



«On explicit cutoff regularization»

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Models in Quantum Field Theory
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Theory of regularization

1) Integral calculus

$$\int_0^1 \frac{ds}{s} \rightarrow \int_\epsilon^1 \frac{ds}{s} = -\ln \epsilon \text{ or } \int_0^1 \frac{ds}{s^{1-\epsilon}} = \frac{1}{\epsilon}$$

2) Theory of generalized functions

Sokhotski–Plemelj theorem
(Sokhotski formula)

3) Quantum field theory: Feynman diagrams and perturbative expansions

We need to deform $G \rightarrow G_\Lambda$ near $x \sim y$

type
dimensional
cutoff
high covariant derivative

deform
~~dimension parameter~~
~~gauge invariance~~
~~operator of second order~~

The main properties

1) Spectral representation: let $A\varphi_\lambda = \lambda^2\varphi_\lambda$ and $AG = \mathbf{1}$, then $G_\Lambda = \mathfrak{J}_\Lambda G$, where

$$\mathfrak{J}_\Lambda(x, y) = \int_{\mathbb{R}_+} d\mu(\lambda) \phi_\lambda(x) \rho(\lambda/\Lambda) \phi_\lambda^*(y);$$

2) Explicit cutoff in coordinate representation near $x \sim y$:

$$|x - y|_\Lambda = \begin{cases} |x - y|, & |x - y| > 1/\Lambda; \\ 1/\Lambda, & |x - y| \leq 1/\Lambda, \end{cases} \quad \text{and} \quad G_\Lambda(x, y) = \frac{1}{4\pi|x - y|_\Lambda^2} (1 + o(1)) \quad \text{in } \mathbb{R}^4;$$

3) Homogenization:

$$\frac{1}{4\pi|x|_\Lambda^2} = \frac{1}{S_3} \int_{\mathbb{S}^3} d\tilde{\sigma}(y) \frac{\omega(y)}{4\pi|y/\Lambda + x|^n}, \quad \frac{1}{S_3} \int_{\mathbb{S}^3} d\tilde{\sigma}(y) \omega(y) = 1;$$

4) Covariance: Does \mathfrak{J}_Λ inherit physical symmetries of A ?

The simplest example

Let $d \in \mathbb{N}$, $x \in \mathbb{R}^d$, $A(x) = -\partial_{x_\mu}\partial_{x^\mu}$, and

$$G(|x|) = \begin{cases} -|x|/2, & d = 1; \\ -\ln(|x|)/2\pi, & d = 2; \\ |x|^{2-d}/(d-2)S_{d-1}, & d \geq 3, \end{cases}$$

then we have

$$G(|x|_\Lambda) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d^d y e^{ix_\mu y^\mu} \hat{G}(|y|) \rho(|y|/\Lambda) = \frac{1}{S_{d-1}} \int_{\mathbb{S}^{d-1}} d\sigma(y) G(|x - y/\Lambda|),$$

where

$$\rho(|t|/\Lambda) = \Gamma(d/2) \left(\frac{|t|}{2\Lambda} \right)^{1-d/2} J_{d/2-1}(|t|/\Lambda).$$

Yang–Mills theory

Yang and Mills (1954)
Trautman (1979)
Babelon and Viallet (1981)

Let G be a compact semisimple Lie group, and \mathfrak{g} is its Lie algebra.

Then, t_a denote the corresponding generators, which satisfy $[t^a, t^b] = f^{abc}t^c$, $\text{tr}(t^a t^b) = -2\delta^{ab}$.

Yang-Mills connection components: $B_\mu(x) = B_\mu^a(x)t^a$.

Components of the field strength: $F_{\mu\nu}(x) = F_{\mu\nu}^a(x)t^a$, $F_{\mu\nu}^a = \partial_\mu B_\nu^a - \partial_\nu B_\mu^a + f^{abc}B_\mu^b B_\nu^c$.

Covariant Laplace operator is $M_{\mu\nu}^{ab}(x; \alpha) = -D_\sigma^{ac}(x)D_\sigma^{cb}(x)\delta_{\mu\nu} - 2\alpha f^{acb}F_{\mu\nu}^c(x)$.

DeWitt(1965)
Seeley(1967)
Luscher(1982)

$$G_{\mu\nu}(x, y) = R_0(x - y)a_{0\mu\nu}(x, y) + R_1(x - y)a_{1\mu\nu}(x, y) + R_2(x - y)a_{2\mu\nu}(x, y) + PS(x, y),$$

$$R_0(x) = \frac{1}{4\pi^2|x|^2}, \quad R_1(x) = -\frac{\ln(|x|^2\mu^2)}{16\pi^2}, \quad R_2(x) = \frac{|x|^2(\ln(|x|^2\mu^2) - 1)}{64\pi^2}.$$

Problem statement

Additional restrictions:

$$B_\mu^a(x) = \frac{s}{2} x^\nu \xi_{\nu\mu}^a, \text{ where } (\xi^a)_{\mu\nu} = \frac{1}{\sqrt{8 \dim \mathfrak{g}}} \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \text{ for all } a \in \{1, \dots, \dim \mathfrak{g}\},$$

then, with $z = x - y$, we have

$$G_{\mu\nu}(x, y) = R_0(z)\delta_{\mu\nu} + 2\alpha s \xi_{\mu\nu} R_1(z) - \frac{\alpha^2 s^2 \delta_{\mu\nu} \xi_{\sigma\beta} \xi_{\sigma\beta}}{2} R_2(z) - \frac{s^2 \delta_{\mu\nu} \xi_{\sigma\beta} \xi_{\sigma\beta} |z|^2 (1 - 2\alpha^2)}{2^9 \pi^2} + o(s^2).$$

We need to calculate $\rho\left(\sqrt{\hat{M}(x, y)/\Lambda}\right) G(x, y)$, where $\rho(r) = \frac{2J_1(r)}{r}$.

Results

$$\begin{aligned}
& \tilde{R}_0(z)\delta_{\mu\nu} + 2\alpha s\xi_{\mu\nu}\left(\bar{R}_0(z) + \tilde{R}_1(z)\right) \\
& + s^2\delta_{\mu\nu}\xi_{\sigma\beta}\xi_{\sigma\beta}\left(\frac{3|z|^2\Lambda^2 - 2\alpha^2}{2^43\Lambda^2}\bar{R}_0(z) + \frac{1 - \alpha^2}{2^5\Lambda^2}\hat{R}_0(z) - \alpha^2\bar{R}_1(z) \right. \\
& \quad \left. - \frac{\alpha^2}{2}\tilde{R}_2(z) - \frac{(|z|^2 + \Lambda^{-2})(1 - 2\alpha^2)}{2^9\pi^2}\right) + \mathcal{O}(s^3).
\end{aligned}$$

$$\tilde{R}_0(x) = \frac{1}{4\pi^2} \begin{cases} |x|^{-2}, & |x| > 1/\Lambda; \\ \Lambda^2, & |x| \leq 1/\Lambda, \end{cases}$$

$$\tilde{R}_1(x) = \frac{1}{4\pi^2} \begin{cases} -\frac{1}{4}\ln(|x|^2\mu^2) - \frac{1}{8}|x|^{-2}\Lambda^{-2}, & |x| > 1/\Lambda; \\ \frac{1}{2}L - \frac{1}{8}|x|^2\Lambda^2, & |x| \leq 1/\Lambda, \end{cases}$$

$$\tilde{R}_2(x) = \frac{1}{4\pi^2} \begin{cases} \frac{1}{16}|x|^2(\ln(|x|^2\mu^2) - 1) + \frac{1}{16}\Lambda^{-2}\ln(|x|^2\mu^2) + \frac{1}{96}|x|^{-2}\Lambda^{-4} + \frac{1}{32}\Lambda^{-2}, & |x| > 1/\Lambda; \\ -\frac{1}{8}\Lambda^{-2}L - \frac{1}{8}|x|^2L + \frac{1}{96}|x|^4\Lambda^2 + \frac{1}{32}|x|^2 - \frac{1}{16}\Lambda^{-2}, & |x| \leq 1/\Lambda. \end{cases}$$

Result for 2-D Sigma-model

Operator is $M^{ab}(x) = -D_{x_\mu}^{ab}\partial_{x^\mu}$. Then, after some transformations and simplifications we get

$$G(x - y) = R_0(x - y) + s^2 m R_1(x - y) + o(s^2)$$

$$\text{with } R_0(x) = -\frac{1}{4\pi} \ln(|x|^2 \mu^2), \quad R_1(x) = \frac{|x|^2 (\ln(|x|^2 \mu^2) - 2)}{16\pi}.$$

$$\tilde{R}_0(x) = \frac{1}{4\pi} \begin{cases} -\ln(|x|^2 \mu^2), & |x| > 1/\Lambda; \\ 2L, & |x| \leq 1/\Lambda, \end{cases}$$

$$\tilde{R}_1(x) = \frac{1}{4\pi} \begin{cases} \frac{1}{4}|x|^2 (\ln(|x|^2 \mu^2) - 2) + \frac{1}{4}\Lambda^{-2} \ln(|x|^2 \mu^2), & |x| > 1/\Lambda; \\ -\frac{1}{2}(L+1)\Lambda^{-2} - \frac{1}{2}L|x|^2, & |x| \leq 1/\Lambda, \end{cases}$$

Many thanks!