

On Charmonium Radiative Transitions and Strong Decays of Charmonium-like States

Gurjav Ganbold

Bogolubov Laboratory of Theoretical Physics, JINR, Dubna
Institute of Physics and Technology, MAS, Ulaanbaatar

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Outline

◆ Radiative transitions of low-lying charmonium states

- Dominant (one-photon) radiative decays of states:
 $J/\Psi(1^{--})$, $\chi_{c0}(0^{++})$, $\chi_{c1}(1^{++})$, $h_c(1^{+-})$, $\chi_{c2}(2^{++})$

◆ Approach (CCQM)

- Numerical results

◆ A Strong Decays of Charmonium-like state

- Four-quark content for $Y(4230)$
- Strong decays: $Y \rightarrow \pi^+ \pi^- J/\Psi$ and $Y \rightarrow K^+ K^- J/\Psi$
- Scalar resonance $f_0(980)$ decays into $\pi^+ \pi^-$ and $K^+ K^-$
- Improved Narrow-width Approximation
- Numerical results

◆ Summary and outlook

Charmonium Radiative Transitions

*in collaboration with T. Gutsche (Tuebingen University),
M. A. Ivanov (BLTP JINR),
V. Lyubovitskij (Tuebingen University)*

State	J P C	Current	Mass (MeV)	Full width (Γ)	Mode	Fraction (Γ_i / Γ)
$\eta_c(^1S_0)$	0 - +	$i \bar{q} \gamma_5 q$	2983.9 ± 0.5	32.0 ± 0.7 MeV	$\gamma + \gamma$	$(1.58 \pm 0.11) \times 10^{-4}$
$J/\Psi(^3S_1)$	1 --	$\bar{q} \gamma_\mu q$	3096.9 ± 0.0006	92.9 ± 2.8 keV	$\gamma + h_c$	$(1.7 \pm 0.4) \times 10^{-4}$
$\chi_{c0}(^3P_0)$	0 ++	$\bar{q} I q$	3414.71 ± 0.30	10.8 ± 0.6 MeV	$\gamma + J/\Psi$	$(1.40 \pm 0.05) \times 10^{-4}$
$\chi_{c1}(^3P_1)$	1 ++	$\bar{q} \gamma_\mu \gamma_5 q$	3510.67 ± 0.05	0.84 ± 0.04 MeV	$\gamma + J/\Psi$	$(34.3 \pm 1.0) \times 10^{-4}$
$h_c(^1P_1)$	1 +-	$i \bar{q} \overleftrightarrow{\partial}_\mu \gamma_5 q$	3525.38 ± 0.11	0.7 ± 0.4 MeV	$\gamma + h_c$	$(51 \pm 6) \times 10^{-4}$
$\chi_{c2}(^3P_2)$	2 ++	$\frac{i}{2} \bar{q} (\gamma_\mu \overleftrightarrow{\partial}_\nu + \gamma_\nu \overleftrightarrow{\partial}_\mu) q$	3556.17 ± 0.07	1.97 ± 0.09 MeV	$\gamma + J/\Psi$	$(19.0 \pm 0.5) \times 10^{-4}$

- ♦ These charmonium states are unusual:
 - the c -quark mass is much larger than the confinement scale (~ 1.5 GeV vs ~ 200 MeV)
 - they have low-lying excited states ($L = 1$, $J^{PC} = 0^{++}, 1^{++}, 1^{+-}, 2^{++}$).
- ♦ Low-lying cc^- mesons
 - have narrow widths,
 - their dominant radiative transitions are one-photon decay modes.
- ♦ Charmonium states below the DD^- threshold have been intensively searched, observed and measured fairly accurately (**LHCb**, **BES-III**, **BELLE**, ...).

◆ ***Small binding energy -> an ideal testing ground to validate model assumptions:***

- Quark potential models
- Lattice simulations QCD
- QCD sum rules
- Effective Lagrangian approaches
- Nonrelativistic effective field theories of QCD
- Constituent quark models
- Bethe-Salpeter approaches
- Light-front quark model
- Coulomb gauge approach

Discrepancies still exist between the theoretical predictions and world data:

- Nonrelativistic potential model [12] and in the Coulomb gauge approach [33] result in widths $\Gamma(J/\psi \rightarrow \gamma\eta_c(1S)) \simeq 2.9$ keV, a factor of 2 larger than the data [34].
- Quark models fail to reproduce the measured branching width $\Gamma(J/\psi \rightarrow \gamma\eta_c)$ and, instead, obtain a significantly larger value [10].
- Constituent quark models describe the radiative transitions of J/ψ , $\psi(2S)$, X_{cJ} , h_c and $\psi(3770)$ [16], but the numerical results differ from the worldwide data.
- Lattice QCD [19,20] carried out on the radiative transition properties of X_{c0} , X_{c1} , however, good descriptions are still not obtained due to technical restrictions.
- Cornwell potential model [30] with a complete factorization of m_c provides numerical results different from the worldwide data.

CCQM approach (in short)

- Hadrons $H(x)$ interact by *quark exchanges*, with hadron-quark coupling g_H .

$$L_{\text{int}} = g_H H(x) J_H(x)$$

- Interpolating quark current (for meson):

$$J_H(x) = \int dx_1 \int dx_2 \delta(x - \omega_1 x_1 - \omega_2 x_2) \cdot \Phi_H(|x_1 - x_2|^2) \cdot \bar{q}(x_2) \Gamma_H q(x_1)$$

- Vertex function (trans. inv.)

$$\Phi_H(-p^2) = \exp\left(\frac{p^2}{\Lambda_H^2}\right)$$

$$\Gamma_P = i\gamma^5; \quad \Gamma_V = \gamma^\mu$$

$$\omega_j = m_j / (m_1 + m_2)$$

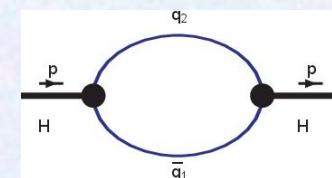
Λ_H ~ hadron “size”

- Quark propagator (in the Schwinger representation):

$$S_m(\hat{p}) = \frac{m + \hat{p}}{m^2 - p^2} = (m + \hat{p}) \cdot \int_0^\infty d\alpha_1 \exp\left[-\alpha_1(m^2 - p^2)\right]$$

- The **compositeness condition** eliminates the bare fields from consideration.

$$Z_H = \langle H_{\text{bare}} | H_{\text{phys}} \rangle^2 = 1 - g_{\text{ren}}^2 \Pi'_H(M_H^2) = 0$$



- Hadronic matrix element containing n quark propagators, m vertices and l quark loops can be written in the general form

$$\Pi(p_1, \dots, p_j) = \int [d^4 k]^\ell \prod_{i_1=1}^m \Phi_{i_1+n}(-k_{i_1+n}^2) \prod_{i_3=1}^n S_{i_3}(\tilde{k}_{i_3} + \tilde{p}_{i_3})$$

- Convert the loop momenta in numerator into derivatives over external momenta:

$$k_i^\mu e^{2kr} = \frac{1}{2} \frac{\partial}{\partial r_i^\mu} e^{2kr},$$

- Move the derivatives outside of the loop integral and calculate

$$\prod_{i=1}^n \int \frac{d^4 k_i}{i\pi^2} e^{k_A k + 2kr} = \prod_{i=1}^n \int \frac{d^4 k_i^E}{\pi^2} e^{-k_E A k_E - 2k_E r_E} = \frac{1}{|A|^2} e^{-r A^{-1} r}$$

- Move the exponent to the left by using identity

$$P \left(\frac{1}{2} \frac{\partial}{\partial r} \right) e^{-r A^{-1} r} = e^{-r A^{-1} r} P \left(\frac{1}{2} \frac{\partial}{\partial r} - A^{-1} r \right)$$

- For any polynomial P (in numerator) make differentiation by using

$$P \left(\frac{1}{2} \frac{\partial}{\partial r} - A^{-1} r \right)$$

$$\left[\frac{\partial}{\partial r_i{}_\mu}, r_j{}_\nu \right] = \delta_{ij} g_{\mu\nu}$$

- The commutations of differential operations are done by a **FORM** program.

$$\Pi = \int_0^\infty d^n \alpha F(\alpha_1, \dots, \alpha_n),$$

- The set of Schwinger parameters α_i can be turned into a simplex by introducing an additional t -integration by introducing the identity

$$\Pi = \int_0^\infty dt t^{n-1} \int_0^1 d^n \alpha \delta \left(1 - \sum_{i=1}^n \alpha_i \right) F(t\alpha_1, \dots, t\alpha_n).$$

- Threshold singularities disappear by introducing a cutoff parameter in t -integration as follows: ($[0, \infty) \rightarrow [0, 1/\lambda^2]$). λ – the infrared confinement scale parameter.

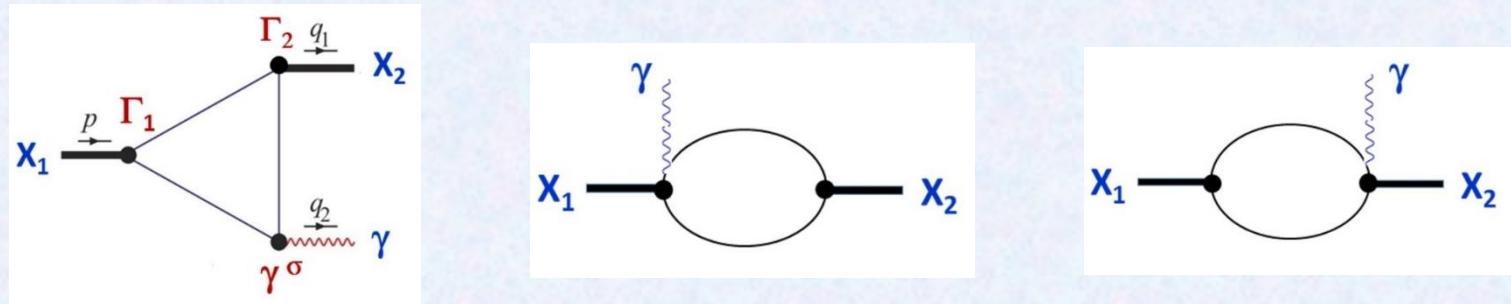
- **Infrared confinement** is introduced to guarantee the absence of all possible *thresholds* corresponding to quark production.

Matrix elements

The invariant matrix element for the one-photon radiative transition $X_1 \rightarrow \gamma X_2$

$$\mathfrak{M}_{X_1 \rightarrow \gamma X_2} = i(2\pi)^4 \delta^4(p - q_1 - q_2) \varepsilon_{X_1} \varepsilon_{X_2} \varepsilon_\gamma T_{X_1 \rightarrow \gamma X_2}(q_1, q_2)$$

In LO, transition amplitude $T_{X_1 \rightarrow \gamma X_2}(q_1, q_2)$ is described by ‘triangle’+‘bubble’ diagrams



The contributions given by the bubble-type diagrams are small and do not exceed the common errors ($\pm 10\%$) of our calculations.

Taking into account the uncertainty of the experimental data, we drop the bubble-type diagrams without loss in accuracy of our estimates.

Transition amplitudes

$$\begin{aligned}
T_{X_1 \rightarrow \gamma X_2}(q_1, q_2) &= g_{X_1} g_{X_2} e_c e N_c \iiint_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \\
&\cdot \int \frac{d^4 k}{(2\pi)^4 i} \exp \left\{ k^2 (\alpha_1 + \alpha_2 + \alpha_3 + s_1 + s_2) + 2k^\nu R^\nu + R_0 \right\} \\
&\cdot \text{tr} \left[\Gamma_2(m_c + \hat{k} + \frac{1}{2}\hat{p}) \Gamma_1(m_c + \hat{k} - \frac{1}{2}\hat{p}) \gamma^\sigma (m_c + \hat{k} - \frac{1}{2}\hat{p} + \hat{q}_2) \right] \\
&= T_{X_1, X_2, \gamma}^{inv}(q_1, q_2) + T_{X_1, X_2, \gamma}^{res}(q_1, q_2).
\end{aligned}$$

$$q_2^\sigma \cdot T_{X_1 \rightarrow \gamma X_2}^{(inv)}(q_1, q_2) = 0.$$

$$\Gamma_1 = \{\gamma^\mu, I, \gamma^\mu \gamma_5, \overset{\leftrightarrow}{\partial}_\nu \gamma^5, i(\gamma^\mu \overset{\leftrightarrow}{\partial}_\nu + \gamma^\nu \overset{\leftrightarrow}{\partial}_\mu)/2\}$$

$$\begin{aligned}
T_{X_1 \rightarrow \gamma X_2}^{(inv)}(q_1, q_2) &= \frac{g_{X_1} g_{X_2} e_c e N_c}{(2\pi)^2} \int_0^{1/\lambda^2} dt \frac{t^2}{(s+t)^2} \iiint_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) \\
&\cdot f_{\Gamma_1, \Gamma_2}(p, q_1, q_2, m_c, s, t, \alpha_1, \alpha_2, \alpha_3) \cdot \exp \left(-t z_0 + \frac{ts}{s+t} z_1 + \frac{s^2}{s+t} z_2 \right), (1)
\end{aligned}$$

$$T_{J/\psi \rightarrow \gamma \eta_c}^{(inv)\rho\sigma} = g_{J/\psi} g_{\eta_c} C(p^2, q_1^2, q_2^2) \epsilon^{q_1 q_2 \rho \sigma},$$

$$T_{\chi_{c0} \rightarrow \gamma J/\psi}^{(inv)\rho\sigma}(q_1, q_2) = g_{\chi_{c0}} g_{J/\psi} d(p^2, q_1^2, q_2^2) \cdot (q_1^\sigma q_2^\rho - g_{\rho\sigma}(q_1 \cdot q_2))$$

$$\begin{aligned} T_{\chi_{c1} \rightarrow \gamma J/\psi}^{(inv)\mu\rho\sigma}(q_1, q_2) &= g_{\chi_{c1}} g_{J/\psi} [\epsilon^{q_2 \mu \sigma \rho}(q_1 \cdot q_2) W_1 + \epsilon^{q_1 q_2 \sigma \rho} q_1^\mu W_2 \\ &\quad + \epsilon^{q_1 q_2 \mu \rho} q_2^\sigma W_3 + \epsilon^{q_1 q_2 \mu \sigma} q_1^\rho W_4 - \epsilon^{q_1 \mu \sigma \rho}(q_1 \cdot q_2) W_4] . \end{aligned}$$

$$T_{h_c \rightarrow \gamma \eta_c}^{(inv)\rho\sigma}(q_1, q_2) = g_{h_c} g_{\eta_c} h(p^2, q_1^2, q_2^2) \cdot (q_2^\rho q_1^\sigma - g_{\rho\sigma}(q_1 \cdot q_2)).$$

$$\begin{aligned} T_{\chi_{c2} \rightarrow \gamma J/\psi}^{(inv)\mu\nu\rho\sigma}(q_1, q_2) &= g_{\chi_{c2}} g_{J/\psi} \left\{ A \cdot \left(g^{\mu\rho} \left[g^{\sigma\nu}(q_1 \cdot q_2) - q_1^\sigma q_2^\nu \right] + g^{\nu\rho} \left[g^{\sigma\mu}(q_1 \cdot q_2) - q_1^\sigma q_2^\mu \right] \right) \right. \\ &\quad \left. + B \cdot \left(g^{\sigma\rho} \left[q_1^\mu q_2^\nu + q_1^\nu q_2^\mu \right] - g^{\mu\sigma} q_1^\nu q_2^\rho - g^{\nu\sigma} q_1^\mu q_2^\rho \right) \right\}, \end{aligned} \quad (40)$$

Modified Vertex for Charmonium

CCQM: The non-local vertex function $\Phi_H(-p^2)$ characterizes the quark distribution inside the hadron. It is unique for the given hadron, each hadron has its own adjustable parameter Λ_H related to the hadron 'size'.

$$\Lambda_X = \{\Lambda_{\eta_c}, \Lambda_{J/\psi}, \Lambda_{\chi^{c0}}, \Lambda_{\chi^{c1}}, \Lambda_{h_c}, \Lambda_{\chi^{c2}}\}$$

These charmonium members have **the same quark content** and possess physical masses in a relative narrow interval $\sim 3 \div 3.5$ GeV.

For this specific case we use the Ansatz: the **charmonium 'size' is proportional to its physical mass**, i.e., $\Lambda_X = \varrho \cdot M_X$ with $\varrho > 0$ - a common adjustable parameter:

$$\varrho \equiv \Lambda_X / M_X$$

Subsequently, we further use the charmonium **vertex function** defined as

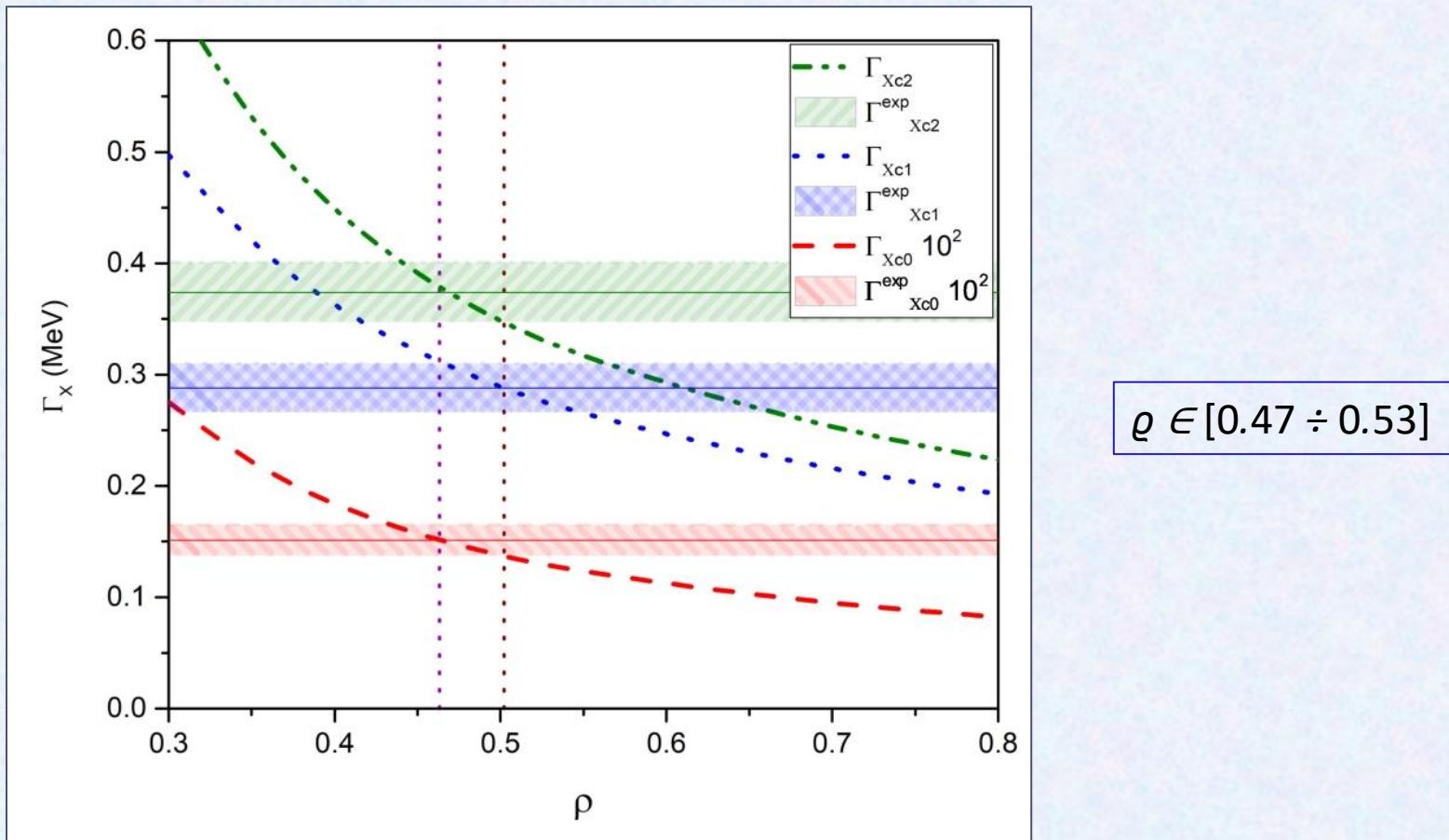
$$\tilde{\Phi}_X(-p^2) = \exp \left(\frac{1}{\varrho^2} \cdot \frac{p^2}{M_X^2} \right)$$

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Numerical results

For further numerical evaluation we keep the basic CCQM parameters:

- ♦ the universal infrared cutoff parameter $\lambda = 0.181 \text{ GeV}$
- ♦ the constituent charm quark mass in the range of $\pm 10\%$ around $m_c = 1.67 \text{ GeV}$.
- ♦ We vary $\varrho > 0$ to fit the latest data for the triplet $\{ X_{c0}, X_{c1}, X_{c2} \}$ from PDG-2021.

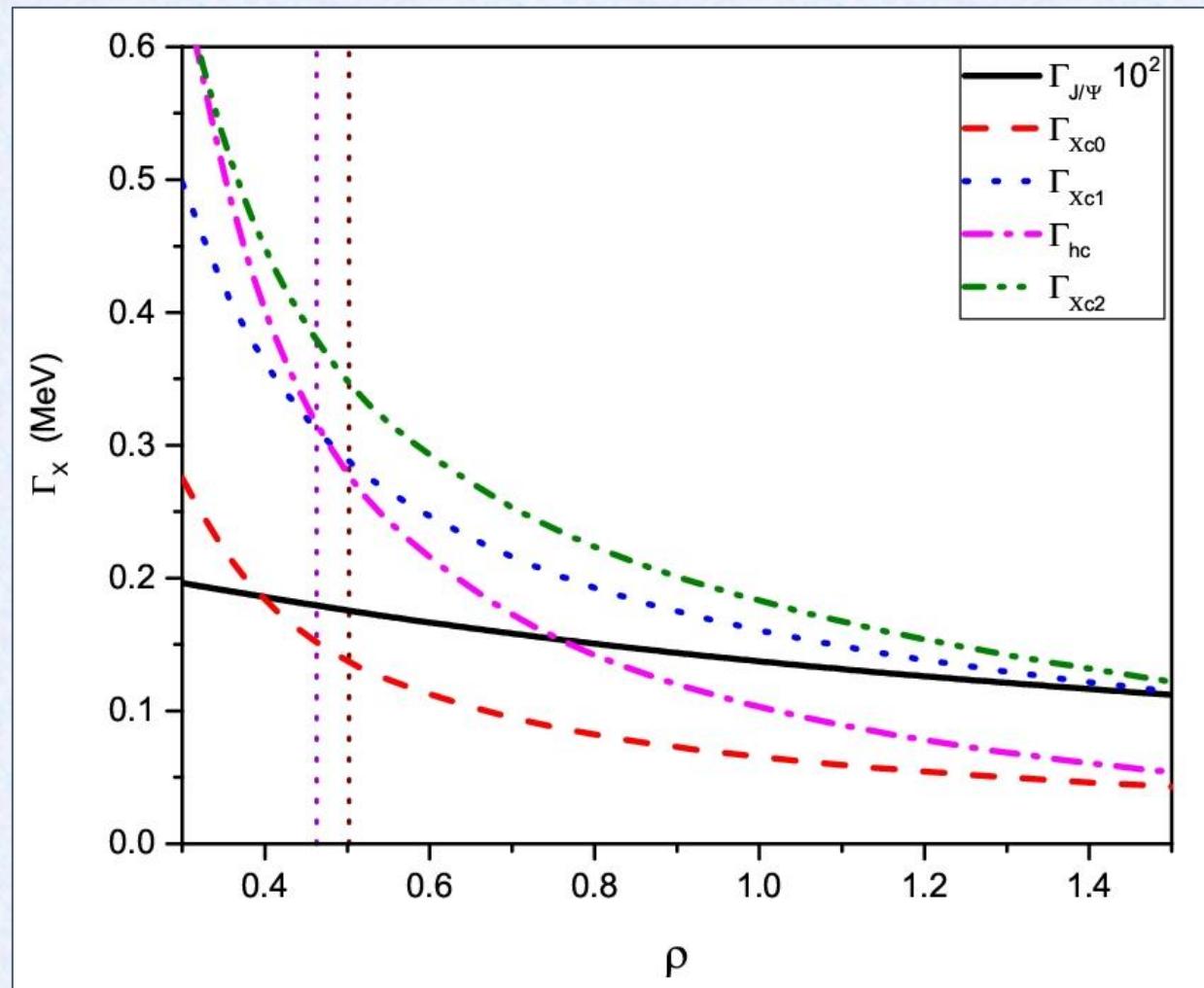


Then we calculate the partial widths of the dominant one-photon radiative decays of the ground ($J/\psi \rightarrow \gamma + \eta_c$) and orbital-excited ($h_c \rightarrow \gamma + J/\psi$) states.

$$\lambda = 0.181 \text{ MeV}$$

$$m_c = 1.80 \text{ GeV}$$

$$\varrho = 0.485$$



Some theoretical predictions of the partial widths (in units of keV)

J^{PC}	Radiative Decay	CCQM $\lambda=0.181$	CCQM $\lambda=0$	PDG-2021	Cornwell potential [30]	Cornwell potential LWL[30]	Lattice QCD [20]	Constit.Q.M [16]
1 --	$\Gamma(J/\psi \rightarrow \gamma + \eta_c)$	1.771	1.771	1.58 ± 0.43			2.64(11)	1.25
0 ++	$\Gamma(\chi_{c0} \rightarrow \gamma + J/\psi)$	142.0	142.0	151 ± 14	118	128		128
1 ++	$\Gamma(\chi_{c1} \rightarrow \gamma + J/\psi)$	296.7	297.0	288 ± 22	315	266		275
1 +-	$\Gamma(h_c \rightarrow \gamma + \eta_c)$	290.8	290.7	357 ± 270			720(50)(20)	587
2 ++	$\Gamma(\chi_{c2} \rightarrow \gamma + J/\psi)$	358.1	356.7	374 ± 27	419	353		467

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- + $\Gamma(\chi_{cJ} \rightarrow \gamma + J/\psi)$ results are close to the recent LHCb data.
- + $\Gamma(J/\psi \rightarrow \gamma + \eta_c) = 1.77$ keV is in agreement with data.
- + $\Gamma(h_c \rightarrow \gamma + \eta_c) = 0.291$ MeV leads to "theoretical full decay width"

$$\Gamma^{theor}(h_c) \simeq (570 \pm 120) \text{ keV.}$$

Strong decays of charmonium-like state Y(4230)

in collaboration with M. A. Ivanov (BLTP JINR)

PDG-2021

State	J P C	Mass (MeV)	Full width (Γ)	Mode	Fraction (Γ_i / Γ)
Y	1 --	4220 ± 15	20 - 100 MeV [~55]	$\pi^+ \pi^- J/\Psi$	seen
				$K^+ K^- J/\Psi$	seen

- ◆ BES-III Collaboration (2017): [Phys.Rev.Lett. 118 (2017) 092001]

$$M_Y = 4222.0 \pm 3.1 \pm 1.4 \text{ MeV}$$

$$\Gamma_Y = 44.1 \pm 4.3 \pm 2.0 \text{ MeV}$$

- ◆ BES-III Collaboration (April 2022): [arXiv:2204.07800]

$$0.02 < \frac{\mathcal{B}(Y \rightarrow K^+ K^- J/\Psi)}{\mathcal{B}(Y \rightarrow \pi^+ \pi^- J/\Psi)} < 0.26$$

- Interpolating quark current (for 4-quark):

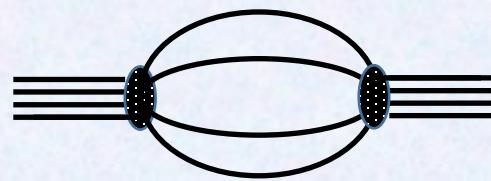
$$J_{4q}^\mu(x_1, x_2, x_3, x_4) = \frac{i}{\sqrt{2}} \left\{ [\bar{q}(x_3)\gamma^5 c(x_1)] \cdot [\bar{c}(x_2)\gamma^\mu q(x_4)] + [\bar{c}(x_2)\gamma^5 q(x_4)] \cdot [\bar{q}(x_3)\gamma^\mu c(x_1)] \right\}$$

- Interaction Vertice (Fourier): $\Phi_Y(-p^2) = \exp\left(\frac{p^2}{\Lambda_Y^2}\right)$ Λ_Y ~ “size” parameter
- Self-energy operator:

$$\begin{aligned} \Pi_Y^{\mu\nu}(p) &= N_c^2 \int \frac{d^4 k_1}{(2\pi)^4 i} \int \frac{d^4 k_2}{(2\pi)^4 i} \int \frac{d^4 k_3}{(2\pi)^4 i} (\Phi_Y[-k^2])^2 \\ &\times \left\{ Tr \left[S_{m3} \left(\hat{k}_3 + \omega_3 \hat{p} \right) \gamma^5 S_{m1} \left(\hat{k}_1 - \omega_1 \hat{p} \right) \gamma^5 \right] \cdot tr \left[S_{m2} \left(\hat{k}_2 + \omega_2 \hat{p} \right) \gamma^\mu \gamma^5 S_{m1} \left(\hat{k}_1 - \omega_1 \hat{p} \right) \gamma^\nu \gamma^5 \right] \right. \\ &\quad \left. + Tr \left[S_{m3} \left(\hat{k}_3 + \omega_3 \hat{p} \right) \gamma^5 S_{m1} \left(\hat{k}_1 - \omega_1 \hat{p} \right) \gamma^5 \right] \cdot tr \left[S_{m2} \left(\hat{k}_2 + \omega_2 \hat{p} \right) \gamma^\mu \gamma^5 S_{m1} \left(\hat{k}_1 - \omega_1 \hat{p} \right) \gamma^\nu \gamma^5 \right] \right\} \end{aligned}$$

- Renormalized Coupling for Y(4230):

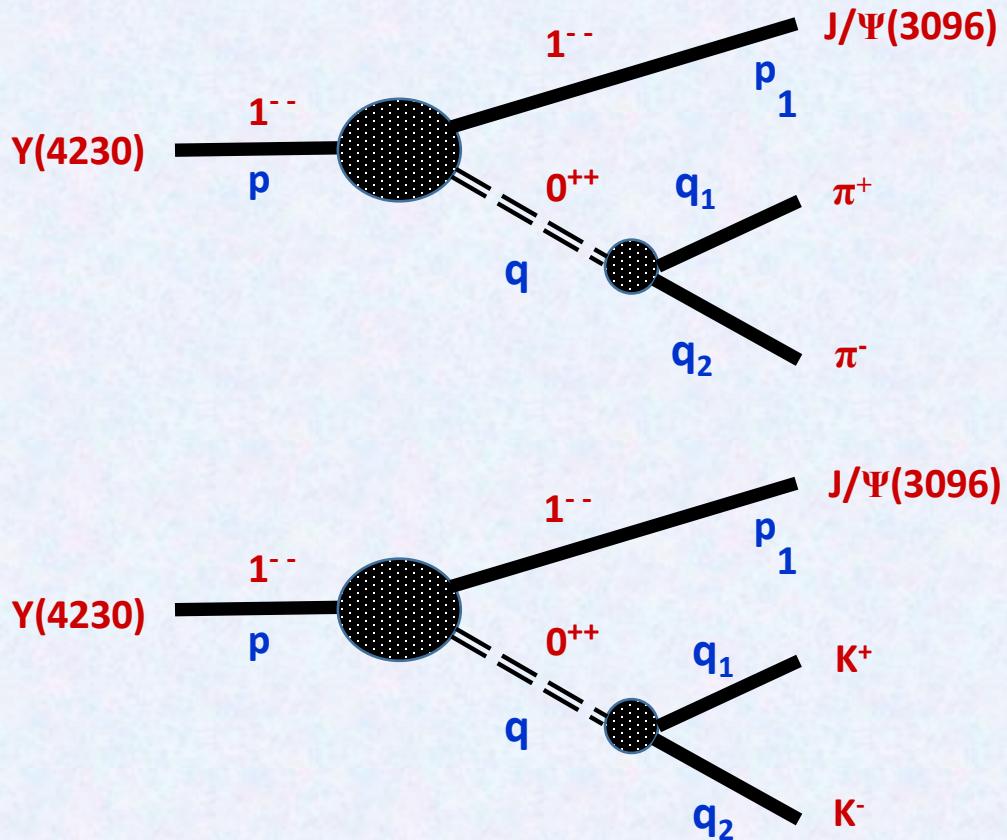
$$\Pi_Y^{\mu\nu}(p) = g_Y^{\mu\nu} \Pi_Y^0(p^2) + p^\mu p^\nu \Pi_Y^{(2)}(p^2)$$



$$g_Y^2 = \frac{d}{dp^2} \Pi_Y^0(p^2) \Big|_{p^2=M_Y^2}$$

$$m_1 = m_2 = m_c, \quad m_3 = m_4 = m_q, \quad q = \{u, d\}$$

Decays $Y \rightarrow \pi^+ \pi^- J/\Psi$ and $Y \rightarrow K^+ K^- J/\Psi$



$$\mathcal{M}_{YVPP}(p^2, p_1^2, q_1^2, q_2^2) \sim \mathcal{M}_{YVS}(p^2, p_1^2, q^2) \cdot \frac{1}{q^2 - M_R^2 + i\Gamma_R M_R} \cdot \mathcal{M}_{SPP}(q^2, q_1^2, q_2^2)$$

Scalar Resonance $f_0(980)$

PDG-2021

State	$J^P C$	Mass (MeV)	Full width (Γ)	Mode	Fraction (Γ_i / Γ)
f_0	0^{++}	990 ± 20	$10 - 100 \text{ MeV } [\sim 55]$	$\pi^+ \pi^-$	dominant
				$K^+ K^-$	seen

◆ Albrecht 20 [BES-III] [Eur.Phys.J. C80 (2020) 453]

Peak width: $\Gamma_{f_0} \approx 55 \text{ MeV}$

◆ Ablikem 05Q [BES-II] [Phys.Rev. D72 (2005) 092002]

$$\frac{\Gamma(f_0 \rightarrow K^+ K)}{\Gamma(f_0 \rightarrow \pi^+ \pi^-) + \Gamma(f_0 \rightarrow K^+ K)} = 0.75 \pm 0.13$$

◆ Aubert 060 [BaBar] [Phys.Rev. D74 (2006) 032003]

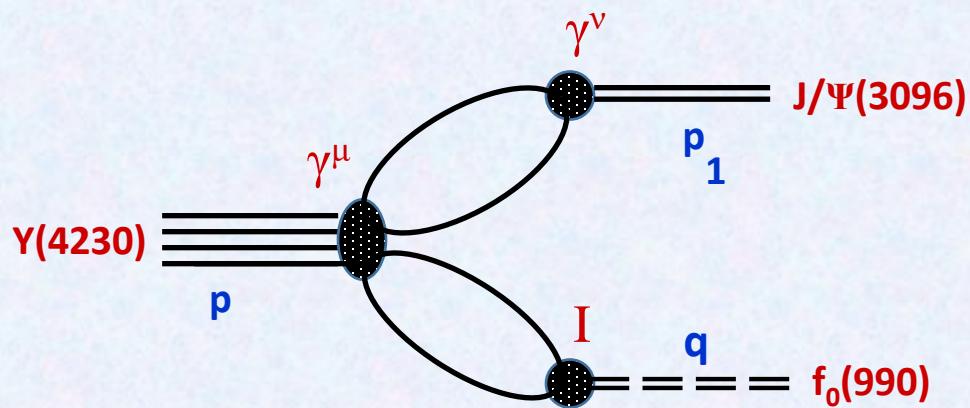
$$\frac{\Gamma(f_0 \rightarrow K^+ K)}{\Gamma(f_0 \rightarrow \pi^+ \pi^-) + \Gamma(f_0 \rightarrow K^+ K)} = 0.52 \pm 0.12$$

$$\frac{\Gamma(f_0 \rightarrow K^+ K)}{\Gamma(f_0 \rightarrow \pi^+ \pi^-) + \Gamma(f_0 \rightarrow K^+ K)} = 0.52 \div 0.84$$

PDG-2021

Decay $\Upsilon \rightarrow V + S$

$$\mathcal{M}_{YVS}(p^2, p_1^2, q^2)$$



$$T_{YVS}^{\mu\nu}(p, p_1, q)$$

$$\begin{aligned}
&= \frac{N_c^2}{2} \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \int_0^\infty d\alpha_3 \int_0^\infty d\alpha_4 \\
&\times \int \frac{d^4 k_1}{(2\pi)^4 i} \int \frac{d^4 k_2}{(2\pi)^4 i} \Phi_Y[-Q^2] \cdot \Phi_V[-u^2] \cdot \Phi_S[-w^2] \cdot \exp \left[-\sum_{i=1}^4 \alpha_i (m_i^2 - k_i^2) \right] \\
&\times \left\{ Tr \left[\gamma^5 (m_1 + \hat{k}_1 - \omega_1 p) \gamma^\nu (m_2 + \hat{k}_2 + \omega_2 p) \gamma^\mu \gamma^5 (m_4 + \hat{k}_4 - \omega_4 p) I (m_3 + \hat{k}_3 + \omega_3 p) \right] \right. \\
&\quad \left. + Tr \left[\gamma^\mu \gamma^5 (m_1 + \hat{k}_1 - \omega_1 p) \gamma^\nu (m_2 + \hat{k}_2 + \omega_2 p) \gamma^5 (m_4 + \hat{k}_4 - \omega_4 p) I (m_3 + \hat{k}_3 + \omega_3 p) \right] \right\}
\end{aligned}$$

$$T_{YVS}^{\mu\nu}(p, p_1, q) = p_1^\mu p^\nu \cdot A(p^2, p_1^2, q^2) - g^{\mu\nu} (p_1 p) \cdot B(p^2, p_1^2, q^2)$$

$$\mathsf{M}_{Y \rightarrow V+S}(p, p_1, q) = i(2\pi)^4 \delta^4(p - p_1 - q) \cdot g_Y g_V g_S \cdot \epsilon_Y^\mu \epsilon_V^\nu T_{YVS}^{\mu\nu}(p, p_1, q)$$

$$\begin{aligned} & |\mathsf{M}_{YVS}(p, p_1, q)|^2 \\ &= g_Y^2 g_V^2 g_S^2 \left(-g^{\mu_1 \mu_2} + \frac{p^{\mu_1} p^{\mu_2}}{p^2} \right) \left(-g^{\nu_1 \nu_2} + \frac{p^{\nu_1} p^{\nu_2}}{p^2} \right) \cdot T_{YVS}^{\mu_1 \nu_1}(p, p_1, q) T_{YVS}^{\mu_2 \nu_2}(p, p_1, q) \\ &= g_Y^2 g_V^2 g_S^2 \left(C_{AA} A^2 + 2C_{AB} AB + C_{BB} B^2 \right) \end{aligned}$$

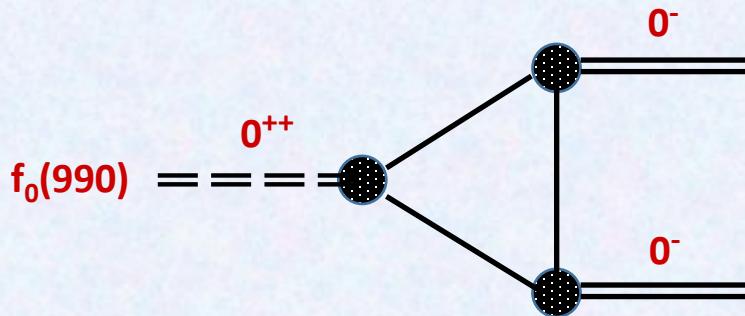
$$\begin{aligned} C_{AA} &= (p \cdot p_1)^2 (\xi - 2 + 1/\xi), & \xi &= (p \cdot p_1)^2 / (p^2 p_1^2), & C_{AA} &= 0.07996 GeV^4, \\ C_{AB} &= (p \cdot p_1)^2 (-\xi + 1), & (p \cdot p_1) &= \frac{1}{2} (p^2 + p_1^2 - q^2), & C_{AB} &= -3.775 GeV^4, \\ C_{BB} &= (p \cdot p_1)^2 (\xi + 2). & \xi &= 1.0216 & C_{BB} &= 527.2 GeV^4 \end{aligned}$$

$$\Gamma(Y \rightarrow V + S) = \frac{1}{8\pi(2S+1)} \frac{|\vec{p}_1^*|}{M_Y^2} \cdot |\mathsf{M}_{YVS}(p, p_1, q)|^2$$

$$|\vec{p}_1^*| = \frac{1}{2M_Y} \lambda^{1/2}(M_Y^2, M_V^2, q^2)$$

Decay S → P + P

$$\mathcal{M}_{SPP}(q^2, q_1^2, q_2^2)$$

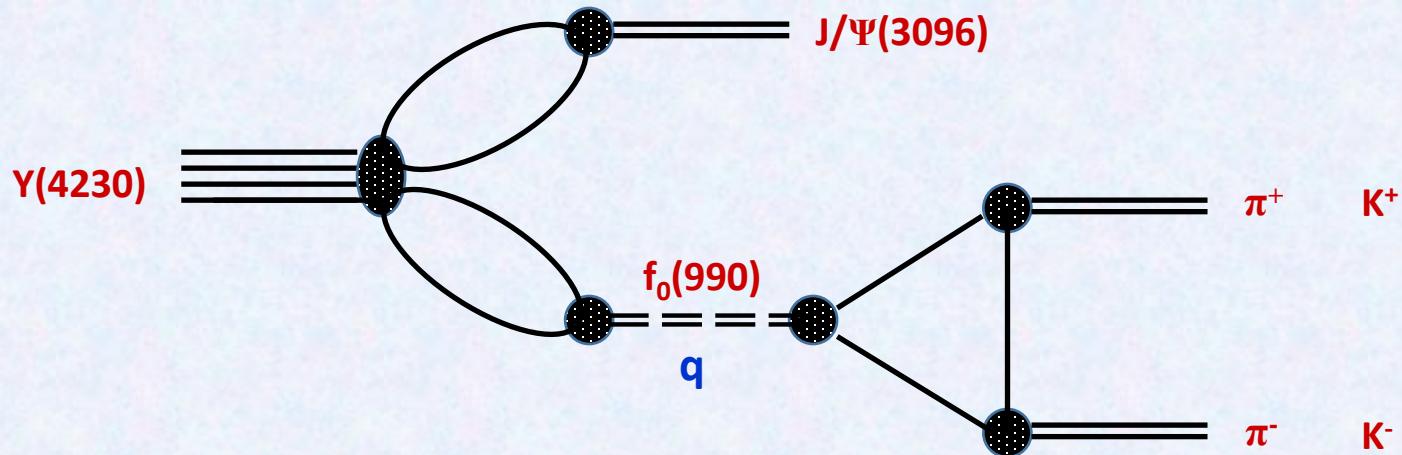


$$\begin{aligned}
 T_{YVS}^{\mu\nu}(p, p_1, q) = & \frac{N_c^2}{2} \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \int_0^\infty d\alpha_3 \int_0^\infty d\alpha_4 \\
 & \times \int \frac{d^4 k_1}{(2\pi)^4 i} \int \frac{d^4 k_2}{(2\pi)^4 i} \Phi_Y[-Q^2] \cdot \Phi_V[-u^2] \cdot \Phi_S[-w^2] \cdot \exp \left[-\sum_{i=1}^4 \alpha_i (m_i^2 - k_i^2) \right] \\
 & \times \left\{ Tr \left[\gamma^5 (m_1 + \hat{k}_1 - \omega_1 p) \gamma^\nu (m_2 + \hat{k}_2 + \omega_2 p) \gamma^\mu \gamma^5 (m_4 + \hat{k}_4 - \omega_4 p) I(m_3 + \hat{k}_3 + \omega_3 p) \right] \right. \\
 & \quad \left. + Tr \left[\gamma^\mu \gamma^5 (m_1 + \hat{k}_1 - \omega_1 p) \gamma^\nu (m_2 + \hat{k}_2 + \omega_2 p) \gamma^5 (m_4 + \hat{k}_4 - \omega_4 p) I(m_3 + \hat{k}_3 + \omega_3 p) \right] \right\}
 \end{aligned}$$

$$T_{YVS}^{\mu\nu}(p, p_1, q) = p_1^\mu p^\nu \cdot A(p^2, p_1^2, q^2) - g^{\mu\nu} (p_1 p) \cdot B(p^2, p_1^2, q^2)$$

Decay $\Upsilon \rightarrow V + P + P$

$$\mathcal{M}_{YVPP}(p^2, p_1^2, q_1^2, q_2^2)$$



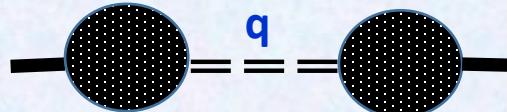
$$\Gamma_{YV\pi\pi} \sim \int_{q_{\min}^2}^{q_{\max}^2} \frac{dq^2}{2\pi} \int d\Omega_{YVS} \left| \mathcal{M}_{YVS}(q^2) \right|^2 \cdot \frac{1}{(q^2 - M_R^2)^2 + \Gamma_R^2 M_R^2} \cdot \int d\Omega_{S\pi\pi} \left| \mathcal{M}_{S\pi\pi}(q^2) \right|^2$$

$$\Gamma_{YVKK} \sim \int_{q_{\min}^2}^{q_{\max}^2} \frac{dq^2}{2\pi} \int d\Omega_{YVS} \left| \mathcal{M}_{YVS}(q^2) \right|^2 \cdot \frac{1}{(q^2 - M_R^2)^2 + \Gamma_R^2 M_R^2} \cdot \int d\Omega_{SKK} \left| \mathcal{M}_{SKK}(q^2) \right|^2$$

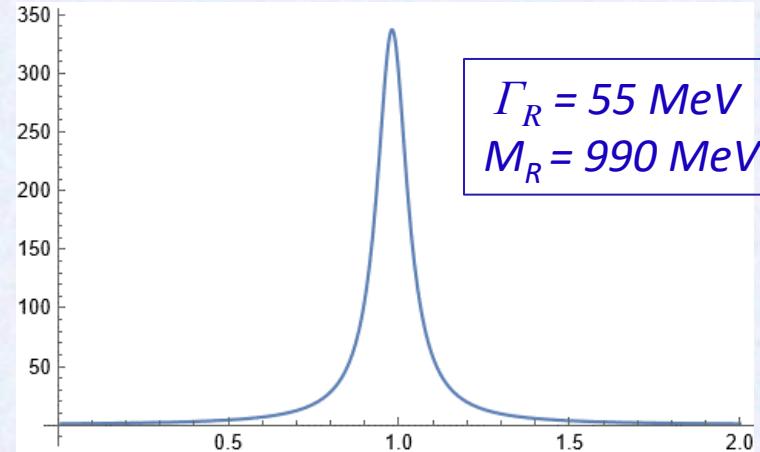
Narrow-width Approximation

$$\Gamma_{YVKK} \sim \int_{q_{\min}^2}^{q_{\max}^2} \frac{dq^2}{2\pi} \int d\Omega_{YVS} |\mathcal{M}_{YVS}(q^2)|^2 \cdot \frac{1}{(q^2 - M_R^2)^2 + \Gamma_R^2 M_R^2} \cdot \int d\Omega_{SKK} |\mathcal{M}_{SKK}(q^2)|^2$$

$$D(q^2) = \frac{1}{(q^2 - M_R^2)^2 + \Gamma_R^2 M_R^2}$$



$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2 + x^2} = \pi \delta(x)$$



$$\int_{q_{\min}^2}^{q_{\max}^2} dq^2 \int d\Omega_1 |\mathcal{M}_1(q^2)|^2 \cdot D(q^2) \cdot \int d\Omega_2 |\mathcal{M}_2(q^2)|^2 \simeq \int d\Omega_1 |\mathcal{M}_1(q^2)|^2 * \int d\Omega_2 |\mathcal{M}_2(q^2)|^2 \cdot \frac{\pi}{M_R \Gamma_R}$$

$$error \sim 0 \left(\frac{\Gamma_R}{M_R} \right)$$

N.Kauer, Phys.Lett. B649 (2007) 413

C.F.Uhlemann, N.Kauer, Nucl.Phys. B814 (2009) 195

$$\begin{aligned}
& \int_{q_{\min}^2}^{q_{\max}^2} \frac{dq^2}{2\pi} \int d\Omega_{YVS} \left| \mathcal{M}_{YVS}(q^2) \right|^2 \cdot D(q^2) \cdot \int d\Omega_{SKK} \left| \mathcal{M}_{SKK}(q^2) \right|^2 \\
& \simeq \int d\Omega_{YVS} \left| \mathcal{M}_{YVS}(q^2) \right|^2 * \frac{\pi}{\Gamma_R M_R} \int d\Omega_{SKK} \left| \mathcal{M}_{SKK}(q^2) \right|^2 + 0 \left(\frac{\Gamma_R}{M_R} \right)
\end{aligned}$$

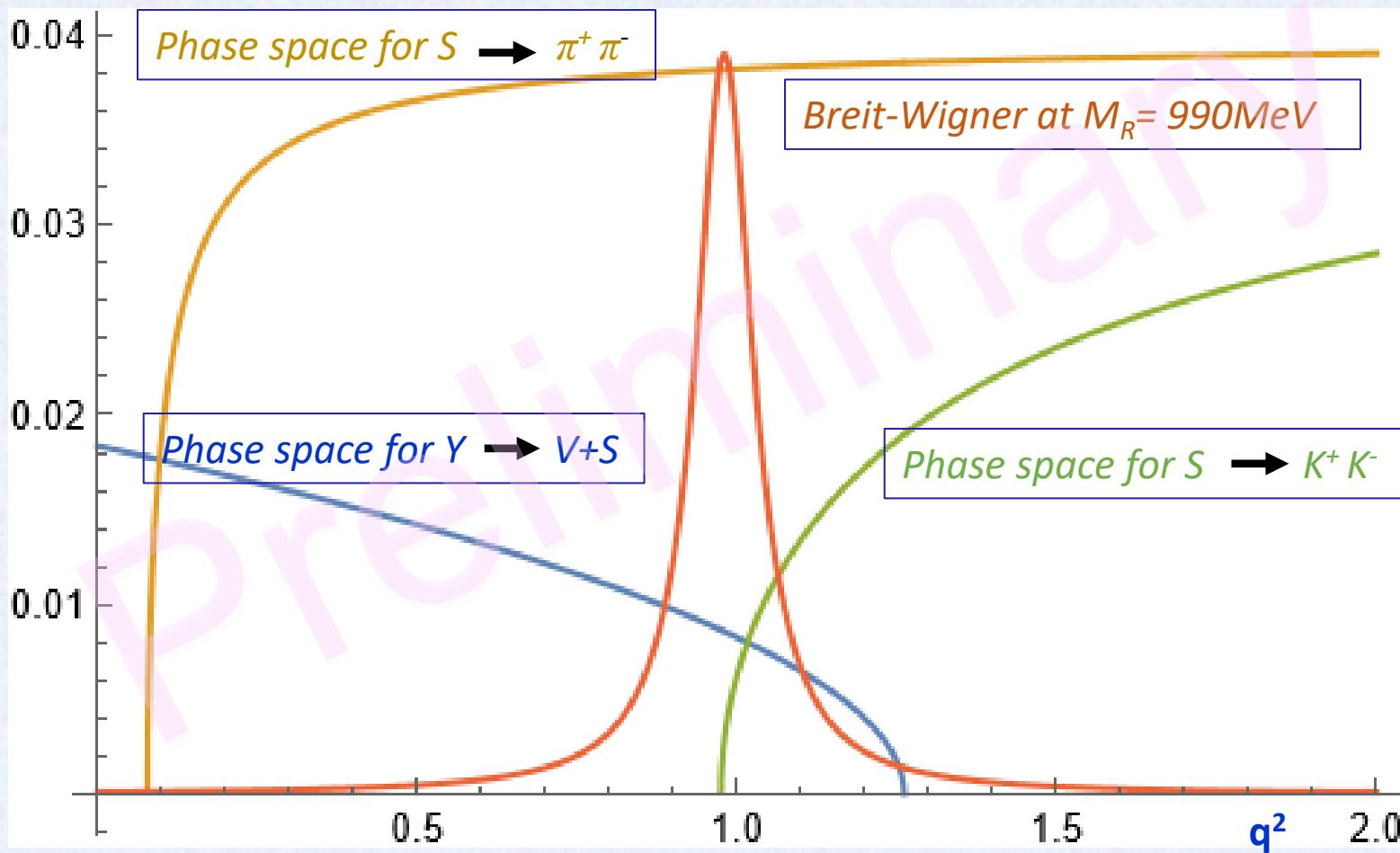
$$\frac{\mathcal{B}(Y \rightarrow K^+ K^- J/\Psi)}{\mathcal{B}(Y \rightarrow \pi^+ \pi^- J/\Psi)} \approx \frac{\mathcal{B}(S \rightarrow K^+ K^-)}{\mathcal{B}(S \rightarrow \pi^+ \pi^-)}$$

PROBLEMS:

- ♦ NWA conventional error ~ (1/3 – 3) times Γ/M
- ♦ NWA error increases for $q^2 \approx M^2$
- ♦ NWA underestimates when phase state is small

$$\begin{aligned}
\Gamma_R &= 55 \text{ MeV} \\
M_R &= 990 \text{ MeV}
\end{aligned}$$

Improved Narrow-width Approximation

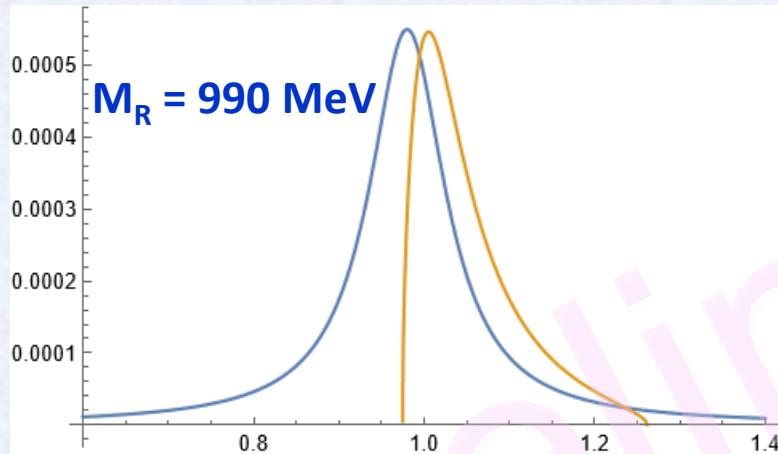


Decay $\Upsilon \rightarrow V + K^+ + K^-$

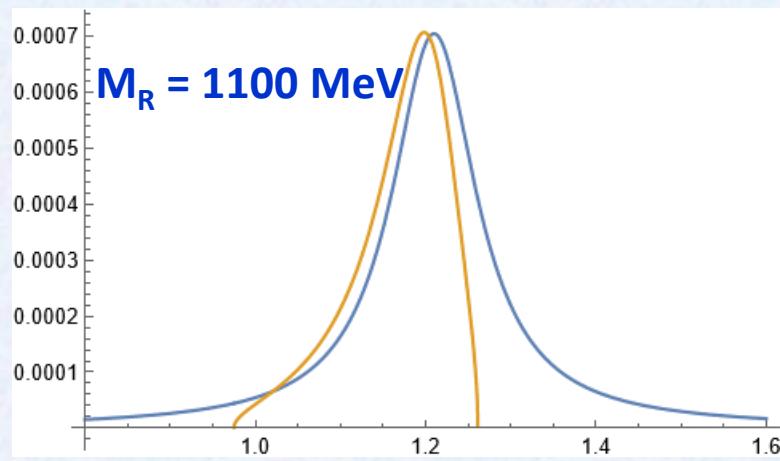
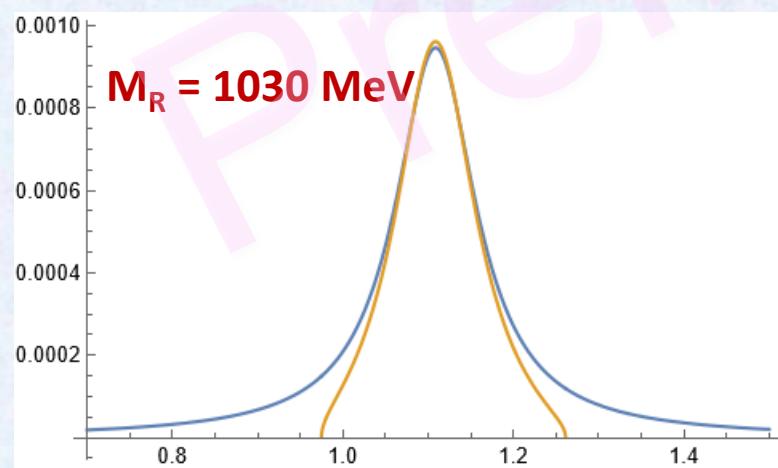
$$f(q^2) \equiv d\Omega_{YVS}(q^2) \cdot \frac{1}{(q^2 - M_R^2)^2 + \Gamma_R^2 M_R^2} \cdot d\Omega_{SKK}(q^2)$$

versus

$$\frac{1}{(q^2 - M_R^2)^2 + \Gamma_R^2 M_R^2}$$



$M_\Upsilon = 4220 \text{ MeV}$
 $M_{J\Psi} = 3097 \text{ MeV}$
 $M_K = 494 \text{ MeV}$

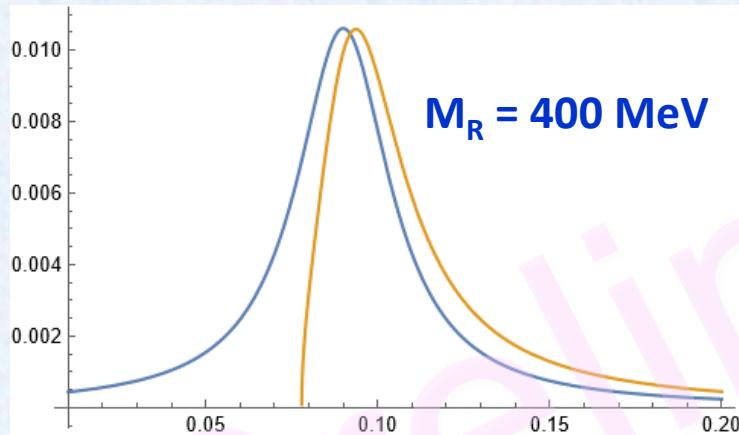


Decay $\Upsilon \rightarrow V + \pi^+ + \pi^-$

$$f(q^2) \equiv d\Omega_{YVS}(q^2) \cdot \frac{1}{(q^2 - M_R^2)^2 + \Gamma_R^2 M_R^2} \cdot d\Omega_{S\pi\pi}(q^2)$$

versus

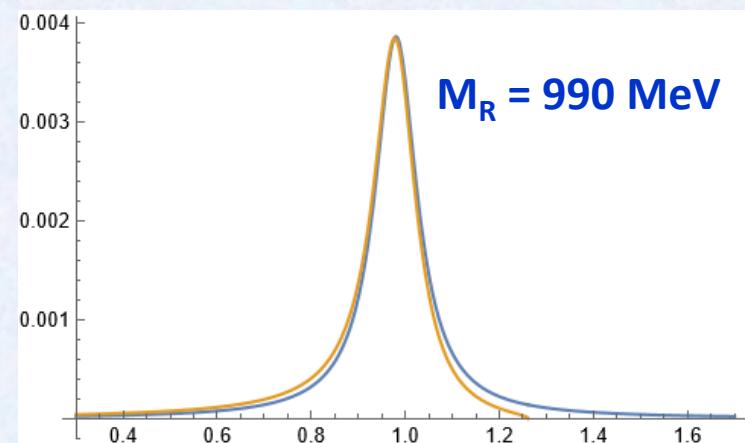
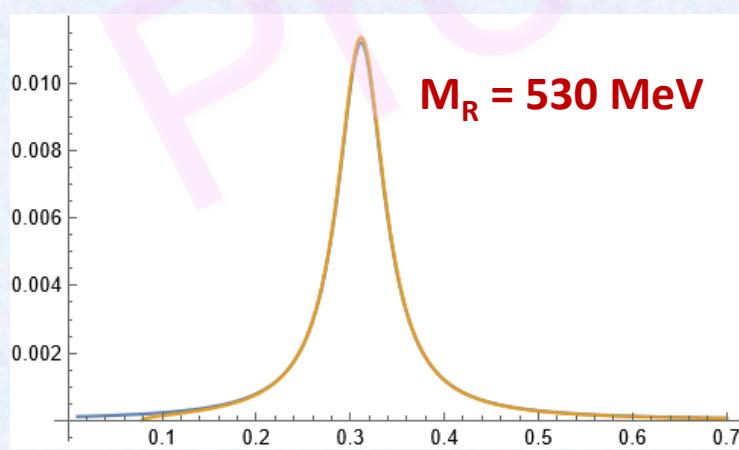
$$\frac{1}{(q^2 - M_R^2)^2 + \Gamma_R^2 M_R^2}$$



$$M_Y = 4220 \text{ MeV}$$

$$M_{J\Psi} = 3097 \text{ MeV}$$

$$M_\pi = 140 \text{ MeV}$$



Numerical results

- Central values
of the **quark masses** and
size parameters (in GeV)

$$\begin{aligned}\lambda &= 0.181, \\ m_{ud} &= 0.241, \quad m_s = 0.428, \\ m_c &= 1.67, \quad m_b = 5.07\end{aligned}$$

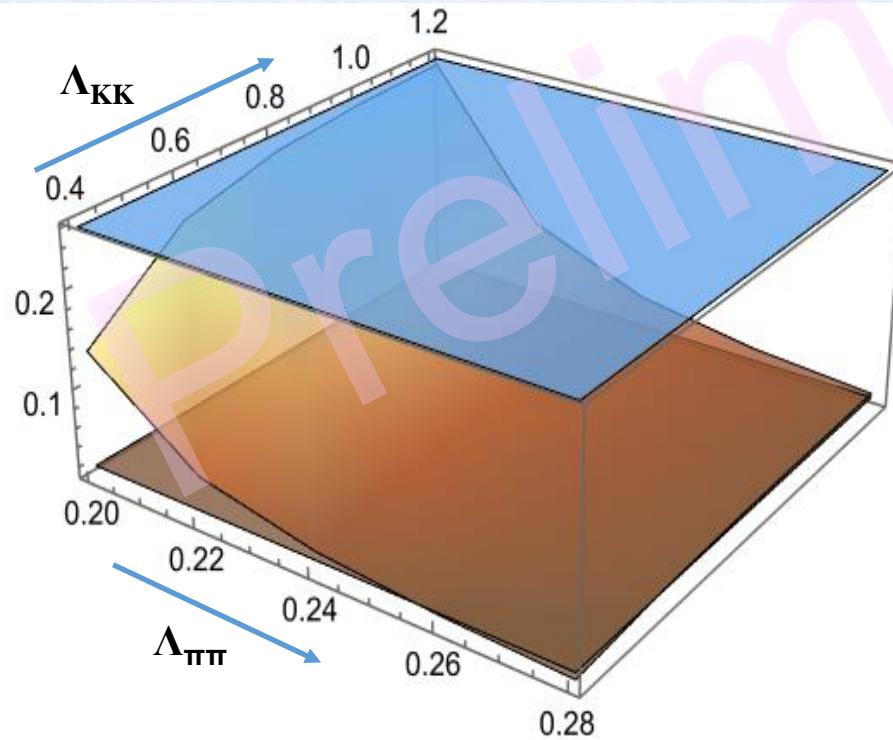
$$\Lambda_\pi \quad \Lambda_K \quad \Lambda_{J/\Psi} \quad \Lambda_Y$$

$$0.71 \quad 1.02 \quad 1.50 \quad 3.30$$

$$\frac{\mathcal{B}(Y \rightarrow K^+ K^- J/\Psi)}{\mathcal{B}(Y \rightarrow \pi^+ \pi^- J/\Psi)} \approx \frac{\int d\Omega_{YVS} \left| \mathcal{M}_{YVS}(M_{K^+ K^-}^2) \right|^2 \cdot \mathcal{B}(S \rightarrow K^+ K^-, M_{K^+ K^-}^2)}{\int d\Omega_{YVS} \left| \mathcal{M}_{YVS}(M_{\pi^+ \pi^-}^2) \right|^2 \cdot \mathcal{B}(S \rightarrow \pi^+ \pi^-, M_{\pi^+ \pi^-}^2)}$$

$$\Lambda_{\pi^+ \pi^-} = 0.20 \div 0.28 \text{ GeV}$$

$$\Lambda_{K^+ K^-} = 0.20 \div 0.28 \text{ GeV}$$



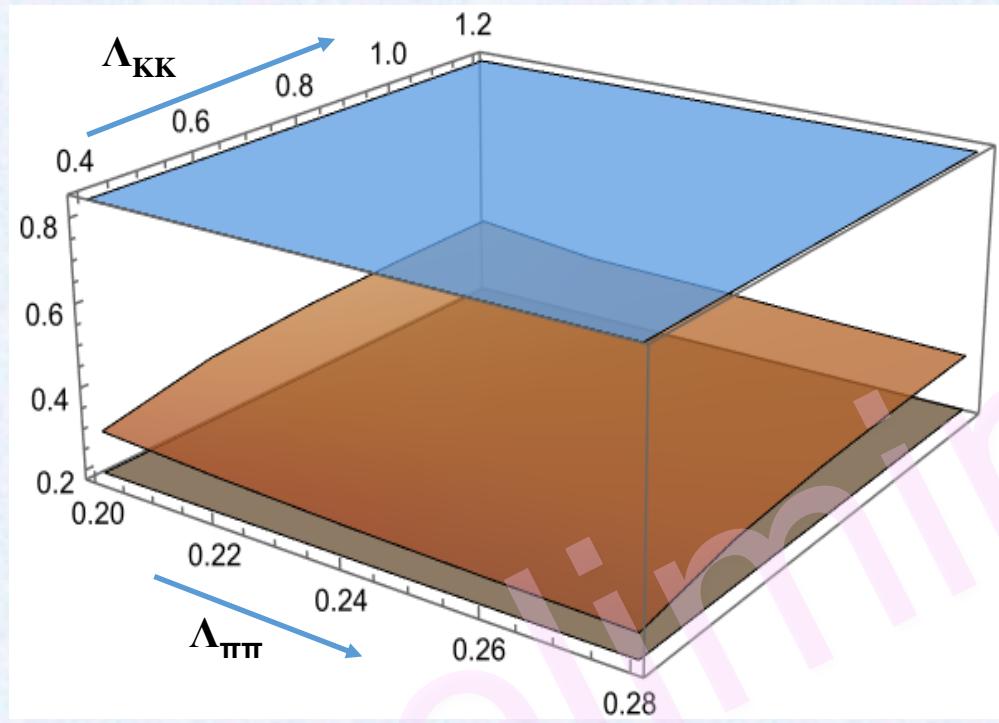
$$\begin{aligned}\Gamma_R &= 55 \text{ MeV} \\ M_{\pi\pi} &= 530 \text{ MeV} \\ M_{KK} &= 1030 \text{ MeV}\end{aligned}$$

Our estimate (preliminary):

$$\frac{\mathcal{B}(Y \rightarrow K^+ K^- J/\Psi)}{\mathcal{B}(Y \rightarrow \pi^+ \pi^- J/\Psi)} = 0.021 \div 0.253$$

BES-III Collaboration (April 2022):
[arXiv:2204.07800]

$$0.02 < \frac{\mathcal{B}(Y \rightarrow K^+ K^- J/\Psi)}{\mathcal{B}(Y \rightarrow \pi^+ \pi^- J/\Psi)} < 0.26$$



Our estimate (preliminary):

$$\frac{\mathcal{B}(S \rightarrow K^+ K^-, M_{K^+ K^-}^2)}{\mathcal{B}(S \rightarrow \pi^+ \pi^-, M_{\pi^+ \pi^-}^2)} = 0.23 \div 0.39$$

$$\Lambda_{\pi^+ \pi^-} = 0.20 \div 0.28 \text{ GeV}$$

$$\Lambda_{K^+ K^-} = 0.40 \div 1.20 \text{ GeV}$$

PDG-2021

Aubert 060 [BaBar]
[Phys.Rev. D74 (2006) 032003]

Ablikem 05Q [BES-II]
[Phys.Rev. D72 (2005) 092002]

$$\frac{\Gamma(f_0 \rightarrow K^+ K)}{\Gamma(f_0 \rightarrow \pi^+ \pi^-)} = 0.19 \div 0.92$$

$$\frac{\Gamma(f_0 \rightarrow K^+ K)}{\Gamma(f_0 \rightarrow \pi^+ \pi^-)} = 0.15 \div 1.49$$

$$\frac{\Gamma(f_0 \rightarrow K^+ K)}{\Gamma(f_0 \rightarrow \pi^+ \pi^-)} = 0.16 \div 0.61$$

Discussion (short)

- ◆ We study the strong decay modes (into $\pi^+\pi^-J/\psi$ and K^+K^-J/ψ) of the exotic charmonium-like vector state Y(4230) by treating it as a four-quark structure within a relativistic covariant confining quark model (CCQM).
- ◆ We consider the four-quark interpolating current as the molecular-type which effectively corresponds to the combination of D and \bar{D}_1 quark currents.
- ◆ Our preliminary estimate within the **Phase-space Improved Narrow-width Approximation** for the ratio of branching decay widths

$$\frac{\mathcal{B}(Y \rightarrow K^+K^-J/\Psi)}{\mathcal{B}(Y \rightarrow \pi^+\pi^-J/\Psi)} = 0.021 \div 0.253$$

is in full agreement with the recent data reported by the BES-III Collaboration.

- ◆ As a by-product, we have also estimated the ratio

$$\frac{\mathcal{B}(f_0 \rightarrow K^+K^-)}{\mathcal{B}(f_0 \rightarrow \pi^+\pi^-)} = 0.23 \div 0.39$$

which does not contradict the existing data.

Summary and Outlook

- ◆ The dominant radiative transitions of the charmonium states $\eta_c(^1S_0)$, $J/\psi(^3S_1)$, $\chi_{c0}(^3P_0)$, $\chi_{c1}(^3P_1)$, $h_c(^1P_1)$ and $\chi_{c2}(^3P_2)$ have been studied within the CCQM.
- ◆ We keep the basic model parameters and introduce one common adjustable parameter ($\varrho = 0.485$) to describe the quark distribution in these charmoniums.
- ◆ Estimated fractional widths for $J/\psi(^3S_1)$ and $h_c(^1P_1)$ are *in good agreement* with the data. We predict the 'theoretical full width' $\Gamma^{theor}(h_c) \simeq (0.57 \pm 0.12)$ MeV compared with latest data $\Gamma^{exp}(h_c) \simeq (0.7 \pm 0.4)$ MeV.
- ◆ We study the strong decays of the exotic charmonium-like vector state $Y(4230)$ by treating it as a four-quark structure of molecular type.
- ◆ Our estimate within the *Phase-space Improved Narrow-width Approximation* for the ratio of branching decay widths is in full agreement with the recent data reported by the BES-III Collaboration (in April 2022).
- ◆ This approach *may be extended* to other sections of hadron physics:
 - light mesons (scalar, isoscalar, ...)
 - radial excitations (charmoniums and bottomoniums)
 - heavy meson and baryon decays
 - exotics (tetraquark, X-Y-Z meson-like objects, ...)