\pi)^d} k^{2m} \exp(-k^{2\nu}\tau) = \frac{\tau^{-\frac{d/2+m}{\nu}}}{(4\pi)^{d/2}} \frac{\Gamma\left(\frac{d/2+m}{\nu}\right)}{\nu\Gamma(d/2)}.\end{aligned}$$

Comparing the right and left sides, we get

$$\begin{aligned}\mathcal{E}_{\nu,\alpha}^{(m)}(0) &= \frac{\Gamma\left(\frac{\alpha+m}{\nu}\right)}{\nu\Gamma(\alpha+m)}, & \mathcal{E}_{\nu,\alpha}(z) &= \frac{1}{\nu} \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{\alpha+m}{\nu}\right)}{\Gamma(\alpha+m)} \frac{z^m}{m!}, \\ \mathcal{E}_{1,\alpha}(z) &= \exp(z).\end{aligned}$$

Plots of $\mathcal{E}_{\nu,\alpha}(-z)$ and $K_{\nu,d}(\tau|x|)$



Fox–Wright Ψ -functions and Fox H -functions

Ψ -functions

$${}_p\Psi_q[(a, A); (b, B); z] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p \Gamma(a_j + A_j k)}{\prod_{i=1}^q \Gamma(b_i + B_i k)} \frac{z^k}{k!},$$

$${}_pF_q[a; b; z] = {}_p\Psi_q[(a, 1); (b, 1); z] \Gamma(b) / \Gamma(a),$$

$$\mathcal{E}_{\nu, \alpha}(z) = \frac{1}{\nu} {}_1\Psi_1 \left[\left(\frac{\alpha}{\nu}, \frac{1}{\nu} \right); (\alpha, 1); z \right].$$

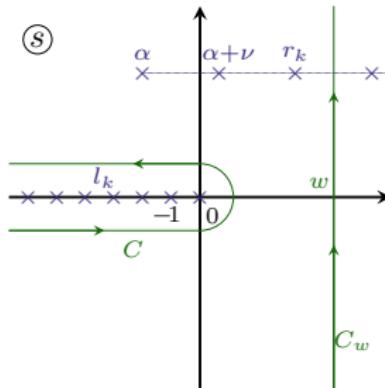
H -functions

$$h_{p,q}^{m,n}[s] = \frac{\prod_{i=1}^m \Gamma(b_i + B_i s) \prod_{j=1}^n \Gamma(1 - a_j - A_j s)}{\prod_{i=m+1}^q \Gamma(1 - b_i - B_i s) \prod_{j=n+1}^p \Gamma(a_j + A_j s)},$$

$$H_{p,q}^{m,n} \left[z \left| \begin{smallmatrix} (a,A) \\ (b,B) \end{smallmatrix} \right. \right] = \frac{1}{2\pi i} \int_C h_{p,q}^{m,n}[s] z^{-s} ds,$$

$${}_p\Psi_q \left[\left. \begin{smallmatrix} (a,A) \\ (b,B) \end{smallmatrix} \right| z \right] = H_{p,q+1}^{1,p} \left[-z \left| \begin{smallmatrix} (1-a,A) \\ (0,1), (1-b,B) \end{smallmatrix} \right. \right].$$

Mellin–Barnes integral representation



Mellin transform

$$\varepsilon_{\nu,\alpha}(s) = \frac{\Gamma\left(\frac{\alpha-s}{\nu}\right)\Gamma(s)}{\nu\Gamma(\alpha-s)},$$

$$\mathcal{E}_{\nu,\alpha}(-z) = \frac{1}{2\pi i} \int_C \varepsilon_{\nu,\alpha}(s) z^{-s} ds, \quad \int_0^\infty z^{s-1} \mathcal{E}_{\nu,\alpha}(-z) dz = \varepsilon_{\nu,\alpha}(s).$$

Asymptotics

non-integer ν

$$\mathcal{E}_{\nu,\alpha}(-z) = \frac{1}{\nu} \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{\alpha+m}{\nu}\right)}{\Gamma(\alpha+m)} \frac{z^m}{m!} = -z^{-\alpha} \sum_m \frac{(-1)^m}{m!} \frac{\Gamma(\alpha+m\nu)}{\Gamma(-m\nu)} z^{-m\nu}.$$

For $\nu > 1/2$ the first series converges, the second is asymptotic, and for $\nu < 1/2$ the opposite is true. For $\nu = 1/2$ we have

$$\mathcal{E}_{\frac{1}{2},\alpha}(z) = \frac{4^\alpha \Gamma\left(\alpha + \frac{1}{2}\right)}{\sqrt{\pi}} (1-4z)^{-\alpha-\frac{1}{2}}, \quad K_{\frac{1}{2},d}(\tau, \mathbf{x}) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}} \frac{\tau}{(\tau^2 + \mathbf{x}^2)^{\frac{d+1}{2}}}.$$

Positive integer $\nu = N$

$$\begin{aligned} \mathcal{E}_{N,\alpha}(-z) &= \frac{N^{-\frac{\alpha}{2N-1}} z^{-\alpha \frac{N-1}{2N-1}}}{\sqrt{2N-1}} \sum_{j=0}^{N-1} \exp \left[-(2N-1)e^{i\varphi_j} \left(\frac{z}{N^2} \right)^{\frac{N}{2N-1}} + i\varphi_j \alpha \right] \\ &\quad \times \sum_{m=0}^{\infty} \frac{E_m}{(2N-1)^m} \left(\frac{N^2}{z} \right)^{\frac{mN}{2N-1}} e^{-i\varphi_j m}. \end{aligned}$$

Other properties of $\mathcal{E}_{\nu,\alpha}(z)$

Differentiation

$$\frac{d^\beta}{dz^\beta} \mathcal{E}_{\nu,\alpha}(z) = \mathcal{E}_{\nu,\alpha+\beta}(z).$$

Connection with Bessel functions

$$\mathcal{C}_\alpha(z) = \sum_{m=0}^{\infty} \frac{1}{\Gamma(\alpha + 1 + m)} \frac{z^m}{m!}, \quad J_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \mathcal{C}_\alpha(-z^2/4),$$

$$K_{\nu,d}(\tau, \mathbf{x}) = \frac{2}{(4\pi)^{d/2}} \int_0^{\infty} k^{d-1} \exp(-k^{2\nu} \tau) \mathcal{C}_{\frac{d}{2}-1}(-k^2 x^2/4) dk,$$

$$\mathcal{E}_{\nu,\alpha}(z) = \frac{1}{\nu} \int_0^{\infty} d\mu \mu^{\alpha/\nu-1} e^{-\mu} \mathcal{C}_{\alpha-1}(z\mu^{1/\nu}),$$

$$\mathcal{E}_{\infty,\alpha}(z) = \mathcal{C}_\alpha(z), \quad K_{\infty,d}(\tau, \mathbf{x}) = \frac{1}{(2\pi x)^{d/2}} J_{d/2}(x).$$

Powers of a Laplace type operator

Direct/inverse Mellin transform trick

$$\hat{K}_F(\tau|x, x') = \frac{\Delta^{1/2}(x, x')}{(4\pi\tau)^{d/2}} \exp\left(-\frac{\sigma(x, x')}{2\tau}\right) \sum_{m=0}^{\infty} \tau^m \cdot \hat{a}_m^F(x, x'),$$

$$\begin{aligned} \hat{G}_{F^s}(x, x') &= \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \hat{K}_F(\tau|x, x') \\ &= \frac{\Delta^{1/2}(x, x')}{(4\pi)^{d/2}} \frac{1}{\Gamma(s)} \sum_{m=0}^{\infty} \Gamma\left(\frac{d}{2} - m - s\right) \left(\frac{\sigma}{2}\right)^{m+s-\frac{d}{2}} \cdot \hat{a}_m^F(x, x'), \end{aligned}$$

$$\begin{aligned} \hat{K}_{F^\nu}(\tau|x, x') &= \frac{1}{2\pi i} \int_{w-i\infty}^{w+i\infty} \tau^{-s} \Gamma(s) \hat{G}_{F^{\nu s}}(x, x') ds \\ &= \frac{\Delta^{1/2}(x, x')}{(4\pi\tau^{1/\nu})^{d/2}} \sum_{m=0}^{\infty} \mathcal{E}_{\nu, \frac{d}{2}-m} \left(-\frac{\sigma}{2\tau^{1/\nu}}\right) \tau^{m/\nu} \cdot \hat{a}_m^F(x, x'). \end{aligned}$$

Powers of a Laplace type operator II

Fegan–Gilkey formula (“functorial” property)

$$[\hat{K}_F(\tau)] = \tau^{-d/2} \sum_{m=0}^{\infty} \tau^m \cdot \hat{E}_m^F, \quad [\hat{K}_{F^\nu}(\tau)] = \tau^{-d/2\nu} \sum_{m=0}^{\infty} \tau^{m/\nu} \cdot \hat{E}_m^{F^\nu},$$
$$\hat{E}_m^{F^\nu} = \frac{\Gamma\left(\frac{d/2-m}{\nu}\right)}{\nu\Gamma(d/2-m)} \hat{E}_m^F.$$

Absence of $\log \tau$ -terms

$$\begin{array}{ccccc} \hat{K}_F(\tau|x, x') & \xrightarrow{\mathcal{M}} & \hat{G}_{F^s}(x, x') & \xrightarrow{F \rightarrow F^\nu} & \hat{G}_{F^{\nu s}}(x, x') \xrightarrow{\mathcal{M}^{-1}} \hat{K}_{F^\nu}(\tau|x, x') \\ \downarrow \sigma \rightarrow 0 & & & & \downarrow \sigma \rightarrow 0 \\ \hat{K}_F(\tau|x, x) & \xrightarrow{\mathcal{M}} & \hat{\zeta}_F(s, x) & \xrightarrow{F \rightarrow F^\nu} & \hat{\zeta}_{F^\nu}(s, x) \xrightarrow{\mathcal{M}^{-1}} \hat{K}_{F^\nu}(\tau|x, x) \end{array}$$

σ -tensors and parallel transport tensor

The definition of $\sigma^a(x, x')$ and $\hat{\mathcal{I}}(x, x')$ (on a manifold with ∇_a)

$$\begin{aligned}\sigma^a{}_b \sigma^b &= \sigma^a, & [\sigma^a] &= 0, & \nabla_{b_n} \cdots \nabla_{b_1} \sigma^a &= \sigma^a{}_{b_1 \dots b_n}, \\ \sigma^a \nabla_a \hat{\mathcal{I}} &= 0, & [\hat{\mathcal{I}}] &= \hat{1}, & \nabla_{b_n} \cdots \nabla_{b_1} \hat{\mathcal{I}} &= \hat{\mathcal{I}}{}_{b_1 \dots b_n}.\end{aligned}$$

Properties

$$\sigma^a{}_{b_1 \dots b_n} \sigma^{b_1} \dots \sigma^{b_n} = 0, \quad n \geq 2, \quad \sigma^{a_1} \dots \sigma^{a_k} \hat{\mathcal{I}}{}_{a_1 \dots a_k} = 0.$$

On a Riemannian manifold with metric g_{ab} we have $\sigma_a \sigma^a = 2\sigma$.

Coincidence limits

$$\begin{aligned}[\sigma^a{}_b] &= \delta^a_b, & [\sigma^a{}_{bc}] &= 0, & [\hat{\mathcal{I}}_a] &= 0, \\ [\sigma^a{}_{bcd}] &= \frac{2}{3} R^a{}_{(cd)b}, & & & [\hat{\mathcal{I}}_{ab}] &= \frac{1}{2} \hat{\mathcal{R}}_{ab}, \\ [\sigma^a{}_{bcde}] &= \frac{3}{2} \nabla_{(c} R^a{}_{de)b}, & & & [\hat{\mathcal{I}}_{abc}] &= \frac{2}{3} \nabla_{(a} \hat{\mathcal{R}}_{b)c}.\end{aligned}$$

“Generalized Fourier transform” method

The problem under consideration

$$\hat{F}(\nabla) = \sum_{k=0}^N \hat{F}_k^{a_1 \dots a_k}(x) \nabla_{a_1} \dots \nabla_{a_k} = \sum_{k=0}^N \hat{F}_k(x) * \nabla^k,$$

$$(\partial_\tau + \hat{F}_x) \hat{K}(\tau|x, x') = 0, \quad \hat{K}(0|x, x') = \frac{1}{\sqrt{g}} \delta(x, x').$$

Substitution of integrals

$$\hat{\delta}(x, x') = \int \frac{d^d k}{(2\pi)^d} \exp(i k_{a'} \sigma^{a'}) \hat{\mathcal{I}}(x, x'),$$

$$\hat{K}(\tau|x, x') = \int \frac{d^d k}{(2\pi)^d} \exp(i k_{a'} \sigma^{a'}) \hat{\mathbf{K}}(\tau, \mathbf{k}|x, x'),$$

Equation for “Fourier image”

The solution via “elongated derivatives”

$$\exp(-ik_{b'}\sigma^{b'})\hat{F}(\nabla_a)\exp(ik_{b'}\sigma^{b'}) = \hat{F}(\nabla_a + ik_{b'}\sigma_a^{b'}),$$

$$\hat{\mathbf{K}}(\tau, \mathbf{k}|x, x') = \exp\left(-\tau \hat{F}(\nabla_a + ik_{b'}\sigma_a^{b'})\right) \hat{\mathcal{I}}(x, x').$$

Cauchy problem for the image

$$\left(\partial_\tau + \hat{F}(\nabla_a + ik_{b'}\sigma_a^{b'})\right) \hat{\mathbf{K}}(\tau, \mathbf{k}|x, x') = 0, \quad \hat{\mathbf{K}}(0, \mathbf{k}|x, x') = \hat{\mathcal{I}}(x, x').$$

Definition of $\llbracket \hat{F} \rrbracket_m$ operators

$$\hat{F}\left(\nabla_a + ik_{b'}\sigma_a^{b'}\right) = \sum_{m=0}^N (i\mathbf{k})^m * \llbracket \hat{F} \rrbracket_m,$$

$$\llbracket \hat{F} \rrbracket_N \equiv \llbracket \hat{F} \rrbracket_N^{b'_1 \dots b'_N} = \hat{F}_N^{a_1 \dots a_N} \sigma_{a_1}^{b'_1} \dots \sigma_{a_N}^{b'_N}, \quad \dim \llbracket \hat{F} \rrbracket_m = N - m.$$

The operator $\hat{\mathbf{T}}(\nabla, \tau, \mathbf{k}|x, x')$

Our ansatz

$$\hat{\mathbf{K}}(\tau, \mathbf{k}) = \exp \left(-\tau(i\mathbf{k})^N * [\![\hat{F}]\!]_N \right) \hat{\mathbf{T}}(\nabla) \hat{\mathcal{I}}.$$

Rearrange operators and exponents once again

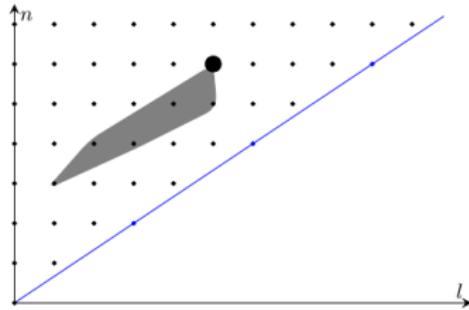
$$\tilde{\nabla}_a = \nabla_a - \tau(i\mathbf{k})^N * \left(\nabla_a [\![\hat{F}]\!]_N \right), \quad [\![\hat{F}]\!]_m(\tilde{\nabla}) = \sum_{n=0}^{N-m} \left(\tau(i\mathbf{k})^N \right)^n * [\![\hat{F}]\!]_{m,n}.$$

Equation for the operator $\hat{\mathbf{T}}(\nabla, \tau, \mathbf{k}|x, x')$

$$\left(\partial_\tau + \hat{\mathbf{F}} \right) \hat{\mathbf{T}}(\nabla, \tau, \mathbf{k}) = 0, \quad \hat{\mathbf{T}}(\nabla, 0, \mathbf{k}) = \hat{1},$$

$$\hat{\mathbf{F}}(\nabla, \tau, \mathbf{k}) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-m} \tau^n (i\mathbf{k})^{m+Nn} \cdot [\![\hat{F}]\!]_{m,n}.$$

Recurrence procedure



Recurrence relations for $\hat{T}_{n,l}(\nabla)$

$$\hat{T}(\nabla, \tau, \mathbf{k}) = \sum_{n=0}^{\infty} \sum_{0 \leq l \leq L_n} \tau^n (i\mathbf{k})^l * \hat{T}_{n,l}(\nabla), \quad L_n(N) = \left(N - \frac{1}{2}\right) n,$$

$$(n+1)\hat{T}_{n+1,l} = - \sum_{p=0}^{N-1} \sum_{q=0}^{N-p} [\![\hat{F}]\!]_{p,q} \hat{T}_{n-q, l-p-Nq},$$

$$\hat{T}_{0,0} = \hat{1}, \quad \hat{T}_{0,l} = 0, \quad l > 0; \quad \dim \hat{T}_{n,l}(\nabla) = Nn - l.$$

Integration over momenta

$$\hat{K}(\tau|x, x') = \sum_{k=0}^{\infty} \tau^n \sum_{0 \leq l \leq L_n} \hat{S}_l(\tau) * \hat{T}_{n,l}(\nabla) \hat{\mathcal{I}}(x, x'),$$

$$\hat{S}_{l, b'_1 \dots b'_l}(\tau|x, x') = \int \frac{d^d k}{(2\pi)^d} (i\mathbf{k})^l \exp \left(-\tau (i\mathbf{k})^N \cdot [\![\hat{F}]\!]_N + i k_{a'} \sigma^{a'} \right)$$

$$= (\det \bar{\sigma}_{b'}^a) \bar{\sigma}_{b'_1}^{a_1} \dots \bar{\sigma}_{b'_l}^{a_l} \int \frac{d^d p}{(2\pi)^d} (ip_{a_1}) \dots (ip_{a_l}) \exp \left(-\tau \hat{F}_N(x, \mathbf{p}) + ip_a \sigma^a \right).$$

Here we have made the substitution $k_{a'} \mapsto p_a = \sigma_a^{b'} k_{b'}$ in order to explicitly extract the operator symbol

$$\hat{F}_N(x, \mathbf{p}) = \hat{F}_N^{a_1 \dots a_N}(x) \times (ip_{a_1}) \dots \times (ip_{a_N}).$$

Minimal operators

$$\hat{F}(\nabla) = \hat{1}(-\square)^M + \hat{P}(\nabla), \quad \hat{F}_N(x, \mathbf{p}) = \hat{1}p^{2M},$$

$$\int \frac{d^d p}{(2\pi)^d} \exp(-\tau p^{2M} + i p_a \sigma^a) = \frac{g^{1/2}(x)}{(4\pi \tau^{1/M})^{d/2}} \mathcal{E}_{M, d/2} \left(-\frac{\sigma}{2\tau^{1/M}} \right),$$

$$\left(\prod_{j=1}^l \bar{\sigma}_{a'_j}^{b_j} \frac{\partial}{\partial \sigma^{b_j}} \right) \mathcal{E}_{M, \frac{d}{2}} \left(-\frac{\sigma^b \sigma_b}{4\tau^{1/M}} \right) = \sum_{p \geq \frac{l}{2}}^l \frac{S_{p, l, a'_1 \dots a'_l}}{(-2\tau^{1/M})^p} \mathcal{E}_{M, \frac{d}{2} + p} \left(-\frac{\sigma}{2\tau^{1/M}} \right).$$

$$\gamma_{a'b'} = \bar{\sigma}_{a'}^c g_{cd} \bar{\sigma}_{b'}^d,$$

$$S_{1,1} = \sigma_{a'},$$

$$S_{1,2} = \gamma_{a'b'},$$

$$S_{2,2} = \sigma_{a'} \sigma_{b'},$$

$$S_{2,3} = 3\gamma_{(a'b'} \sigma_{c')},$$

$$S_{2,4} = 3\gamma_{(a'b'} \gamma_{c'd')}$$

$$S_{3,3} = \sigma_{a'} \sigma_{b'} \sigma_{c'},$$

$$S_{3,4} = 6\gamma_{(a'b'} \sigma_{c'} \sigma_{d')},$$

Our main results

Double functional series

$$\hat{K}(\tau|x, x') = \frac{\Delta^{-1}(x, x')}{(4\pi\tau^{1/M})^{d/2}} \sum_{m=-\infty}^{\infty} \tau^{\frac{m}{M}} \sum_{n \geq N_m}^{\infty} \mathcal{E}_{M, \frac{d}{2} + Mn - m} \left(-\frac{\sigma}{2\tau^{1/M}} \right) \hat{b}_{m,n}(x, x'),$$

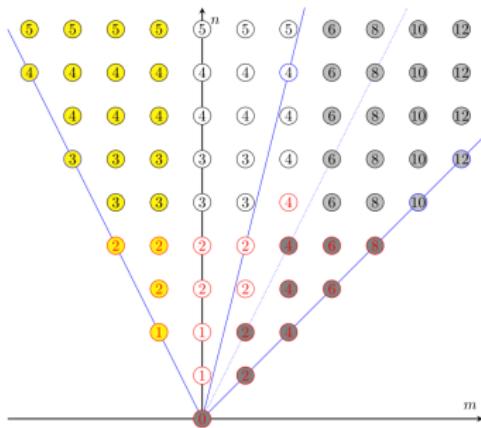
$$\hat{b}_{m,n}(x, x') = \frac{1}{(-2)^{Mn-m}} \sum_{l=Mn-m}^{\lfloor L_{m,n} \rfloor} S_{Mn-m, l}(x, x') * \hat{T}_{n,l}(\nabla) \hat{\mathcal{I}}(x, x'),$$

$$N_m(M) = \begin{cases} \frac{m}{M}, & m > 0, \\ \frac{2|m|}{2M-1}, & m < 0, \end{cases}, \quad L_{m,n} = 2Mn - \max \left\{ 2m, \frac{n}{2} \right\}.$$

Three step algorithm:

- ① Compute operators $\llbracket \hat{F} \rrbracket_{p,q}$
- ② Recurrently find $\hat{T}_{n,l}(\nabla)$ up to the required order
- ③ Convolve obtained operators with $S_{p,l}$

Laplace type operator — spurious coefficients

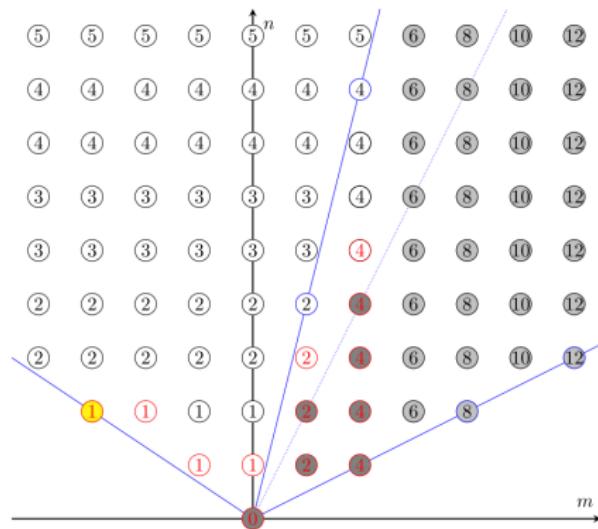


Examples of coefficients:

$$\hat{b}_{-1,3} = \frac{1}{12} \sigma^{a'_1} \sigma^{a'_2} \sigma^{b_1} \sigma^{b_2} (\sigma_{a'_1}{}^c b_1 \sigma_{a'_2}{}^{cb_2} + \sigma_{a'_1}{}^{b_1}{}^c \sigma_{a'_2}{}^{cb_2} + \sigma_{a'_1}{}^{b_1}{}^c \sigma_{a'_2}{}^{b_2 c}) \hat{\mathcal{I}} \equiv 0,$$

$$\begin{aligned} \hat{b}_{0,2} = & \frac{1}{2} \sigma^{b'} \sigma^c (\sigma_{b'}{}^{ac} + \sigma_{b'}{}^{ca}) \hat{\mathcal{I}}_a + \frac{1}{8} \left(\sigma^{a'} (4\sigma_{a'}{}^b{}^b + \sigma^{b'} (2\sigma_{a'}{}^{cd} \sigma_{b'}{}^{cd} + \sigma_{a'}{}^c \sigma_{b'}{}^{d c})) \right. \\ & \left. + 2\sigma^b (\sigma_{a'}{}^{bc}{}^c + \sigma_{a'}{}^c{}^b) \right) + 4\sigma^a \bar{\sigma}_{cb'} (\sigma^{b'}{}_a{}^c + \sigma^{b'}{}_c{}^a) \hat{\mathcal{I}}. \end{aligned}$$

Coefficients for 4th order operator



An example of a non-trivial coefficient

$$\hat{F}(\nabla) = \square^2 + \hat{\Omega}^{abc} \nabla_a \nabla_b \nabla_c + \hat{D}^{ab} \nabla_a \nabla_b + H^a \nabla_a + \hat{P},$$

$$\hat{b}_{-1,1}(x, x') = \frac{1}{8} (4\sigma\sigma_{a'}\square\sigma^{a'} + \hat{\Omega}^{abc}\sigma_a\sigma_b\sigma_c) \hat{\mathcal{I}}(x, x').$$

Coefficients E_2 and E_4 without $\hat{\Omega}^{abc}$

$$\hat{E}_2(x) = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(\frac{d/2-1}{2}\right)}{2\Gamma(\frac{d}{2}-1)} \left\{ \frac{1}{2d} \hat{D} + \frac{\hat{1}}{6} R \right\}.$$

$$\begin{aligned} \hat{E}_4(x) = & \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(\frac{d}{4}\right)}{4\Gamma\left(\frac{d}{2}\right)} \left\{ (d-2) \left(\frac{\hat{1}}{90} R_{abcd}^2 - \frac{\hat{1}}{90} R_{ab}^2 + \frac{\hat{1}}{36} R^2 + \frac{1}{6} \hat{\mathcal{R}}_{ab}^2 + \frac{\hat{1}}{15} \square R \right) \right. \\ & - \frac{1}{3} \hat{D}^{ab} R_{ab} + \frac{1}{6} \hat{D} R + \frac{1}{d+2} \left(\frac{1}{2} \hat{D}_{ab} \hat{D}^{ab} + \frac{1}{4} \hat{D}^2 - \frac{2}{3} (d+1) \nabla_a \nabla_b \hat{D}^{ab} \right. \\ & \quad \left. \left. + \frac{1}{6} (d+4) \square \hat{D} \right) - 2 \hat{P} + \nabla_a \hat{H}^a \right\}. \end{aligned}$$

Our results:

- For differential operators of general form, using the “generalized Fourier transform”, a recurrence procedure for finding the expansion of the heat kernel in powers of the background dimension is developed.
- When applied to higher-order minimal operators, it leads to a double GEF function series. The coefficients of the series are efficiently calculated using a three-step algorithm (even outside the coincidence limit).
- A feature of this expansion is the presence of nontrivial coefficients at negative powers of τ . This is the reason why DeWitt’s method cannot be directly applied to higher order operators.
- A program has been written in Wolfram Mathematica that allows to calculate coefficients.
- Expansions for powers of operators are obtained. In particular, it gives a new proof of the Fegan–Gilkey formula.