Evaluation of massive helicity amplitudes using 4-component spinors

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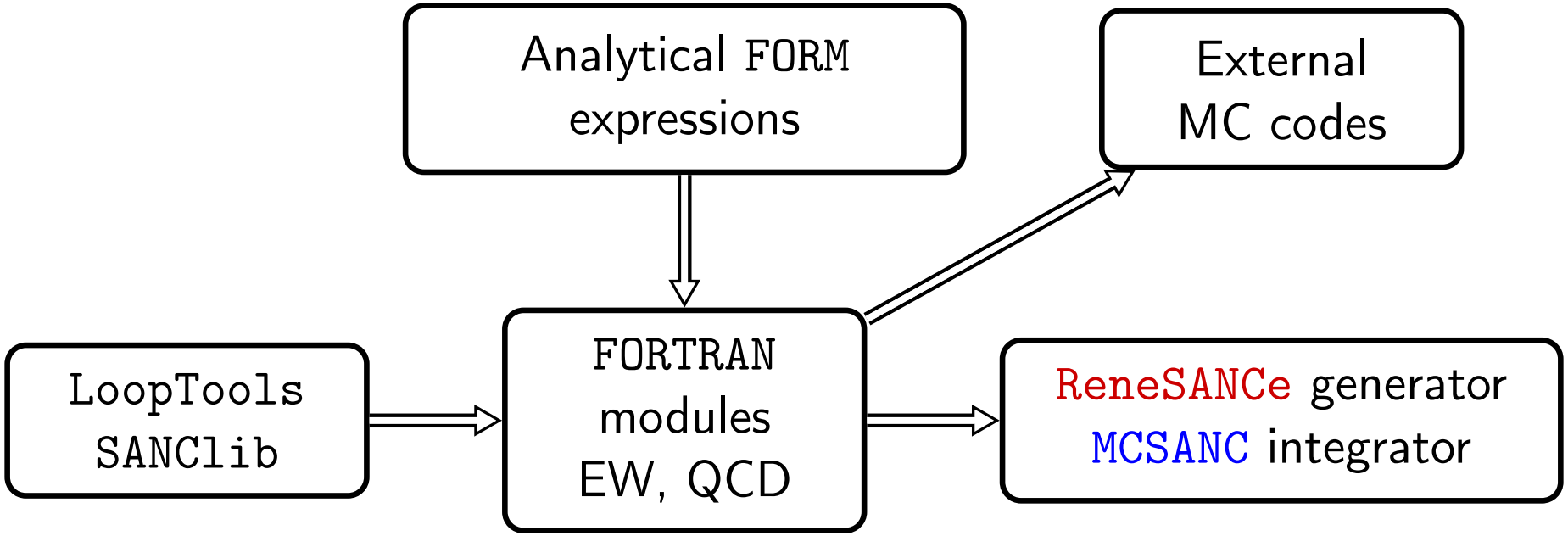
Outline

- SANC framework
- Motivation
- Clifford algebra
- Dirac spinors
- Examples of amplitudes
- Numerical comparison
- Conclusion and plans

The SANC framework and products



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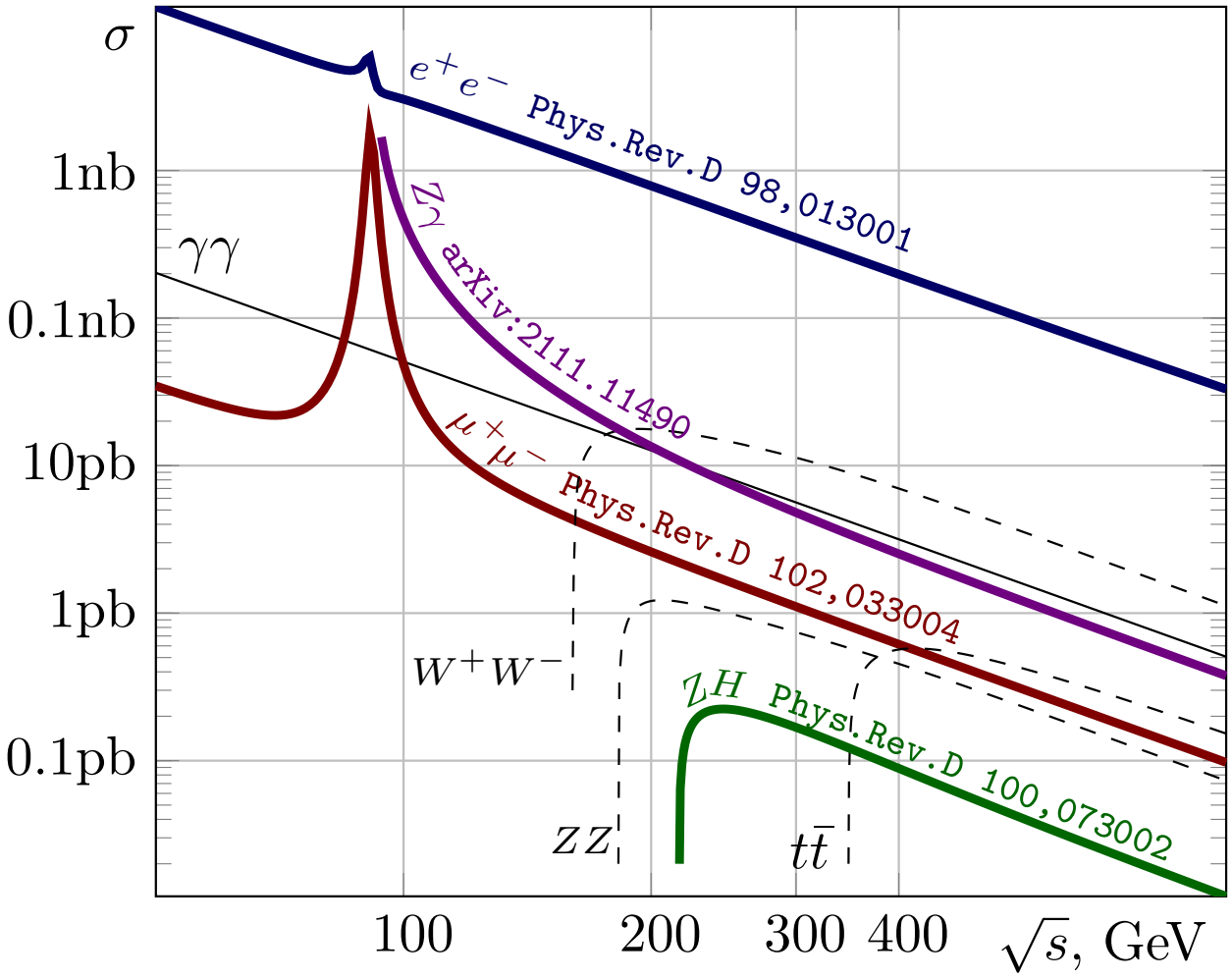
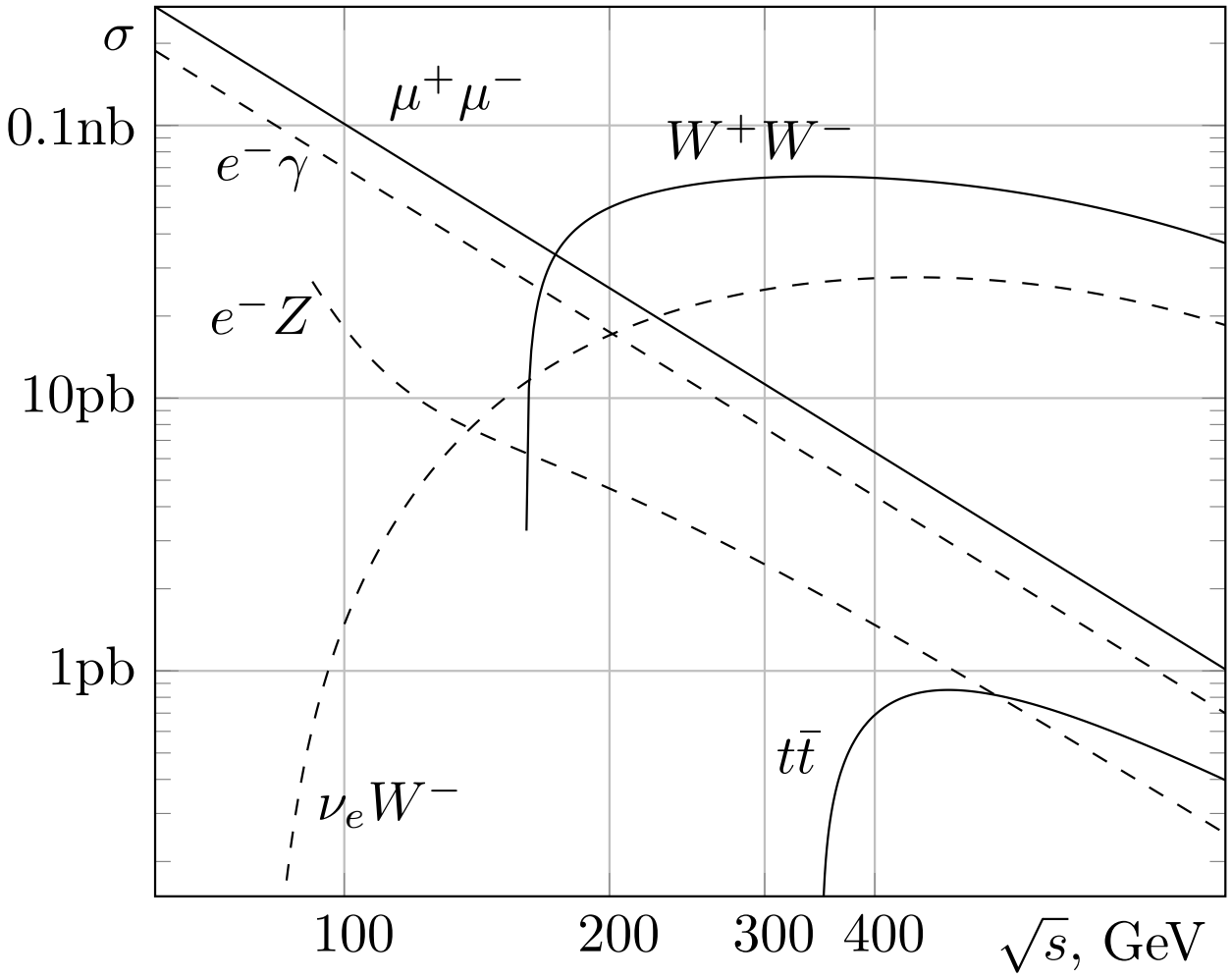
Basic processes of SM for e^+e^- annihilation



The cross sections are given for polar angles between $10^o < \theta < 170^o$ in the final state.

Basic processes of SM for $e^{\pm}\gamma$ and $\gamma\gamma$ initial state



The cross sections are given for polar angles between $10^o < \theta < 170^o$ in the final state

Physics an low Q^2 and final state polarization

BESIII experiment at the BEPCII accelerator

Institute of High Energy Physics (IHEP) (Beijing)

- e^+e^- beams
- $\sqrt{s}$ from 2 to 4.63 GeV
- $L = 10^{33} cm^{-2} c^{-1}$

The project of the Super Charm-Tau factory

Budker Institute of Nuclear Physics of the SB RAS (Novosibirsk)

- e^+e^- beams
- $\sqrt{s}$ from 2 to 5 GeV
- $L = 10^{35} cm^{-2} c^{-1}$
- longitudinal polarization of the electrons

Our possibility

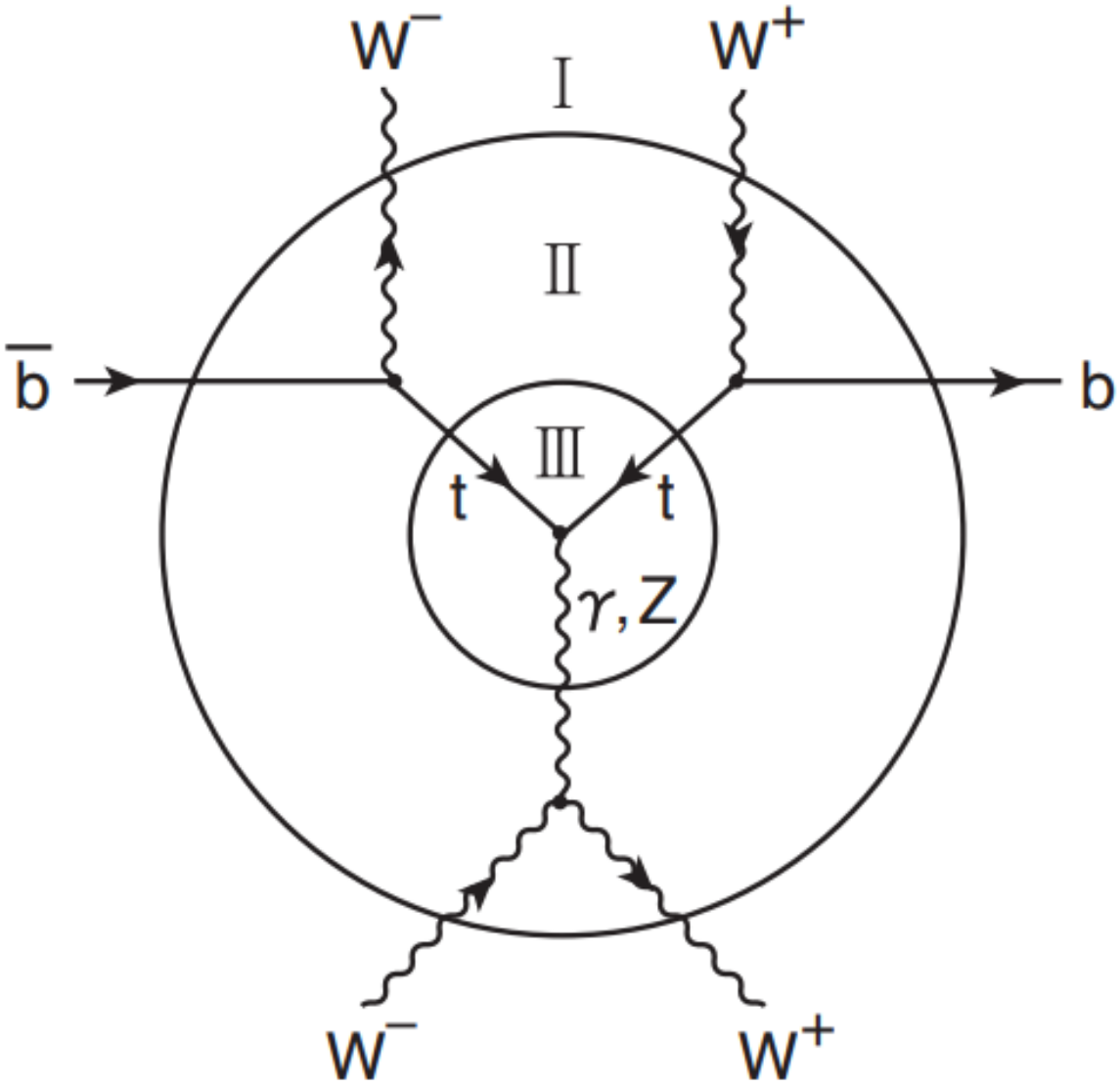
- Luminosity
- Full phase-space
- Polarization of final particles

Two types of HA methods

Non-covariant HA

- Dirac-matrix multiplication in some reference frame (Klein–Nishina formula obtained by this approach);
- Evaluation of "Polarization vectors" of virtual particles (used by `HELAS` and `SHERPA.AMEGIC++`)
- Applied to solve Dyson-Schwinger Equations (DSE) numerically by `ALPGEN` and `WHIZARD/O'Mega`.
- Extended to 1-loop by `Madgraph` and `RECOLA`

HELAS scheme



Two types of HA methods

Covariant HA

- Reference frame is not fixed;
- Pioneered by N. Fedorov at Minsk in 1956;
- Becomes popular after works of `CALCUL` group in 1980s;
- Extended by Bern-Dixon-Kosower to d-dimensions and thus to loop integrands;
- Nowadays used by `GoSam`/`Golem` and `FormCalc` packages for automated 1-loop calculations;
- Little-group formulation

Typical amplitude for $\bar{q}q \to ggg$

$$
\begin{aligned}
M^{+--+-} &= -[4|2|3\rangle \left[\frac{m[14][42]\langle 53\rangle + (15)[42][4|2|3\rangle - [14](32)[4|1|5\rangle}{8\, p_5\cdot p_1\; p_2\cdot p_3\; p_3\cdot p_4\; [54]}\right] \\
&+ \langle 35\rangle \left[\frac{m[14][42]\langle 53\rangle + (15)[42][4|2|3\rangle - [14](32)[4|1|5\rangle}{8\, p_2\cdot p_3\; p_3\cdot p_4\; p_4\cdot p_5}\right] \\
&+ \frac{\langle 35\rangle^2}{\langle 34\rangle\langle 45\rangle (p_1+p_2)^2}\left[\frac{[14](32)+(13)[42]}{[54]} + \frac{[14](52)+(15)[42]}{[34]}\right].
\end{aligned}
$$

$$
\begin{aligned}
M^{+-++-} &= -[4|1|5\rangle \left[\frac{-m(15)(52)[43] + (15)[32][4|1|5\rangle - [14](52)[3|2|5\rangle}{8\, p_5\cdot p_1\; p_2\cdot p_3\; p_4\cdot p_5\; \langle 53\rangle}\right] \\
&+ [43][4|1|5\rangle \left[\frac{[14](52)+(15)[42]}{4\, p_5\cdot p_1\; p_3\cdot p_4 \langle 53\rangle [54]}\right] \\
&- [43]\left[\frac{[14](52)[3|1|5\rangle + (15)[32][4|1|5\rangle - m(15)(52)[43]}{4\, p_5\cdot p_1\; p_3\cdot p_4\; p_4\cdot p_5}\right] \\
&- \frac{[43]^2\langle 35\rangle}{2\, p_3\cdot p_4 [54](p_1+p_2)^2}\left[\frac{[14](52)+(15)[42]}{\langle 53\rangle} + \frac{[13](52)+(15)[32]}{\langle 54\rangle]}\right].
\end{aligned}
$$

Ozeren, K.J., и W.J. Stirling. EPJ C 48, №. 1 (2006): 159–68.

Limitations and complications

Collinear singularities

- HAs are especially elegant for massless particles;
- In collinear regions there may be "singular terms" of the from $\frac{m^2}{(2pk)^2} \sim \delta(2pk)$ giving nonzero contribution but missing in massless amplitude;
- Conclusion: Massless HAs $\neq$ polarized matrix elements.

HA of massive particle is Lorentz-convariant (not *invariant*)

- spin-quantization axis should be specified;
- The helicity states are defined relative to reference frame;
- Auxilary vectors needed to decompose massive momenta into sum of massless vectors;

Clifford algebra of Dirac matrices

Multivector components and grade

$$\langle A\rangle_0 \equiv \langle A\rangle \equiv \frac{\mathrm{Tr}[A]}{\mathrm{Tr}[1]} = \frac{1}{4}\,\mathrm{Tr}[A],$$

$$\langle A\rangle_1 \equiv \gamma_\mu \langle \gamma^\mu A\rangle,$$

$$\langle A\rangle_2 \equiv -\frac{1}{2!}\gamma_{\mu\nu}\langle \gamma^{\mu\nu} A\rangle,$$

$$\langle A\rangle_3 \equiv -\frac{1}{3!}\gamma_{\mu\nu\rho}\langle \gamma^{\mu\nu\rho} A\rangle = -\gamma_\mu\gamma_5\langle \gamma^\mu\gamma_5 A\rangle = \gamma_5\langle \gamma_5 A\rangle_1,$$

$$\langle A\rangle_4 \equiv \frac{1}{4!}\gamma_{\mu\nu\rho\sigma}\langle \gamma^{\mu\nu\rho\sigma} A\rangle = \gamma_5\langle \gamma_5 A\rangle$$

$$\gamma^{\mu\ldots\nu} = \gamma^{[\mu}\cdots\gamma^{\nu]} = \frac{1}{k!}\gamma^\mu\wedge\cdots\wedge\gamma^\nu$$

Inner, Wedge product and Contraction

$$A \cdot B \equiv \langle AB \rangle,$$

$$A \wedge B = \sum_{r,s} \Big\langle \langle A \rangle_r \langle B \rangle_s \Big\rangle_{s+r},$$

$$A \rfloor B = \sum_{r,s} \langle \langle A \rangle_r \langle B \rangle_s \rangle_{s-r}$$

$$(a_1 \wedge \cdots \wedge a_n) \cdot (b_n \wedge \cdots \wedge b_1) = \begin{vmatrix} a_1 \cdot b_1 & \dots & a_1 \cdot b_n \\ \vdots & \ddots & \vdots \\ a_n \cdot b_1 & \dots & a_n \cdot b_n \end{vmatrix}$$

Maxwell's bivector

It is well known, that polarization vector ε of the photon is gauge-dependent quantity, whereas Maxwell's bivector $\mathbf{F}$ of field strength is gauge-independent and in this sense physical quantity. Maxwell equation is equivalent to relation

$$k\mathbf{F} = 0, \qquad \mathbf{F} = k \wedge \varepsilon, \qquad k^2 = 0$$

Invariance with respect to gauge transformation $\varepsilon \to \varepsilon + Ck$ is evident.

Maxwell's bivector in spinor notation

$$\begin{aligned} \varepsilon^+_\mu(k, g_+) &= \frac{[g_+|\sigma_\mu|k\rangle}{\sqrt{2}[k|g_+]}, \\ \varepsilon^-_\mu(k, g_-) &= \frac{\langle g_-|\sigma_\mu|k]}{\sqrt{2}\langle k|g_-\rangle} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{F}^+ &= k\varepsilon^+ = \sqrt{2}\begin{bmatrix} |k\rangle\langle k| & \\ & 0 \end{bmatrix}, \\ \mathbf{F}^- &= k\varepsilon^- = \sqrt{2}\begin{bmatrix} 0 & \\ & |k][k| \end{bmatrix} \end{aligned}$$

Polarization vector in axial gauge

Having some auxiliary vector g we can fix a gauge with condition

$$\varepsilon = \frac{g\rfloor\mathbf{F}}{g\cdot k}, \qquad \varepsilon\cdot k = 0, \qquad \varepsilon\cdot g = 0$$

Gauge transformation

Direct verification shows that polarization vectors in two gauges are differ in vector proportional to k:

$$\begin{aligned}
\varepsilon^{g_1} - \varepsilon^{g_2} &= \frac{g_1\rfloor\mathbf{F}}{g_1\cdot k} - \frac{g_2\rfloor\mathbf{F}}{g_2\cdot k} = \frac{\big(g_1(k\cdot g_2) - (k\cdot g_1)g_2\big)\rfloor\mathbf{F}}{(g_1\cdot k)(g_2\cdot k)} \\
&= -\frac{\big(k\rfloor g_1\wedge g_2\big)\rfloor\mathbf{F}}{(g_1\cdot k)(g_2\cdot k)} = -\frac{k\wedge\big(g_1\wedge g_2\big)\rfloor\mathbf{F} - \big(g_1\wedge g_2\big)\rfloor(k\wedge\mathbf{F})}{(g_1\cdot k)(g_2\cdot k)} \\
&= -\frac{\langle g_1 g_2\mathbf{F}\rangle}{(g_1\cdot k)(g_2\cdot k)}k
\end{aligned}$$

Eikonal factor in terms of Maxwell's bivector

$$\frac{2\varepsilon_4 \cdot p_2}{2p_2 \cdot p_4} - \frac{2\varepsilon_4 \cdot p_1}{2p_1 \cdot p_4} = 4\frac{(p_1 \cdot p_4)(p_2 \cdot \varepsilon_4) - (p_2 \cdot p_4)(p_1 \cdot \varepsilon_4)}{z_{14}z_{24}}$$

$$= 4\frac{\begin{vmatrix} p_1 \cdot p_4 & p_2 \cdot p_4 \\ p_1 \cdot \varepsilon_4 & p_2 \cdot \varepsilon_4 \end{vmatrix}}{z_{14}z_{24}} = 4\frac{(p_1 \wedge p_2) \cdot (\varepsilon_4 \wedge p_4)}{z_{14}z_{24}}$$

$$= -\frac{4\langle p_1 p_2 \mathbf{F}_4 \rangle}{z_{14}z_{24}} = -\frac{\mathrm{Tr}[p_1 p_2 \mathbf{F}_4]}{z_{14}z_{24}}$$

$$= \frac{2\varepsilon_4^{p_1} \cdot p_2}{2p_2 \cdot p_4} = \frac{4\langle p_1 \mathbf{F}_4 \rangle_1 \cdot p_2}{z_{14}z_{24}}$$

with

$$z_{14} \equiv 2p_1 \cdot p_4, \qquad z_{24} \equiv 2p_2 \cdot p_4$$

Scalar QED example: $\phi^+\phi^-\gamma\gamma \to 0$

$$\phi^+(p_1) + \phi^-(p_2) + \gamma(p_3) + \gamma(p_4) \to 0$$

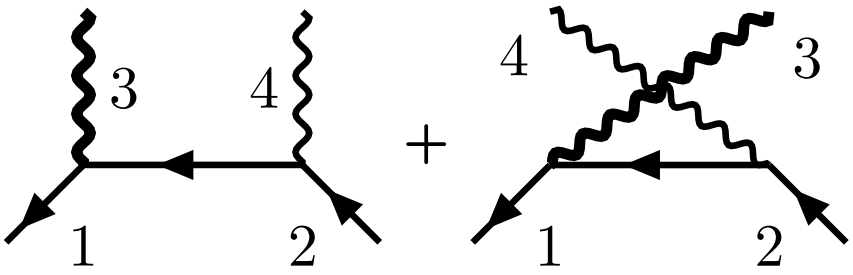
$$\mathcal{A} = \frac{(2p_1 \cdot \varepsilon_3)(2p_2 \cdot \varepsilon_4)}{z_{13}} + \frac{(2p_1 \cdot \varepsilon_4)(2p_2 \cdot \varepsilon_3)}{z_{14}} + 2\varepsilon_3 \cdot \varepsilon_4$$

$$\begin{aligned}
\mathcal{A} &= -\frac{\mathrm{Tr}[p_4 p_2 \mathbf{F}_3](2p_2 \cdot \varepsilon_4) - z_{24}\,\mathrm{Tr}[\varepsilon_4 p_2 \mathbf{F}_3]}{z_{14} z_{24}} \\
&= -\frac{\big(2p_4 \cdot \langle p_2 \mathbf{F}_3 \rangle_1\big)(2p_2 \cdot \varepsilon_4) - z_{24}\big(2\varepsilon_4 \cdot \langle p_2 \mathbf{F}_3 \rangle_1\big)}{z_{14} z_{24}} \\
&= -\frac{\mathrm{Tr}[p_2 \langle p_2 \mathbf{F}_3 \rangle_1 \mathbf{F}_4]}{z_{13} z_{14}} = -\frac{m^2\,\mathrm{Tr}[\mathbf{F}_3 \mathbf{F}_4]}{2 z_{13} z_{14}} + \frac{\mathrm{Tr}[p_2 \mathbf{F}_3 p_2 \mathbf{F}_4]}{2 z_{13} z_{14}}
\end{aligned}$$

$$\mathcal{A} = \frac{\mathrm{Tr}[(p_2 + m)\mathbf{F}_3 (p_2 - m)\mathbf{F}_4]}{2 z_{13} z_{14}}$$

$e^+e^-Z\gamma^* \to 0$

$$e^+(p_1) + e^-(p_2) + Z(p_3) + \gamma^*(p_4) \to 0$$

$\mathcal{A} =$ 

For virtual photon $p_4^2 \neq 0$. Vector e_4 does not contain γ_5 ! We also relax property $e_4 \cdot p_4 \neq 0$.

$$\mathcal{A} = \bar{v}_1 e_3 \frac{1}{p_{24} - m} e_4 u_2 + \bar{v}_1 e_4 \frac{1}{p_{23} - m} e_3 u_2$$

$$m_1 = m_2 = m, \quad e_3 = \varepsilon_3 (g_V + g_A \gamma_5)$$

$$P_1 = p_1 + \frac{p_4}{2}, \qquad P_2 = p_2 + \frac{p_4}{2}, \qquad \mathbf{F}_4 = p_4 \wedge e_4$$

$$Z_{14} = 2P_1 \cdot p_4, \qquad Z_{24} = 2P_2 \cdot p_4, \qquad P_1 + P_2 + p_3 = 0$$

$$p_{24}^2 - m^2 = 2p_2 \cdot p_4 + p_4^2 = (2p_2 + p_4) \cdot p_4 = 2P_2 \cdot p_4 = Z_{24}$$

Because of $p_4 e_4 = p_4 \cdot e_4 + p_4 \wedge e_4$:

$$\begin{aligned}
(p_{24} + m)e_4 u_2 &= (2e_4 \cdot p_2 + p_4 \cdot e_4 + p_4 \wedge e_4)u_2 \\
&= (e_4 \cdot [2p_2 + p_4] + \mathbf{F}_4)u_2 = (2e_4 \cdot P_2 + \mathbf{F}_4)u_2, \\
\bar{v}_1 e_4 (p_{23} + m) &= \bar{v}_1 e_4 (-p_{14} + m) \\
&= \bar{v}_1(-e_4 \cdot [2p_1 + p_4] + \mathbf{F}_4) = \bar{v}_1(-2e_4 \cdot P_1 + \mathbf{F}_4)
\end{aligned}$$

$$\frac{e_4 \cdot P_2}{p_4 \cdot P_2} - \frac{e_4 \cdot P_1}{p_4 \cdot P_1} = -\frac{\mathrm{Tr}[P_1 P_2 \mathbf{F}_4]}{Z_{14} Z_{24}}$$

Scalarized amplitude

$$\mathcal{A} = -\frac{\mathrm{Tr}[P_1 P_2 \mathbf{F}_4]}{Z_{14} Z_{24}} \bar{v}_1 e_3 u_2 + \frac{\bar{v}_1 \mathbf{F}_4 e_3 u_2}{Z_{14}} + \frac{\bar{v}_1 e_3 \mathbf{F}_4 u_2}{Z_{24}}$$

This result can be obtained by formal using of axial gauge $g_4 = P_2$ for virtual photon:

$$\varepsilon_4 = \frac{P_2 \rfloor \mathbf{F}_4}{P_2 \cdot p_4} \qquad \varepsilon_4 \cdot P_2 = 0 \qquad \varepsilon_4 \cdot P_1 = \frac{\overline{\mathrm{Tr}}[P_1 P_2 \mathbf{F}_4]}{P_2 \cdot p_4}$$

$e^+(p_1) + e^-(p_2) + \gamma^*(p_3) + \gamma^*(p_4) \to 0$

$$P_{23} = p_2 + \frac{p_3}{2}, \qquad P_{24} = p_2 + \frac{p_4}{2}$$

$$Z_{24} = 2P_{24} \cdot p_4, \qquad Z_{23} = 2P_{23} \cdot p_3$$

After scalarization

$$\mathcal{A} = \bar{v}_1 e_3 \frac{2e_4 \cdot P_{24} + \mathbf{F}_4}{Z_{24}} u_2 + \bar{v}_1 e_4 \frac{2e_3 \cdot P_{23} + \mathbf{F}_3}{Z_{23}} u_2$$

Now with axial gauges $g_4 = P_{24}$ and $g_3 = P_{23}$ we arrive at answer:

$$\mathcal{A} = \bar{v}_1 \frac{P_{23} \langle \mathbf{F}_3 \mathbf{F}_4 \rangle_{0,4} - \langle \mathbf{F}_3 P_{23} \mathbf{F}_4 \rangle_1}{Z_{23} Z_{24}} u_2$$

$e^+e^-\gamma\gamma\gamma \to 0$

$$e^+(p_1) + e^-(p_2) + \gamma(p_3) + \gamma(p_4) + \gamma(p_5) \to 0$$

$$\mathcal{A} = \mathcal{A}^3 + \mathcal{A}^4 + \mathcal{A}^5$$

$$\mathcal{A} = -\frac{\mathrm{Tr}[p_1 p_2 \mathbf{F}_3]}{z_{13} z_{23} z_{24} z_{25}} \bar{v}_1 \Big\{ \langle \mathbf{F}_4 p_2 \mathbf{F}_5 \rangle_1 - p_2 \langle \mathbf{F}_4 \mathbf{F}_5 \rangle_{0,4} \Big\} u_2$$

$$+ \frac{\bar{v}_1 \mathbf{F}_3 \Big\{ \langle \mathbf{F}_4 p_2 \mathbf{F}_5 \rangle_1 - p_2 \langle \mathbf{F}_4 \mathbf{F}_5 \rangle_{0,4} \Big\} u_2}{z_{13} z_{25} z_{24}}$$

Dirac matrices in 6-dimensions

Dirac matrices in 6-dimensions

In $d = 6$ dimensions we have 8×8-matrices

$$\Gamma^M{}_\Theta{}^\Omega = \begin{bmatrix} & \acute{\gamma}^M{}_\alpha{}^{\dot\beta} \\ \grave{\gamma}^M{}_{\dot\alpha}{}^\beta & \end{bmatrix}, \qquad \begin{aligned} &\Gamma^M\Gamma^N + \Gamma^N\Gamma^M = 2g^{MN}, \\ &g^{MN} = \mathrm{diag}[g^{\mu\nu}, 1, -1] \end{aligned}$$

$$\begin{aligned} \acute{\gamma}_M &= \{\gamma_\mu, \gamma_5, +1\}, \\ \grave{\gamma}_M &= \{\gamma_\mu, \gamma_5, -1\}, \end{aligned} \quad \Gamma^7 = i\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^5\Gamma^6, \quad \Gamma^7{}_\Theta{}^\Omega = \begin{bmatrix} \delta_\alpha{}^\beta & \\ & -\delta_{\dot\alpha}{}^{\dot\beta} \end{bmatrix}$$

$$\Gamma^{MN} \equiv \Gamma^{[M}\Gamma^{N]} = \begin{bmatrix} \acute{\!\!\gamma}^{MN}{}_\alpha{}^\beta & \\ & \grave{\!\!\gamma}^{MN}{}_{\dot\alpha}{}^{\dot\beta} \end{bmatrix}, \qquad \begin{aligned} \acute{\!\!\gamma}^{MN} &= \acute{\gamma}^{[M}\grave{\gamma}^{N]}, \\ \grave{\!\!\gamma}^{MN} &= \grave{\gamma}^{[M}\acute{\gamma}^{N]}, \end{aligned}$$

Spinor metric

$$\epsilon^{\Theta\Omega} = \epsilon^{\Omega\Theta} = \begin{bmatrix} & \epsilon^{\alpha\dot{\beta}} \\ \epsilon^{\dot{\alpha}\beta} & \end{bmatrix}, \qquad \epsilon_{\Theta\Omega} = \epsilon_{\Omega\Theta} = \begin{bmatrix} & \epsilon_{\alpha\dot{\beta}} \\ \epsilon_{\dot{\alpha}\beta} & \end{bmatrix}$$

Components of spinor metric are

$$\epsilon^{\alpha\dot{\beta}} = \epsilon^{\dot{\beta}\alpha} = \epsilon_{\alpha\dot{\beta}} = \epsilon_{\dot{\beta}\alpha} = \begin{bmatrix} \epsilon^{AB} & 0 \\ 0 & \epsilon_{\dot{A}\dot{B}} \end{bmatrix}$$

In $d = 5$

Using $\acute{\gamma}^6{}_\alpha{}^{\dot{\beta}} =$ "1" we may not distinguish between dotted and undotted $d = 5$-spinor indexes.

(Levi-Civita) totally antisymmetric spinor

$$\epsilon^{\alpha\beta\gamma\delta} = \quad 3\epsilon^{[\alpha\beta}\epsilon^{\gamma\delta]} = \quad \epsilon^{\alpha\beta}\epsilon^{\gamma\delta} - \epsilon^{\alpha\gamma}\epsilon^{\beta\delta} + \epsilon^{\alpha\delta}\epsilon^{\beta\gamma}, \qquad \epsilon^{1234} = 1$$

Dirac spinors

$$|u\rangle = u_\alpha{}^a = \begin{pmatrix} u_A{}^a \\ u^{\dot{A}a} \end{pmatrix} = \begin{pmatrix} |u^a\rangle \\ |u^a] \end{pmatrix},$$

$$\langle u| = u_a{}^\alpha = \begin{pmatrix} u_a{}^A & -u_{a\,\dot{A}} \end{pmatrix} = \begin{pmatrix} \langle u_a| & -[u_a| \end{pmatrix}$$

Projection operator

$$\acute{p} \equiv |u\rangle\langle u| = |u^a\rangle\langle u_a| = \not{p} + m\omega_+ + \tilde{m}\omega_-, \qquad p^2 = m\tilde{m}$$

Dual projector and dual spinors

$$\grave{p}^{\gamma\delta} \equiv -\frac{1}{2}\acute{p}_{\alpha\beta}\epsilon^{\alpha\beta\gamma\delta}, \quad \grave{p} = |u^{\dot{a}}]\![u_{\dot{a}}| = \acute{p} - m - \tilde{m} = \not{p} - \tilde{m}\omega_+ - m\omega_-$$

Dirac equation

$$\epsilon^{\alpha\beta\gamma\delta}u_\alpha{}^a u_\beta{}^b u_\gamma{}^c = -\epsilon^{ab}\grave{p}^{\gamma\delta}u_\gamma{}^c \equiv 0 \quad \Rightarrow \quad \grave{p}|u^c\rangle \equiv 0 \quad \Rightarrow \quad [\![u_{\dot{a}}|u^b\rangle = 0$$

Spinor product matrices

$$\langle p|q]\,[q|p\rangle = 2\acute{p}\cdot\grave{q} \qquad \Leftrightarrow \qquad \langle p_a|q^{\dot{a}}]\,[q_{\dot{a}}|p^b\rangle = (2\acute{p}\cdot\grave{q})\delta_a^b$$

Inverse little-group matrices

$$\frac{1}{\langle p|q]} = \frac{[q|p\rangle}{2\acute{p}\cdot\grave{q}}, \qquad \frac{1}{[q|p\rangle} = \frac{\langle p|q]}{2\acute{p}\cdot\grave{q}}$$

Schouten identity for Dirac spinors

$$|p\rangle\frac{1}{[q|p\rangle}[q| + |q\rangle\frac{1}{[p|q\rangle}[p| = \mathbf{1}$$

Decompose $|u\rangle$ as linear combination of $|p\rangle$ and $|q\rangle$

$$|u\rangle = |p\rangle\frac{1}{[q|p\rangle}[q|u\rangle + |q\rangle\frac{1}{[p|q\rangle}[p|u\rangle$$

Maxwell's bivector

$$\acute{k}\,\dddot{F} = 0, \qquad \dddot{F} = \grave{k} \wedge \acute{\varepsilon}, \qquad \acute{k}\grave{k} = k^2 = 0$$

Polarization vector in axial gauge

$$\acute{\varepsilon} = \frac{\acute{g} \rfloor \dddot{F}}{g \cdot k}, \qquad \acute{\varepsilon} \cdot \grave{k} = 0, \qquad \acute{\varepsilon} \cdot \grave{g} = 0$$

Expression in terms of spinors

$$\dddot{F}^a_{\dot{a}} = \sqrt{2}|k^a\rangle\!\rangle \otimes [\![k_{\dot{a}}|$$

$$\mathbf{F}^+ = \dddot{F}^0_{\dot{0}} = \sqrt{2}\begin{bmatrix} |k\rangle\langle k| & \\ & 0 \end{bmatrix}, \qquad \mathbf{F}^- = \dddot{F}^1_{\dot{1}} = \sqrt{2}\begin{bmatrix} 0 & \\ & |k][k| \end{bmatrix}$$

Examples of process calculation

On-shell process $e^+e^-Z\gamma \to 0$ in $d = 6$

$$e^+(p_1) + e^-(p_2) + Z(p_3) + \gamma(p_4) \to 0$$

$$\begin{aligned} \acute{p}_1 &= \not{p}_1 - m, \quad \acute{p}_3 = \not{p}_3 + 0, \quad z_{14} = 2\acute{p}_1 \cdot \grave{p}_4 = z_{23}, \\ \acute{p}_2 &= \not{p}_2 + m, \quad \acute{p}_4 = \not{p}_4 + 0, \quad z_{24} = 2\acute{p}_2 \cdot \grave{p}_4 = z_{14}, \end{aligned} \qquad \acute{\acute{F}}_4 = \acute{p}_4 \grave{\varepsilon}_4$$

Feynmann rules give us

$$\mathcal{A} = \langle 1|e_3 \frac{1}{\grave{p}_{24}} \grave{\varepsilon}_4|2\rangle + \langle 1|\grave{\varepsilon}_4 \frac{1}{\grave{p}_{23}} e_3|2\rangle = \frac{\langle 1|e_3 \acute{p}_{24} \grave{\varepsilon}_4|2\rangle}{z_{24}} + \frac{\langle 1|\grave{\varepsilon}_4 \acute{p}_{23} e_3|2\rangle}{z_{23}}$$

Simplification with Dirac spinors

$$(\acute{p}_2 + \acute{p}_4)\grave{\varepsilon}_4|2\rangle = |2\rangle\langle 2|\grave{\varepsilon}_4|2\rangle + \acute{\acute{F}}_4|2\rangle = |2\rangle(2\acute{p}_2 \cdot \grave{\varepsilon}_4) + \acute{\acute{F}}_4|2\rangle$$

Gauge-invariant form for $e^+e^-Z\gamma \to 0$ in $d=6$

$$\mathcal{A} = -\frac{\mathrm{Tr}[\acute{p}_1 \grave{p}_2 \overset{\prime\prime}{F}_4]}{z_{14}z_{24}}\langle 1|e_3|2\rangle + \frac{\langle 1|e_3 \overset{\prime\prime}{F}_4|2\rangle}{z_{24}} + \frac{\langle 1|\overset{\prime\prime}{F}_4 e_3|2\rangle}{z_{14}}$$

Ward identity is satisfied by *each* term in the expression.

$$\frac{\mathcal{A}}{\sqrt{2}} = -\frac{1}{\langle 1|4]}\langle 1|2]\frac{1}{\langle 4|2]}\otimes\langle 1|e_3|2\rangle + \langle 1|e_3|4\rangle\otimes\frac{1}{\langle 2|4]} + \frac{1}{[4|1\rangle}\otimes\langle 4|e_3|2\rangle$$

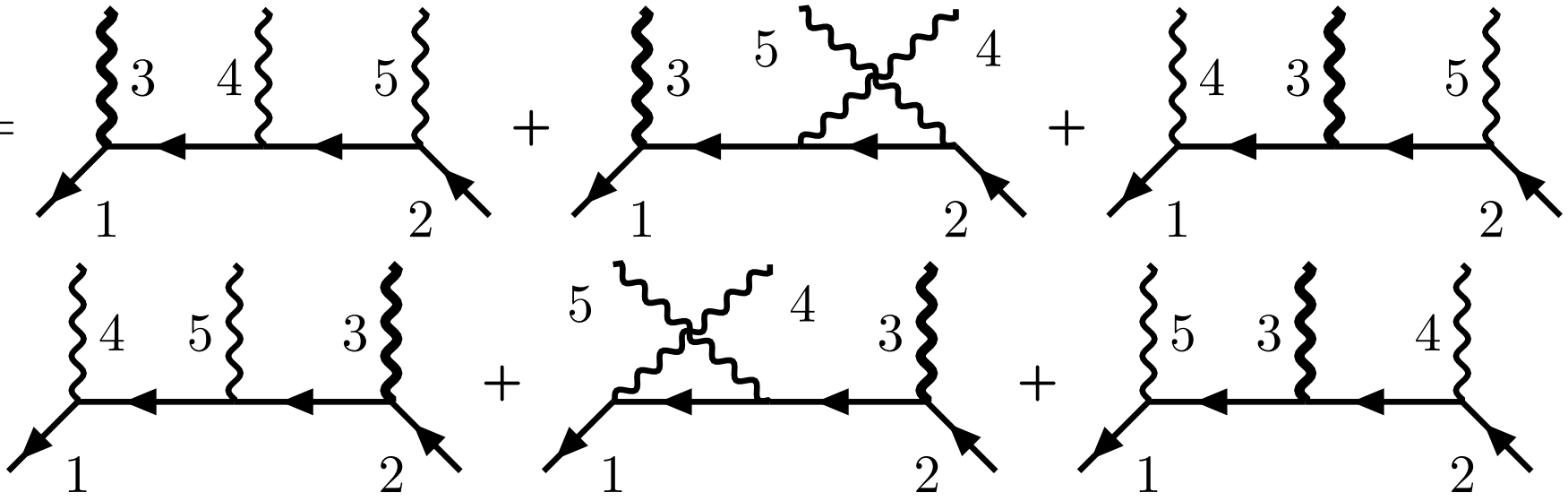
On-shell $e^+e^-\gamma\gamma \to 0$ in $d=6$

$$e^+(p_1) + e^-(p_2) + \gamma(p_3) + \gamma(p_4) \to 0$$

$$\mathcal{A}/2 = -\frac{1}{\langle 4|1|3\rangle}\otimes\Big(\langle 1|3]\otimes[4|2\rangle - \langle 1|4]\otimes[3|2\rangle\Big)$$
$$+ \frac{1}{[4|1\rangle}[4|3\rangle\otimes[3|4\rangle\otimes\frac{1}{\langle 2|4]} + \frac{1}{[3|1\rangle}[3|4\rangle\otimes[4|3\rangle\otimes\frac{1}{\langle 2|3]}$$

$$e^+(p_1) + e^-(p_2) + Z(p_3) + \gamma(p_4) + \gamma(p_5) \to 0$$

$\mathcal{A} =$



$$\mathcal{A} = \langle 1|e_3 \frac{1}{\not{p}_{13}} \not{\varepsilon}_4 \frac{1}{\not{p}_{25}} \not{\varepsilon}_5|2\rangle + \langle 1|\not{\varepsilon}_4 \frac{1}{\not{p}_{14}} e_3 \frac{1}{\not{p}_{25}} \not{\varepsilon}_5|2\rangle + \langle 1|\not{\varepsilon}_4 \frac{1}{\not{p}_{14}} \not{\varepsilon}_5 \frac{1}{\not{p}_{23}} e_3|2\rangle$$
$$+ \langle 1|e_3 \frac{1}{\not{p}_{13}} \not{\varepsilon}_5 \frac{1}{\not{p}_{24}} \not{\varepsilon}_4|2\rangle + \langle 1|\not{\varepsilon}_5 \frac{1}{\not{p}_{15}} e_3 \frac{1}{\not{p}_{24}} \not{\varepsilon}_4|2\rangle + \langle 1|\not{\varepsilon}_5 \frac{1}{\not{p}_{15}} \not{\varepsilon}_4 \frac{1}{\not{p}_{23}} e_3|2\rangle$$

Scalarized amplitude

$$
\begin{aligned}
\mathcal{A} = \Bigg(& -\frac{\mathrm{Tr}[\acute{p}_1 \grave{p}_2 \acute{\acute{F}}_4]\,\mathrm{Tr}[\acute{p}_1 \grave{p}_2 \acute{\acute{F}}_5]}{z_{14} z_{24} z_{15} z_{25}} + \frac{\mathrm{Tr}[\acute{p}_2 \grave{\grave{F}}_4 \grave{p}_2 \acute{\acute{F}}_5]}{z_{24} z_{245} z_{25}} + \frac{\mathrm{Tr}[\acute{p}_1 \grave{\grave{F}}_4 \grave{p}_1 \acute{\acute{F}}_5]}{z_{14} z_{145} z_{15}} \Bigg) \langle 1|e_3|2\rangle \\
& + \left(\frac{\mathrm{Tr}[\acute{p}_1 \grave{p}_2 \acute{\acute{F}}_4]}{z_{14} z_{24}} + \frac{\mathrm{Tr}[\acute{p}_2 \grave{p}_5 \acute{\acute{F}}_4]}{z_{24} z_{245}} \right) \frac{\langle 1|e_3 \acute{\acute{F}}_5|2\rangle}{z_{25}} \\
& + \left(\frac{\mathrm{Tr}[\acute{p}_1 \grave{p}_2 \acute{\acute{F}}_4]}{z_{14} z_{24}} - \frac{\mathrm{Tr}[\acute{p}_1 \grave{p}_5 \acute{\acute{F}}_4]}{z_{14} z_{145}} \right) \frac{\langle 1|\grave{\grave{F}}_5 e_3|2\rangle}{z_{15}} \\
& + \left(\frac{\mathrm{Tr}[\acute{p}_1 \grave{p}_2 \acute{\acute{F}}_5]}{z_{15} z_{25}} + \frac{\mathrm{Tr}[\acute{p}_2 \grave{p}_4 \acute{\acute{F}}_5]}{z_{25} z_{245}} \right) \frac{\langle 1|e_3 \acute{\acute{F}}_4|2\rangle}{z_{24}} \\
& + \left(\frac{\mathrm{Tr}[\acute{p}_1 \grave{p}_2 \acute{\acute{F}}_5]}{z_{15} z_{25}} - \frac{\mathrm{Tr}[\acute{p}_1 \grave{p}_4 \acute{\acute{F}}_5]}{z_{15} z_{145}} \right) \frac{\langle 1|\grave{\grave{F}}_4 e_3|2\rangle}{z_{14}} \\
& - \frac{\langle 1|e_3 \acute{\acute{F}}_4 \acute{\acute{F}}_5|2\rangle}{z_{25} z_{245}} - \frac{\langle 1|e_3 \acute{\acute{F}}_5 \acute{\acute{F}}_4|2\rangle}{z_{24} z_{245}} - \frac{\langle 1|\grave{\grave{F}}_4 e_3 \acute{\acute{F}}_5|2\rangle}{z_{14} z_{25}} \\
& - \frac{\langle 1|\grave{\grave{F}}_5 e_3 \acute{\acute{F}}_4|2\rangle}{z_{15} z_{24}} - \frac{\langle 1|\grave{\grave{F}}_4 \grave{\grave{F}}_5 e_3|2\rangle}{z_{14} z_{145}} - \frac{\langle 1|\grave{\grave{F}}_5 \grave{\grave{F}}_4 e_3|2\rangle}{z_{15} z_{145}}
\end{aligned}
$$

$e^+e^-Z\gamma\gamma \to 0$ in $d=6$

$$
\begin{aligned}
\mathcal{A} = -\Big(\mathcal{S}_4\otimes\mathcal{S}_5 + \frac{\mathcal{Y}_{154}\otimes\tilde{\mathcal{Y}}_{145}}{z_{145}} + \frac{\mathcal{Y}_{254}\otimes\tilde{\mathcal{Y}}_{245}}{z_{245}}\Big)\otimes\mathcal{B} \\
+\left[\Big(\mathcal{S}_4 + \frac{\mathcal{Y}_{245}\cdot\mathcal{E}_{54}}{z_{245}}\Big)\otimes\mathcal{C}_{25} - \frac{\mathcal{E}_{54}}{z_{245}}\otimes\mathcal{C}_{24}\right]\otimes\mathcal{G}_{15} \\
+\left[\Big(\mathcal{S}_4 - \frac{\mathcal{Y}_{145}\cdot\mathcal{E}_{54}}{z_{145}}\Big)\otimes\mathcal{C}_{15} - \frac{\mathcal{E}_{45}}{z_{145}}\otimes\mathcal{C}_{14}\right]\otimes\mathcal{G}_{25} \\
+\left[\Big(\mathcal{S}_5 + \frac{\mathcal{Y}_{254}\cdot\mathcal{E}_{45}}{z_{245}}\Big)\otimes\mathcal{C}_{24} - \frac{\mathcal{E}_{45}}{z_{245}}\otimes\mathcal{C}_{25}\right]\otimes\mathcal{G}_{14} \\
+\left[\Big(\mathcal{S}_5 - \frac{\mathcal{Y}_{154}\cdot\mathcal{E}_{45}}{z_{145}}\Big)\otimes\mathcal{C}_{14} - \frac{\mathcal{E}_{54}}{z_{145}}\otimes\mathcal{C}_{15}\right]\otimes\mathcal{G}_{24} \\
-\mathcal{C}_{14}\otimes\mathcal{C}_{25}\otimes\mathcal{H}_{45} - \mathcal{C}_{15}\otimes\mathcal{C}_{24}\otimes\mathcal{H}_{54}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B} &= \langle 1|e_3|2\rangle, \\
\mathcal{G}_{15} &= \langle 1|e_3|5\rangle, \\
\mathcal{H}_{45} &= \langle 4|e_3|5\rangle, \\
\mathcal{E}_{45} &= [\![4|5\rangle, \\
\mathcal{C}_{15} &= \frac{1}{[\![5|1\rangle}, \\
\mathcal{S}_5 &= \frac{1}{\langle 1|5]\!]}\langle 1|2]\!]\frac{1}{\langle 5|2]\!]}, \\
\mathcal{Y}_{154} &= \frac{1}{\langle 1|5]\!]}\langle 1|4]\!], \\
\tilde{\mathcal{Y}}_{145} &= \frac{1}{[\![1|4\rangle}[\![1|5\rangle
\end{aligned}
$$

Tuned comparison

Setup for tuned comparison

We performed a tuned comparison of our results for polarized Born and hard Bremsstrahlung with the results WHIZARD and CalcHEP programs.

Initial parameters

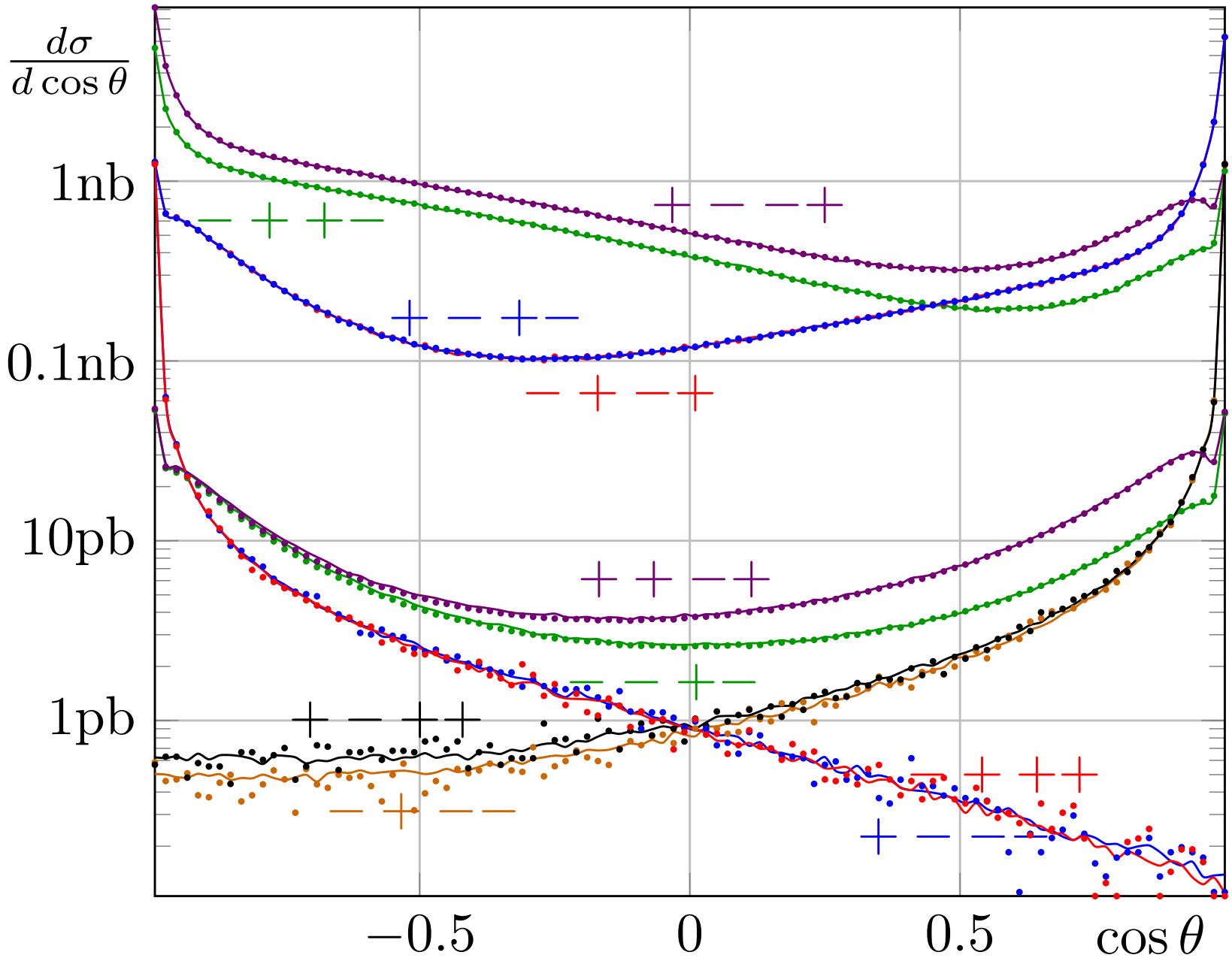
$$\alpha^{-1}(0) = 137.03599976, \quad M_W = 80.4514 \text{ GeV}, \quad \Gamma_W = 2.0836 \text{ GeV},$$
$$M_H = 125.0 \text{ GeV}, \quad M_Z = 91.1876 \text{ GeV}, \quad \Gamma_Z = 2.49977 \text{ GeV},$$
$$m_e = 0.5109990 \text{ MeV}, \quad m_\mu = 0.105658 \text{ GeV}, \quad m_\tau = 1.77705 \text{ GeV},$$
$$m_d = 0.083 \text{ GeV}, \quad m_s = 0.215 \text{ GeV}, \quad m_b = 4.7 \text{ GeV},$$
$$m_u = 0.062 \text{ GeV}, \quad m_c = 1.5 \text{ GeV}, \quad m_t = 173.8 \text{ GeV}.$$

with cuts $|\cos\theta| < 0.9, \quad E_\gamma > 1$ GeV

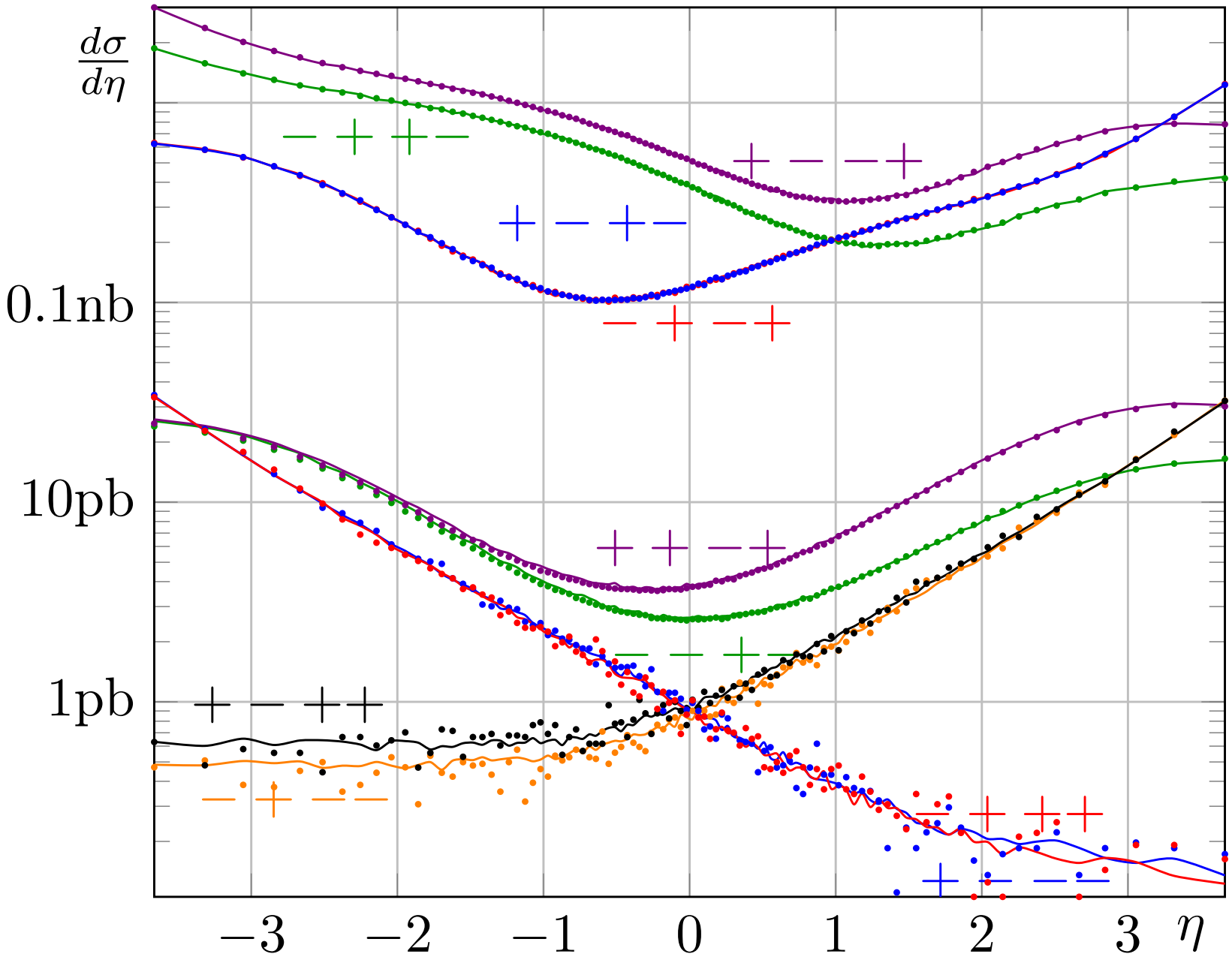
WHIZARD and CalcHEP

- W. Kilian, T. Ohl, J. Reuter, Eur.Phys.J.C71 (2011) 1742,
- A.Belyaev, N.Christensen,A.Pukhov, Comp. Phys. Comm. 184 (2013), pp. 1729-1769

ReneSANCe vs. WHIZARD (dots): all-polarized $e^+e^- \to \tau^+\tau^-\gamma$



ReneSANCe vs. WHIZARD (dots): all-polarized $e^+e^- \to \tau^+\tau^-\gamma$



Conclusion

- Applying extended set of Clifford-algebra operations we obtained *explicitly gauge-invariant form* of amplitudes for some processes.
  - Expressions contain only Maxwell bivector.
  - Relations to scalar QED and photon power expansion are transparent.
- Generalized form of axial-type gauge is proposed.
  - Massive gauge-fixing vectors are allowed.
  - Simplification of "amplitude" with off-shell photons is possible.
- Spinor formalism in $d = 6$ dimensions is applied to obtain modular form of amplitude.
  - Formalism is implemented as `C++14` library.
  - Allowed pseudo-mass term $\mu\gamma_5$ can be useful to deal with 1-loop integrands.

Future plans

- Application of the formalism to virtual part.