Schwinger-Keldysh diagram technique for instantonic systems

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Why QFT?

- QFT — the best predictability power
- Main quantity to calculate — correlation function
- Correlation function:
    - should by translated into observable quantity, e.g. scattering amplitude/response function
    - usually calculated using perturbation theory
    - perturbative calculation = diagram technique — significantly depends on the problem

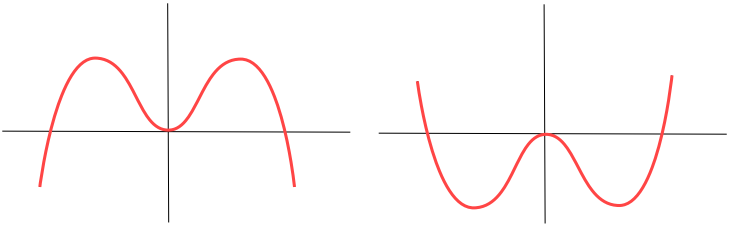
Types of diagram technique

1. Scattering problem — in-out correlation functions, Feynman technique
2. Solid state — in-in correlation functions, Schwinger-Keldysh technique
3. In-in thermal state correlation functions allows simplification — analytic continuation from imaginary-time (Matsubara) correlation functions. Difficulties
   - Only few types of correlation functions can be obtained by analytic continuation
   - Numerical analytic continuation is ill-posed problem

In this talk

We want to calculate real-time correlation functions in instantonic systems

$$\langle \phi(t_1) \ldots \phi(t_n) \rangle = \frac{1}{Z} \operatorname{tr}\left[e^{-\beta H} \phi(t_1) \ldots \phi(t_n) \right]$$



Plan of the talk

- Instantonic correlation functions in **imaginary time**
- Instantonic correlation functions in **real time**
- Further comments and applications

Instantons

Arise in different places in QM and QFT

1. Tunneling processes
2. False vacuum decay
3. Structure of vacuum (periodic potentials, Yang-Mills, etc)
   (Callan, Dashen, Gross; ...)
4. Initial states of Universe
   (Halliwell, Myers; Barvinsky, Kamenschik)

Imaginary time correlation functions I

- Consider Euclidean quantum mechanics, described by the partition function

$$Z = \mathrm{Tr}\, e^{-\beta \hat{H}} = \int\limits_{\text{periodic}} \mathcal{D}x\ e^{-S[x]}, \qquad S[x] = \int_0^\beta d\tau\ \left[\frac{\dot{x}^2}{2} + V(x)\right]$$

- Zero mode issue! Expand the action about a saddle point x_c

$$\ddot{x}_c(\tau) - V'(x_c) = 0, \qquad x(0) = x(\beta),$$

and go to the integration over perturbation η, i.e. $x = x_c + \eta$

$$Z = e^{-S[x_c]} \int\limits_{\text{periodic}} \mathcal{D}\eta\ \exp\left\{-S^{(2)}[x_c, \eta] - S^{\text{int}}[x_c, \eta]\right\}$$

Imaginary time correlation functions II

where $S^{(2)}[x_c, \eta]$ is quadratic part of $S[x_c + \eta]$

$$S^{(2)}[x_c, \eta] = \frac{1}{2} \int d\tau \, d\tau' \; \eta(\tau) \, K(\tau, \tau') \, \eta(\tau'),$$
$$K(\tau, \tau') = [-\partial_\tau^2 + V''(x_c)] \, \delta(\tau - \tau').$$

Taking derivative of e.o.m.

$$[-\partial_\tau^2 + V''(x_c)]\dot{x}_c(\tau) = 0,$$

we observe that $\dot{x}_c$ — zero mode of K.

- Reason — translation invariance of the action in Euclidean time

$$S[x^{\tau_0}] = S[x], \qquad x^{\tau_0}(\tau) = x(\tau + \tau_0).$$

Integration over zero-mode is non-Gaussian!

Imaginary time correlation functions III

- Solution — gauge-fixing

$$1 = \frac{1}{\sqrt{2\pi\xi}} \int_0^\beta d\tau_0 \, \frac{d\chi[x^{\tau_0}]}{d\tau_0} \, \exp\Big\{-\frac{1}{2\xi}\chi[x^{\tau_0}]^2\Big\},$$

where

$$\chi[x] = \frac{1}{\|\dot{x}_c\|} \int_0^\beta d\tau \, \dot{x}_c(\tau) x(\tau)$$

- Partition function takes the form

$$Z = \frac{\beta}{\sqrt{2\pi\xi}} \int \mathcal{D}x \; J[x] \; e^{-S_\xi[x]},$$

$$S_\xi[x] = S[x] + \frac{1}{2\xi}\chi[x]^2, \qquad J[x] = \frac{d\chi[x^{\tau_0}]}{d\tau_0}\bigg|_{\tau_0=0}$$

Imaginary time correlation functions IV

- Perturbative expansion for Z originates from

$$Z = \frac{\beta\|\dot{x}_c\|}{\sqrt{2\pi\xi}} e^{-S[x_c]} \int \mathcal{D}\eta \Big[1 + \frac{1}{\|\dot{x}_c\|} \int_0^\beta d\tau\, \eta_0(\tau)\dot{\eta}(\tau)\Big] e^{-S^{(2)}_\xi[x_c,\eta] - S^{\mathsf{int}}[x_c,\eta}$$

$$S^{(2)}_\xi[x_c,\eta] = \frac{1}{2}\int d\tau\, d\tau'\ \eta(\tau)\, K_\xi(\tau,\tau')\, \eta(\tau'),$$

$$K_\xi(\tau,\tau') = [-\partial_\tau^2 - V''(x_c)]\,\delta(\tau-\tau') + \frac{1}{\xi\|\dot{x}_c\|^2}\dot{x}_c(\tau)\dot{x}_c(\tau').$$

- Inserting $x(t)$ to path integral, we get a puzzle!

$$\langle x(t)\rangle = x_c(t) + \text{ corrections.}$$

Imaginary time correlation functions V

- Generalize the partition function Z to the generating functional

$$Z[j] = \mathrm{Tr}\Big[e^{-\beta\hat{H}} T_\tau \exp\Big(\int_0^\beta d\tau\, j(\tau)\hat{x}(\tau)\Big)\Big],$$

in terms of which the n-point correlation function reads

$$D(\tau_1, \dots \tau_n) = \mathrm{Tr}\Big[e^{-\beta\hat{H}} T_\tau\Big(\hat{x}(\tau_1)\dots\hat{x}(\tau_n)\Big)\Big] = \frac{1}{Z}\frac{\delta^n Z[j]}{\delta j(\tau_1)\dots\delta j(\tau_n)}\bigg|_{j=}$$

Path integral representation of $Z[j]$ has the form

$$Z[j] = \mathrm{Tr}\, e^{-\beta\hat{H}} = \int_{\text{periodic}} \mathcal{D}x\; e^{-S[x]+\int_0^\beta d\tau\, j(\tau)x(\tau)}$$

Imaginary time correlation functions VI

- The calculation of $Z[j]$ repeats those of Z, except that the integration over τ_0, originated from the partition of unity

$$Z[j] = \frac{1}{\sqrt{2\pi\xi}\|\dot{x}_c\|} \int \mathcal{D}x \; J[x] \; e^{-S_\xi[x]} \times \int_0^\beta d\tau_0 \; e^{\int_0^\beta d\tau \, j(\tau) x(\tau - \tau_0)},$$

- Now, observables are gauge-invariant!

$$D(\tau_1, \ldots \tau_n) \propto \int \mathcal{D}x \left[\int_0^\beta d\tau_0 \, x(\tau_1 - \tau_0) \ldots x(\tau_n - \tau_0) \right] J[x] \; e^{-S_\xi[x]}$$

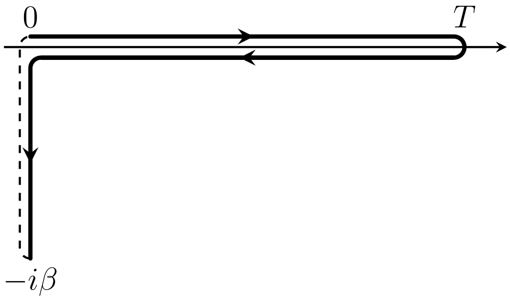
Some comments

- In general, obtained correlators cannot be analytically continued to real times **(Evans 1992; Baier, Niegawa 1994)**
- Imaginary time instantonic correlators first arise in QCD context **(Polyakov 1976; Callan, Dashen, Gross 1976)**
- Loop calculations a rarely known (for partition function only!) **(Lowe, Stone 1978; Bezoglov, Onischenko 2017; Shuryak, Turbiner 2018)**
- Recent interest in resurgence methods community **(Dunne, Unsal et al)**

Schwinger-Keldysh path integral I

- Schwinger-Keldysh correlation function generating functional

$$Z[j_+, j_-] = \mathrm{Tr}\left[e^{-\beta\hat{H}} T_C \exp\left(i\int_0^T d\tau\, j_+(\tau)\hat{x}(\tau) - i\int_0^T d\tau\, j_-(\tau)\hat{x}(\tau)\right)\right]$$



Schwinger-Keldysh path integral II

- Path integral form

$$Z[j_+, j_-] = \int\limits_{x_e(0)=x_e(\beta)} \mathcal{D}x_e \int\limits_{\substack{x_\pm(0)=x_e(\mp 0)\\ x_+(T)=x_-(T)}} \mathcal{D}x_+\,\mathcal{D}x_-$$

$$\exp\left\{-S_e[x_e] + iS[x_+] - iS[x_-] + i\int_0^T dt\, j_+ x_+ - i\int_0^T dt\, j_- x_-\right\}$$

- Real-time contour breaks the symmetry $\tau \mapsto \tau + \tau_0$! However, zero-mode is still present

$$\eta_0(z) \propto \partial_z x_c(z), \qquad z \in C$$

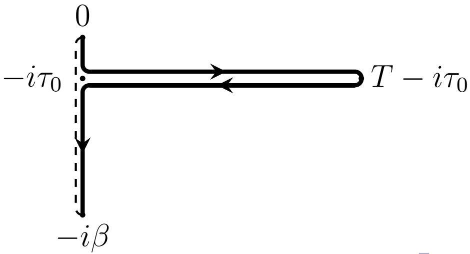
- We cannot perform a finite shift along imaginary axis! What is the corresponding symmetry?

Real-time correlation functions I

- Path integral is independent of gluing point!

$$Z_{\tau_0}[j_+, j_-] = \int\limits_{x_e(0)=x_e(\beta)} \mathcal{D}x_e \int\limits_{\substack{x_\pm(0)=x_e(\tau_0\mp 0)\\ x_+(T)=x_-(T)}} \mathcal{D}x_+ \mathcal{D}x_-$$

$$\exp\left\{-S_e[x_e] + iS[x_+] - iS[x_-] + i\int_0^T dt\, j_+ x_+ - i\int_0^T dt\, j_- x_-\right\}$$



Real-time correlation functions II

- Independence of τ_0. Then just average over it!

$$Z = Z_{\tau_0} = \frac{1}{\beta} \int_0^\beta d\tau_0 \, Z_{\tau_0}$$

- Treating the integration over τ_0 on the same footing as x_e, x_+, x_-, we observe that Z

$$x_e(\tau) \ \mapsto \ x_e(\tau + \tau_1), \quad x_\pm(t) \mapsto x_\pm(t), \quad \tau_0 \ \mapsto \ \tau_0 - \tau_1.$$

Real-time correlation functions III

- Repeating a gauge-fixing procedure, one obtains

$$Z[j_+, j_-] = \frac{\beta}{\sqrt{2\pi\xi}} \int_0^\beta d\tau_0 \int \mathcal{D}[x]_{\tau_0} \; J[x]$$
$$\exp\left\{ -S_e[x_e] + iS[x_+] - iS[x_-] - \frac{1}{2\xi}(\chi)^2 + i\int_0^T dt\, j_+ x_+ - i\int_0^T dt\, j_- x_- \right\}.$$

here

$$\chi = \frac{1}{\|\dot{\bar{x}}\|}\left[\int_0^\beta d\tau\, \partial_\tau \bar{x}_e(\tau)\, x_e(\tau) + \int_0^T dt\, \partial_t \bar{x}^{\tau_0}(t) \left(x_+(t) - x_-(t)\right)\right]$$

$$J = \frac{1}{\|\dot{\bar{x}}\|}\left[\int_0^\beta d\tau\, \partial_\tau \bar{x}_e(\tau)\, \dot{x}_e(\tau) + i\int_0^T dt\, \partial_t^2 \bar{x}^{\tau_0}(t) \left(x_+(t) - x_-(t)\right)\right.$$

Real-time correlation functions IV

- Generating functional becomes

$$Z[j_+,j_-]=\frac{e^{-S_e[\bar{x}_e]}}{\sqrt{2\pi\xi}}\int\limits_0^\beta d\tau_0 \int \mathcal{D}\eta_e\, \mathcal{D}\eta_+\, \mathcal{D}\eta_-$$

$$\times\left[\|\dot{\bar{x}}\|+\int_0^\beta d\tau\ \partial_\tau\eta_{0\,e}^{\tau_0}(\tau)\,\partial_\tau\eta_e(\tau)-\int_0^T dt\ \partial_t\eta_0^{\tau_0}(t)\left(\eta_+(t)-\eta_-(t)\right)\right]$$

$$\times\exp\left\{-S_e^{(2)}[\bar{x}_e,\eta_e]+iS^{(2)}[\bar{x}^{\tau_0},\eta_+]-iS^{(2)}[\bar{x}^{\tau_0},\eta_-]-\frac{1}{2\xi}\left(\chi[\bar{x}_e^{\tau_0}+\eta_e,\bar{x}^{\tau_0}+\eta_\pm;\bar{x}^{\tau_0}]\right)^2\right.$$

$$+\int_0^\beta d\tau\, j_e(\tau)\eta_e(\tau)+i\int_0^T dt\, j_+(t)\left(\bar{x}^{\tau_0}(t)+\eta_+(t)\right)-i\int_0^T dt\, j_-(t)\left(\bar{x}^{\tau_0}(t)+\eta_-(t)\right)$$

$$\left.\left.-S_e^{\text{int}}[\bar{x}_e,\eta_e]+iS^{\text{int}}[\bar{x}^{\tau_0},\eta_+]-iS^{\text{int}}[\bar{x}^{\tau_0},\eta_-]\right\}\right|_{j_e=0},$$

Real-time correlation functions V

- Equation on Green's function

$$
\begin{bmatrix} -\partial_\tau^2 + V''(\bar{x}_e) & & \\ & -\partial_t^2 - V''(\bar{x}^{\tau_0}) & \\ & & -\partial_t^2 - V''(\bar{x}^{\tau_0}) \end{bmatrix} \begin{bmatrix} \eta_e(\tau) \\ \eta_+(t) \\ \eta_-(t) \end{bmatrix}
$$

$$
+\frac{1}{\xi} \begin{bmatrix} \eta^{\tau_0}_{0\,e}(\tau) \\ \eta^{\tau_0}_{0+}(t) \\ \eta^{\tau_0}_{0-}(t) \end{bmatrix} \left[\int_0^\beta d\tau\, \eta^{\tau_0}_{0\,e}(\tau) \quad i\int_0^T dt\, \eta^{\tau_0}_{0+}(t) \quad -i\int_0^T dt\, \eta^{\tau_0}_{0-}(t) \right] \begin{bmatrix} \eta_e(\tau) \\ \eta_+(t) \\ \eta_-(t) \end{bmatrix}
$$

$$
= \begin{bmatrix} j_e(\tau) \\ j_+(t) \\ j_-(t) \end{bmatrix},
$$

Real-time correlation functions VI

- Algorithm
  1. Fix τ_0, defining background
  2. Calculate correlation function for fixed τ_0
  3. Average over τ_0

Discussion and TODO

Done

- Consistent perturbation theory for real-time correlators is constructed
- Dependence on difference of real-time points can be proven
- Complex backgrounds contributes to real-time correlation functions!

ToDo

- Resonant states?


Thank you for your attention!