CBK-relation and $\{\beta\}$-expansion for the static potential $V_{QQ}(\vec{q}^{\,2})$.

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Kataev, Molokoedov, work in progress

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Outline

- CBK-relation: one-fold factorization of β-function
- CBK-relation: expansion in powers of the conformal anomaly $\beta(a_s)/a_s$
- Representation of the PT series for Adler D-function and Bjorken polarized sum rule in powers of $\beta(a_s)/a_s$
- Representation of the PT series for the static potential in powers of $\beta(a_s)/a_s$
- <u>Unpolarized</u> Bjorken sum rule: $\{\beta\}$-expansion
- Outlook, conclusion

Adler function, R-ratio, Bjorken polarized sum rule

$$D^{(M)}(a_s(Q^2)) = d_R \left(\sum_f Q_f^2\right) D_{NS}^{(M)}(a_s(Q^2)) + d_R \left(\sum_f Q_f\right)^2 D_{SI}^{(M\geq 3)}(a_s(Q^2))$$

$$C^{(M)}(a_s(Q^2)) = C_{NS}^{(M)}(a_s(Q^2)) + d_R \sum_f Q_f C_{SI}^{(M\geq 4)}(a_s(Q^2))$$

$$D\left(Q^2\right) = Q^2 \int_0^\infty \frac{R\left(s\right)}{\left(s+Q^2\right)^2}\, ds$$

$$\boxed{a_s = \alpha_s/\pi}$$

$M \geq 1$ is the order of PT

$$\boxed{D_{NS}^{(M)}(a_s(Q^2)) = 1 + \sum_{m=1}^{M} d_m a_s^m(Q^2)}$$

$$\boxed{C_{NS}^{(M)}(a_s(Q^2)) = 1 + \sum_{m=1}^{M} c_m a_s^m(Q^2)}$$

$$r_1 = d_1, \quad r_2 = d_2,$$

$$r_3 = d_3 - \frac{\pi^2}{3} d_1 \beta_0^2, \quad r_4 = d_4 - \pi^2 \left(d_2 \beta_0^2 + \frac{5}{6} d_1 \beta_1 \beta_0\right),$$

$$r_5 = d_5 - \pi^2 \left(2 d_3 \beta_0^2 + \frac{7}{3} d_2 \beta_0 \beta_1 + \frac{1}{2} d_1 \beta_1^2 + d_1 \beta_0 \beta_2\right) + \frac{\pi^4}{5} d_1 \beta_0^4$$

$$\beta_0 = \frac{11}{12} C_A - \frac{1}{3} T_f n_f \quad , \quad \beta_1 = \frac{17}{24} C_A^2 + \frac{5}{12} C_A T_f n_f - \frac{1}{4} C_F T_f n_f$$

The CBK relation

$$D_{NS}^{(M)}(a_s)C_{NS}^{(M)}(a_s) = 1+\Delta_{csb}^{(N)}(a_s)= 1+\left(\frac{\beta^{(N)}(a_s)}{a_s}\right)K^{(N)}(a_s)= 1+\left(\frac{\beta^{(N)}(a_s)}{a_s}\right)\sum_{i\geqslant 1}K_i a_s^i$$

The CBK was firstly detected in $\overline{\text{MS}}$-scheme at $\mathcal{O}(a_s^3(Q^2))$ level by *Broadhurst, Kataev in (93)*:

$SU(N_c)$ QCD + conformal symmetry *Crewther (72)* study of AVV diagram :

$$D_{NS}^{(3)}C_{NS}^{(3)} = 1 + \left(\frac{\beta^{(2)}(a_s)}{a_s}\right)(K_1 a_s + K_2 a_s^2) + \mathcal{O}(a_s^4) = 1 - \beta_0 K_1 a_s^2 - (\beta_0 K_2 + \beta_1 K_1)a_s^3 + \mathcal{O}(a_s^4)$$

It was confirmed later on at $\mathcal{O}(a_s^4)$ level *Baikov, Chetyrkin, Kühn, (10)* :

$$D_{NS}^{(4)}C_{NS}^{(4)} = 1 + \left(\frac{\beta^{(3)}(a_s)}{a_s}\right)(K_1 a_s + K_2 a_s^2 + K_3 a_s^3) + \mathcal{O}(a_s^5)$$

$$K_1 = \left(-\frac{21}{8} + 3\zeta_3\right)C_F,$$

$$K_2 = \left(\frac{397}{96} + \frac{17}{2}\zeta_3 - 15\zeta_5\right)C_F^2 + \left(-\frac{629}{32} + \frac{221}{12}\zeta_3\right)C_F C_A + \left(\frac{163}{24} - \frac{19}{3}\zeta_3\right)C_F T_F \underline{n_f}, \ldots$$

Proof indications of the validity of the CBK were given in *(97) by Crewther*

and in *(03) by Braun, Korchemsky, Müller*. At a_s^4 level confirmed by *Chetyrkin (22)* in QCD-like m

Then the CBK will be fulfilled in the gauge-dependent MOM-like schemes

in Landau gauge $\xi = 0$ in all orders of PT as well *(Garkusha, Kataev, Molokoedov (18))*.

Another representation: two-fold guess for CS breaking term

The conformal symmetry breaking term $\Delta^{(N)}_{csb}(a_s) = \left(\frac{\beta^{(N)}(a_s)}{a_s}\right) K^{(N)}(a_s)$

can be represented in the following form up to $O(a_s^4)$ at least *(Kataev, Mikhailov (10, 12))*:

$$\boxed{\Delta^{(N)}_{csb}(a_s) = \sum_{n=1}^{N} \left(\frac{\beta(a_s)}{a_s}\right)^n P_n(a_s) = \sum_{n=1}^{N} \left(\frac{\beta(a_s)}{a_s}\right)^n \sum_{r\geq 1} P_n^{[r]} a_s^r}$$

where coefficients $P_n^{[r]}$ are uniquely defined and do not depend on $T_F n_f$-structures and contain C_F and C_A quadratic Casimir operator of $SU(N_c)$ group (exception - light-by-light scattering effects).

For e.g.

$$\Delta^{(2)}_{csb}(a_s) = \left(\frac{\beta^{(2)}(a_s)}{a_s}\right) P_1^{(2)}(a_s) + \left(\frac{\beta^{(1)}(a_s)}{a_s}\right)^2 P_2^{(1)}(a_s)$$

$$= (-\beta_0 a_s - \beta_1 a_s^2)(P_1^{[1]} a_s + P_1^{[2]} a_s^2) + \beta_0^2 a_s^2 P_2^{[1]} a_s$$

$$= -\beta_0 P_1^{[1]} a_s^2 + (-\beta_0 P_1^{[2]} - \beta_1 P_1^{[1]} + \beta_0^2 P_2^{[1]}) a_s^3$$

$$P_1^{[1]} = \left(-\frac{21}{8} + 3\zeta_3\right) C_F,$$

$$P_1^{[2]} = \left(\frac{397}{96} + \frac{17}{2}\zeta_3 - 15\zeta_5\right) C_F^2 + \left(-\frac{47}{48} + \zeta_3\right) C_F C_A,$$

$$P_2^{[1]} = \left(\frac{163}{8} - 19\zeta_3\right) C_F, \quad \ldots$$

Two-fold expansion for Adler function

The double sum expression for $\Delta_{csb}(a_s)$ motivated *(Cvetič, Kataev (16))* to propose similar representation for Adler (and Bjorken) function:

$$\boxed{\begin{aligned} D_{NS}^{(M)}(a_s) &= 1 + D_0^{(M)}(a_s) + \sum_{n\geq 1}\left(\frac{\beta^{(N)}(a_s)}{a_s}\right)^n D_n^{(N)}(a_s) \\ &= 1 + D_0^{(M)}(a_s) + \sum_{n\geq 1}\left(\frac{\beta^{(N)}(a_s)}{a_s}\right)^n \sum_{r\geq 1} D_n^{[r]} a_s^r \end{aligned}}$$

At the four-loop level :

$$D_n(a_s) = \sum_{r=1}^{4-n} a_s^r \sum_{k=1}^{r} D_n^{[r]}[k, r-k] C_F^k C_A^{r-k}$$

$$+a_s^4 \delta_{n0} \underbrace{\left(D_0^{(4)}[F,A] \frac{d_F^{abcd} d_A^{abcd}}{d_R} + D_0^{(4)}[F,F] \frac{d_F^{abcd} d_F^{abcd}}{d_R} n_f \right)}_{light-by-light\ scattering\ effects}$$

In $SU(N_c)$ QCD in $\overline{\text{MS}}$-scheme coefficients $D_n^{[r]}[k, r-k]$ are unambiguously determined from the corresponding system of linear equations at the $\mathcal{O}(a_s^4)$ level at least. The same is true for the Bjorken polarized sum rule.

Supposing that two-fold expansion for Adler function is valid at the five-loop level, one can obtain:

$$D^{(5)}_{NS}(a_s) = 1 + D^{(5)}_0(a_s) + \left(\frac{\beta^{(4)}(a_s)}{a_s}\right) D^{(4)}_1(a_s) + \left(\frac{\beta^{(3)}(a_s)}{a_s}\right)^2 D^{(3)}_2(a_s)$$
$$+ \left(\frac{\beta^{(2)}(a_s)}{a_s}\right)^3 D^{(2)}_3(a_s) + \left(\frac{\beta^{(1)}(a_s)}{a_s}\right)^4 D^{(1)}_4(a_s)$$

This representation is in full agreement with $\{\beta\}$-expansion, proposed by *(Mikhailov (07))* :

$$d_1 = d_1[0], \qquad d_2 = \beta_0 d_2[1] + d_2[0],$$
$$d_3 = \beta_0^2 d_3[2] + \beta_1 d_3[0,1] + \beta_0 d_3[1] + d_3[0],$$
$$d_4 = \beta_0^3 d_4[3] + \beta_1\beta_0 d_4[1,1] + \beta_2 d_4[0,0,1] + \beta_0^2 d_4[2] + \beta_1 d_4[0,1] + \beta_0 d_4[1] + d_4[0]$$

At the $\mathcal{O}(a_s^5)$ level the analogous expansion will have the following form :

$$d_5 = \beta_0^4 d_5[4] + \beta_1\beta_0^2 d_5[2,1] + \beta_0^3 \underline{d_5[3]} + \beta_2\beta_0 d_5[1,0,1] + \beta_1^2 d_5[0,2] + \beta_1\beta_0 d_5[1,1]$$
$$+\beta_0^2 \underline{d_5[2]} + \beta_3 d_5[0,0,0,1] + \beta_2 d_5[0,0,1] + \beta_1 d_5[0,1] + \beta_0 \underline{d_5[1]} + \underline{d_5[0]}$$

Comparing these representations, we lead to the following equalities:

$$D^{(1)}_1 = -d_2[1] = -d_3[0,1] = -d_4[0,0,1] = -d_5[0,0,0,1],$$
$$D^{(2)}_1 = -d_3[1] = -d_4[0,1] = -d_5[0,0,1],$$
$$D^{(3)}_1 = -d_4[1] = -d_5[0,1],$$
$$D^{(1)}_2 = d_3[2] = d_4[1,1]/2 = d_5[0,2] = d_5[1,0,1]/2,$$
$$D^{(2)}_2 = d_4[2] = d_5[1,1]/2, \qquad D^{(1)}_3 = -d_4[3] = -d_5[2,1]/3.$$

Known $\{\beta\}$-coefficients: three-loop level

Coefficients	Color structures	
$d_1[0]$	C_F	$\frac{3}{4}$
$d_2[0]$	C_F^2	$-\frac{3}{32}$
	$C_F C_A$	$\frac{1}{16}$
$d_2[1]$	C_F	$\frac{33}{8} - 3\zeta_3$
$d_3[0]$	C_F^3	$-\frac{69}{128}$
	$C_F^2 C_A$	$-\frac{101}{256} + \frac{33}{16}\zeta_3$
	$C_F C_A^2$	$-\frac{53}{192} - \frac{33}{16}\zeta_3$
$d_3[1]$	C_F^2	$-\frac{111}{64} - 12\zeta_3 + 15\zeta_5$
	$C_F C_A$	$\frac{83}{32} + \frac{5}{4}\zeta_3 - \frac{5}{2}\zeta_5$
$d_3[0,1]$	C_F	$\frac{33}{8} - 3\zeta_3$
$d_3[2]$	C_F	$\frac{151}{6} - 19\zeta_3$

Known $\{\beta\}$-coefficients: four-loop level

Coefficients	Color structures	
$d_4[0]$	C_F^4	$\frac{4157}{2048} + \frac{3}{8}\zeta_3$
	$C_F^3 C_A$	$-\frac{3509}{1536} - \frac{73}{128}\zeta_3 - \frac{165}{32}\zeta_5$
	$C_F^2 C_A^2$	$\frac{9181}{4608} + \frac{299}{128}\zeta_3 + \frac{165}{64}\zeta_5$
	$C_F C_A^3$	$-\frac{30863}{36864} - \frac{147}{128}\zeta_3 + \frac{165}{64}\zeta_5$
	$\frac{d_F^{abcd} d_A^{abcd}}{d_R}$	$\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5$
	$\frac{d_F^{abcd} d_F^{abcd}}{d_R} n_f$	$-\frac{13}{16} - \zeta_3 + \frac{5}{2}\zeta_5$
$d_4[1]$	C_F^3	$-\frac{785}{128} - \frac{9}{16}\zeta_3 + \frac{165}{2}\zeta_5 - \frac{315}{4}\zeta_7$
	$C_F^2 C_A$	$-\frac{3737}{144} + \frac{3433}{64}\zeta_3 - \frac{99}{4}\zeta_3^2 - \frac{615}{16}\zeta_5 + \frac{315}{8}\zeta_7$
	$C_F C_A^2$	$-\frac{2695}{384} - \frac{1987}{64}\zeta_3 + \frac{99}{4}\zeta_3^2 + \frac{175}{32}\zeta_5 - \frac{105}{16}\zeta_7$
$d_4[0,1]$	C_F^2	$-\frac{111}{64} - 12\zeta_3 + 15\zeta_5$
	$C_F C_A$	$\frac{83}{32} + \frac{5}{4}\zeta_3 - \frac{5}{2}\zeta_5$
$d_4[2]$	C_F^2	$-\frac{4159}{384} - \frac{2997}{16}\zeta_3 + 27\zeta_3^2 + \frac{375}{2}\zeta_5$
	$C_F C_A$	$\frac{14615}{256} + \frac{39}{16}\zeta_3 - \frac{9}{2}\zeta_3^2 - \frac{185}{4}\zeta_5$
$d_4[0,0,1]$	C_F	$\frac{33}{8} - 3\zeta_3$
$d_4[1,1]$	C_F	$\frac{151}{3} - 38\zeta_3$
$d_4[3]$	C_F	$\frac{6131}{36} - \frac{203}{2}\zeta_3 - 45\zeta_5$

Comment-observation

Numerical values for $d_1[0]$, $d_2[0]$, $d_3[0]$, $d_4[0]$ agree with the $O(a_s^4)$ results *by Brodsky, Wu*(12) provided they are using not $\{\beta\}$-expansion , but the expansion of d_2, d_3 and d_4 in powers of n_f and follow $O(a_s^3)$ considerations by *by Grunberg, Kataev*(92)

The relation between $V_{\rm cusp}$ and $V_{Q\bar{Q}}$: analogy with CBK-relation

The closed Wilson loop: $W = \frac{1}{N}\langle 0|\mathrm{Tr}\left(P\exp\left(i\oint_C dx^\mu A_\mu(x)\right)\right)|0\rangle,$

Contour C consists of two straight line segments along directions v_1^μ and v_2^μ ($v_1^2 = v_2^2 = 1$), forming a cusp and extending to infinity where the contour is closed. The Euclidean cusp angle $\cos\phi = v_1 \cdot v_2$.

$$\log W = \log Z + \text{finite}, \quad \Gamma_{\rm cusp}(\phi, \alpha_s) = \frac{\partial \log Z}{\partial \log \mu},$$

$$\Gamma_{\rm cusp}(\pi - \delta, \alpha_s) = -C_F \alpha_s \frac{V_{\rm cusp}(\alpha_s)}{\delta} + \mathcal{O}\left(\alpha_s^4 \frac{\log\delta}{\delta}\right), \quad \delta \ll 1$$

$$V_{\rm cusp}^{(M)}(a_s) - V_{Q\bar{Q}}^{(M)}(a_s) = \frac{\beta^{(M-1)}(a_s)}{a_s} C^{(M-1)}(a_s), \quad C^{(r)}(a_s) = \sum_{k=1}^{r} C_k^{(r)} a_s^k,$$

proved for $M = 2$ [Grozin, Henn, Korchemsky, Marquard, 1510.07803] and for $M = 3$ convincing arguments are given [Brüser, Grozin, Henn, Stahlhofen, 1902.05076]

We study $\quad V_{\rm cusp}^{(M)}(a_s) - V_{Q\bar{Q}}^{(M)}(a_s) = \sum_{n=1}^{M} \left(\frac{\beta^{(M-n+1)}(a_s)}{a_s}\right)^n \mathscr{V}_n^{(M-n+1)}(a_s),$

$$\mathscr{V}_n^{(r)}(a_s) = \sum_{k=1}^{r} \mathscr{V}_{n,k}^{(r)} a_s^{k-1}, \quad \mathscr{V}_{1,1}^{(1)} = \mathscr{V}_{1,1}^{(2)} = \mathscr{V}_{1,1}^{(3)} = \cdots = 0,$$

$$V_{\rm cusp}^{(3)}(a_s) - V_{Q\bar{Q}}^{(3)}(a_s) = a_s^2\left(\frac{10}{9}C_A\beta_0 - \frac{28}{9}\beta_0^2\right) + a_s^3\left(\beta_0\left(C_F C_A\left(\frac{16357}{1152} - \frac{209}{24}\zeta_3 - \frac{11}{240}\pi^4\right) - C_A^2 C_{1,3}^{AA}\right)\right.$$

$$\left. + \frac{10}{9}C_A\beta_1 - 2\cdot\frac{28}{9}\beta_0\beta_1 + \beta_0^2\left(C_F\left(\frac{19}{2}\zeta_3 - \frac{1487}{96} + \frac{\pi^4}{20}\right) + C_A\left(\frac{1879}{144} - \frac{89}{12}\zeta_3 - \frac{\pi^4}{8}\right)\right) + \beta_0^3\left(2\zeta_3 - \frac{134}{27}\right)\right)$$

$\{\beta\}$-expansion for the static $Q\bar{Q}$ potential

Analytical results for the three-loop static potential in the $\overline{\text{MS}}$-scheme *[Lee, Smirnov A., Smirnov V., Steinhauser, 1608.02603]*

$$V(\vec{q}^{\,2}) = -\frac{4\pi C_F \alpha_s(\vec{q}^{\,2})}{\vec{q}^{\,2}}\left[1 + a_1 a_s(\vec{q}^{\,2}) + a_2 a_s^2(\vec{q}^{\,2}) + \left(a_3 + \frac{\pi^2 C_A^3 L}{8}\right) a_s^3(\vec{q}^{\,2})\right],$$

$$V_{Q\bar{Q}}(a_s) = 1 + a_1 a_s + a_2 a_s^2 + a_3 a_s^3$$

$$V_{Q\bar{Q}}^{(M)}(a_s) = 1 + V_0^{(M)}(a_s) + \sum_{n=1}^{M}\left(\frac{\beta^{(M-n+1)}(a_s)}{a_s}\right)^n V_n^{(M-n+1)}(a_s),$$

$$V_0^{(M)}(a_s) = \sum_{k=1}^{M} V_{0,k}^{(M)} a_s^k, \qquad V_n^{(r)}(a_s) = \sum_{k=1}^{r} V_{n,k}^{(r)} a_s^{k-1}.$$

At the fixed numbers n and k, $V_{n,k}^{(r)} \equiv V_{n,k}^{(r+1)} \equiv V_{n,k}^{(r+2)} \equiv \ldots$.

This representation is in full agreement with its $\{\beta\}$-expansion structure, namely:

$$\begin{aligned}
a_1 &= \beta_0 a_1[1] + a_1[0],\\
a_2 &= \beta_0^2 a_2[2] + \beta_1 a_2[0,1] + \beta_0 a_2[1] + a_2[0],\\
a_3 &= \beta_0^3 a_3[3] + \beta_1\beta_0 a_3[1,1] + \beta_2 a_3[0,0,1] + \beta_0^2 a_3[2] + \beta_1 a_3[0,1] + \beta_0 a_3[1] + a_3[0],
\end{aligned}$$

supplemented by the following equalities:

$$\begin{aligned}
&a_1[1] = a_2[0,1] = a_3[0,0,1] = -V_{1,1}^{(1)},\\
&a_2[1] = a_3[0,1] = -V_{1,2}^{(2)},\\
&a_2[2] = a_3[1,1]/2 = V_{2,1}^{(2)}.
\end{aligned}$$

Coefficients	Group structures	Numbers
$a_1[1]$	—	$\frac{5}{3}$
$a_1[0]$	C_A	$-\frac{2}{3}$
$a_2[2]$	—	$\frac{25}{9}$
$a_2[1]$	C_F	$\frac{35}{16} - 3\zeta_3$
	C_A	$-\frac{217}{72} + \frac{7}{2}\zeta_3$
$a_2[0]$	$C_F C_A$	$-\frac{385}{192} + \frac{11}{4}\zeta_3$
	C_A^2	$\frac{133}{144} - \frac{11}{4}\zeta_3 + \frac{\pi^2}{4} - \frac{\pi^4}{64}$
$a_3[3]$	—	$\frac{125}{27}$
$a_3[2]$	C_F	$\frac{5471}{288} - \frac{39}{2}\zeta_3$
	C_A	$-\frac{7943}{576} + \frac{69}{4}\zeta_3 + \frac{\pi^4}{15}$
$a_3[1]$	C_F^2	$-\frac{571}{192} - \frac{19}{8}\zeta_3 + \frac{15}{2}\zeta_5$
	$C_F C_A$	$-\frac{70069}{3456} + \frac{49}{2}\zeta_3 - \frac{15}{4}\zeta_5$
	C_A^2	$\frac{2491}{288} - \frac{309}{16}\zeta_3 - \frac{1091}{128}\zeta_5 - \frac{171}{128}\zeta_3^2 + \frac{9}{4}s_6 - \frac{761}{53760}\pi^6$ $+\pi^4\left(\frac{9}{640} + \frac{5}{192}\log 2 - \frac{3}{64}\log^2 2\right)$ $+\pi^2\left(-\frac{17}{576} + \frac{19}{64}\zeta_3 + \frac{1}{16}\log 2 + \frac{21}{32}\zeta_3 \log 2 + \frac{3}{2}\alpha_4\right)$

Blue terms differ slightly from those given in *[Brodsky, Mojaza, Wu (14) arXiV:1304.4631]*: 5471/288 vs 5171/288; 7943/576 ≈ 13.879 vs 2981/192 ≈ 15.526; 70069/3456 vs 66769/3456. D (14)spite application of the different $\{\beta\}$-motivated expansion procedure ($\mathcal{R}_\delta$) by BMW, the rest coefficients are the same!

Coefficients	Group structures	Numbers
$a_3[0]$	$C_F^2 C_A$	$\frac{6281}{2304} + \frac{209}{96}\zeta_3 - \frac{55}{8}\zeta_5$
	$C_F C_A^2$	$\frac{3709}{3456} - \frac{379}{96}\zeta_3 + \frac{55}{16}\zeta_5$
	C_A^3	$-\frac{19103}{27648} + \frac{181}{48}\zeta_3 + \frac{1431}{512}\zeta_5 + \frac{55}{512}\zeta_3^2 + \frac{3}{16}s_6 - \frac{21097}{1935360}\pi^6$ $+\pi^4\left(\frac{211}{23040} - \frac{15}{256}\log 2 - \frac{61}{2304}\log^2 2\right)$ $+\pi^2\left(-\frac{191}{768} + \frac{841}{768}\zeta_3 - \frac{955}{576}\log 2 + \frac{203}{384}\zeta_3 \log 2 + \frac{5}{3}\alpha_4\right)$
	$\frac{d_F^{abcd} d_F^{abcd}}{N_A} n_f$	$\frac{5}{96}\pi^6 + \pi^4\left(-\frac{23}{24} + \frac{1}{6}\log 2 - \frac{1}{2}\log^2 2\right)$ $+\pi^2\left(\frac{79}{36} - \frac{61}{12}\zeta_3 + \log 2 + \frac{21}{2}\zeta_3 \log 2\right)$
	$\frac{d_F^{abcd} d_A^{abcd}}{N_A}$	$\frac{1511}{2880}\pi^6 + \pi^4\left(-\frac{39}{16} + \frac{35}{12}\log 2 + \frac{31}{12}\log^2 2\right)$ $+\pi^2\left(\frac{929}{72} - \frac{827}{24}\zeta_3 - 74\alpha_4 + \frac{461}{6}\log 2 - \frac{217}{4}\zeta_3 \log 2\right)$

$$\alpha_4 = \mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\log^4 2}{4!} \quad \text{with polylogarithmic function} \quad \mathrm{Li}_n(x) = \sum_{k=1}^{\infty} x^k k^{-n},$$

$$s_6 = \zeta_6 + \zeta_{-5,-1} \quad \text{with} \quad \zeta_6 = \frac{\pi^6}{945} \quad \text{and multiple zeta value} \quad \zeta_{-5,-1} = \sum_{k=1}^{\infty}\sum_{i=1}^{k-1} \frac{(-1)^{i+k}}{ik^5}.$$

The term $a_3[0]$ contains $d_F^{abcd} d_A^{abcd}/N_A$ and $d_F^{abcd} d_F^{abcd} n_f/N_A$ contributions of the light-by-light scattering type that satisfies the correct transition to the QED limit. The result of *[Brodsky, Mojaza, Wu (14) arXiV:1304.4631]* is modified to totally analytical expression In their paper they used results of semi-analytical calculations by *Smirnov, Smirnov and Steinhauser (08)(10)*

Is it possible to use decomposition in powers of conformal anomaly for PT series of any observables in QCD?

We strongly suspect that the answer is yes.

For this, consider the RG-invariant quantity - the unpolarized Bjorken sum rule, which at this moment is not included in any known equation containing a factorized β-function (à la CBK-relation or relation between $V_{\rm cusp}$ and $V_{Q\bar{Q}}$) and study its possible $\{\beta\}$-expansion in powers of conformal anomaly:

$$\int_0^1 dx(F_1^{\bar{\nu}p}(x,Q^2) - F_1^{\nu p}(x,Q^2)) = 1 + F_0^{(M)}(a_s) + \sum_{n=1}^{M-1}\left(\frac{\beta^{(M-n)}(a_s)}{a_s}\right)^n F_n^{(M-n)}(a_s);$$

$$\int_0^1 dx F_1^{\bar{\nu}p-\nu p}(x,Q^2) = 1 + F_{0,1}^{(4)}a_s + \left(F_{0,2}^{(4)} - \beta_0 F_{1,1}^{(3)}\right)a_s^2 + \left(F_{0,3}^{(4)} - \beta_0 F_{1,2}^{(3)} - \beta_1 F_{1,1}^{(3)} + \beta_0^2 F_{2,1}^{(2)}\right)a_s^3$$

$$+\left(F_{0,4}^{(4)} - \beta_0 F_{1,3}^{(3)} - \beta_1 F_{1,2}^{(3)} - \beta_2 F_{1,1}^{(3)} + \beta_0^2 F_{2,2}^{(2)} + 2\beta_0\beta_1 F_{2,1}^{(2)} - \beta_0^3 F_{3,1}^{(1)}\right)a_s^4$$

Using the results of the $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_s^3)$ analytical calculations *[Chetyrkin, Gorishnii, Larin, Tkachov, Phys.Lett.B 137 (84)]*; *[Larin, Tkachov, Vermaseren, Phys.Rev.Lett. 66 (91)]*:

$$F_{0,1} = -\frac{C_F}{2}, \quad F_{1,1} = \frac{4}{3}C_F, \quad F_{0,2} = \frac{11}{16}C_F^2 - \frac{1}{24}C_F C_A,$$

$$F_{2,1} = -\frac{155}{36}C_F, \quad F_{1,2} = \left(\frac{431}{96} - \frac{1}{2}\zeta_3\right)C_F^2 + \left(-\frac{35}{144} + \frac{7}{2}\zeta_3 - 5\zeta_5\right)C_F C_A,$$

$$F_{0,3} = \left(-\frac{313}{64} - \frac{47}{4}\zeta_3 + \frac{35}{2}\zeta_5\right)C_F^3 + \left(\frac{687}{128} + \frac{125}{8}\zeta_3 - \frac{95}{4}\zeta_5\right)C_F^2 C_A + \left(-\frac{463}{288} - \frac{137}{24}\zeta_3 + \frac{115}{2}\zeta_5\right)C_F C_A^2$$

Using the unpublished results of Baikov, Chetyrkin, Kühn of the analytical four-loop $SU(3)$ calculations, presented by K.G. Chetyrkin in 2015 at the Conference "Calculations for Modern and Future Colliders" in Dubna `https://indico.cern.ch/event/368497/contributions/1787078/attachments/1134035/1621959/01_calc15.pdf`, we find

$$F_{3,1} = \frac{1780}{81}, \quad F_{2,2} = \frac{78155}{2592} + \frac{87}{2}\zeta_3 + 12\zeta_3^2 - 110\zeta_5,$$
$$F_{1,3} = -\frac{97247}{2592} - \frac{61153}{648}\zeta_3 - \frac{965}{54}\zeta_5 + \frac{296}{27}\zeta_3^2 - \frac{49}{4}\zeta_7 - 6B,$$
$$F_{0,4} = -\frac{8139161}{124416} - \frac{308489}{1296}\zeta_3 + \frac{239665}{1296}\zeta_5 - \frac{2927}{108}\zeta_3^2 + \frac{253757}{2592}\zeta_7 - \frac{33}{2}B + B \cdot n_f,$$

where $B \sim d_F^{abcd} d_F^{abcd}/d_R$-term is the light-by-light scattering type contribution to the Bjorken <u>unpolarized</u> sum rule.

Conclusion

- The BCK -realtions for Adler function and polarized Bjorken sum rule are reminded.
- The BCK-type cancellations of conformal-symmetry terms and factorization of QCD beta -function BUT in region of momentum transferred $(0 + q_1^2 + q_2^2 + q_3^2)$ are discussed for the static potential (more careful considerations are in progress)
- It is shown that the analytical expressions for $\{\beta\}$-expansion by Mikhailov (07) while combined with with R_δ-approach by Brodsky, Wu and Mojaza (14) agree with considered here results of expansions in multiple-power β-function represntation (used for D-function and Bjp-sum rule by Cvetic and Kataev (16) and Garkusha, Kataev and Mlokoedov (22))
- Guess is made that this multiple β-function represntation in QCD which distinguish scale-independet terms from the proportional to β_k-ones ones may be true for any RG-invariant quantity. Analysis is made for still unpublished a_s^4-results for Bjorken <u>unpolarized</u> DIS sum rule
- The multiple-power β-function representations may be helpful for getting parts of high-order PT coefficients for RG-invariant quantities.