# Tachyon condensation in a chromomagnetic background field and the groundstate of QCD

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In an attempt toward a better understanding of the vacuum of QCD I propose a condensation of the tachyonic mode in SU(2). In the Savvidy vacuum, this mode is known to be unstable. In an approximation where the gluon fields are reduced to the tachyonic mode, which can be considered as a complex scalar field in (1+1)-dimensions, I apply the methods known from the Higgs model and finite temperature field theory. The symmetry is spontaneously broken by a condensate of tachyons, i.e. of the unstable mode. As a result, I obtain a stable vacuum state with energy below zero. The energy of this state is a minimum in two parameters, the chromomagnetic background field, and the condensate. Raising the temperature, I observe a phase transition, and a restoration of the symmetry.[1]

[1] based on arxiv 2207.08711

# Plan

- The chromomagnetic vacuum and the problem with its stability
- Field theory with the tachyonic mode
- Symmetry breaking by a condensate of tachyons
- Symmetry restoration in the Hartree approximation
- Conclusions

# The chromomagnetic vacuum and the problem with its stability

consider QCD: $\mathcal{L}_{QCD} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \text{quarks},$

with the field strength $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$

($a$-color index of SU(N))

and a color magnetic abelian background field, $A^a_\mu \to A^a_\mu + B^a_\mu$,

with $B^a_\mu = \frac{B}{2}\delta^{a3}\left(-g_{\mu 1}x_2 + g_{\mu 2}x_1\right)$

In QCD, the effective potential following from the generalization of the Heisenberg-Euler Lagrangian of QED reads

$$V_{eff} = \frac{B^2}{2} + \frac{11N(gB)^2}{96\pi^2}\left(\ln\frac{(gB)^2}{\mu^4} - \frac{1}{2}\right) - i\frac{(gB)^2}{8\pi}$$

the real part has a minimum below zero at finite B,

$$gB_{|_{min}} = \mu^2 \exp\left(-\frac{24\pi^2}{11g^2}\right), \quad V_{eff\,|_{min}} = -\frac{11\mu^4}{96\pi^2}\exp\left(-\frac{48\pi^2}{11g^2}\right),$$

This field will be spontaneously created and provide a new groundstate of QCD. This is the so-called *Savvidy vacuum*

# The origin of the instability

The spectrum, belonging to the linearized part of the above Lagrangian, reads $p_0^2 = p_3^2 + gB(2n + 1 + 2s)$

$n$ numbers the Landau levels, $s = \pm 1$ is the spin projection, it is twice that of an electron from the larger spin of the gluon

for $n = 0$, $s = -1$, we have $p_0^2 = p_3^2 - gB$,

thus a one particle energy below zero, which is responsible for the imaginary part in the effective potential

it can be viewed as a negative mass square: $p_0^2 = p_3^2 - m^2$,

this mode is called a 'tachyonic state', also the 'unstable state'

<u>Comment 1</u>: The contribution from the stable states, i.e., all states except for the tachyoninc one, also result in a negative effective potential (without any imaginary part), formally substituting '11'$\rightarrow$'5' in front of the logarithm

<u>Comment 2</u>: The spontaneous generation of the magnetic field in the color magnetic background is a symmetry breaking. As it was shown, the symmetry is NOT restored at high temperature.

# Attempts to remove the instability

- *Copenhagen vacuum* [1]
- In [2] (and successors) the idea was spelled out that the self-interaction of the tachyonic mode, which is a consequence of the non-Abelian structure of the theory, should remove the imaginary part like it happens with the quartic oscillator in quantum mechanics.
- In [3], an attempt was undertaken to sum ring (*daisy*) diagrams using the gluon polarization tensor in some tractable approximation.

[1] J. Ambjorn and P. Olesen. On the Formation of a Random Color Magnetic Quantum Liquid in QCD.
*Nucl. Phys. B*, 170(1):60–78, 1980
[2] Curt A. Flory. Covariant Constant Chromomagnetic Fields and Elimination of the One Loop Instabilities.
1983.
Preprint, SLAC-PUB3244, 1983
[3] Vladimir Skalozub and Michael Bordag. Color ferromagnetic vacuum state at finite temperature.
*Nucl. Phys. B*, 576:430–44, 2000

# Adding an $A_0$-field

The common conclusion in [2] and [3] was that one may disregard the imaginary part, keeping the real part without changes

- In [4] an attempt was undertaken to find a minimum of the effective potential when in addition to the chromomagnetic background also a constant $A_0$-field (Polyakov loop) is present. This work was motivated by a lattice calculation, [5], where a minimum in the $(A_0, B)$-plane was seen (with no imaginary part). However, in the two-loop calculation done in [4], a very unnatural behavior of the real part of the effective potential in the $(A_0, B)$-plane was found. This behavior puts into question the above-mentioned conclusion that when summing ring diagrams, the real part stays in place.

[4] M. Bordag and V. Skalozub. Effective potential of gluodynamics in background of Polyakov loop and colormagnetic field.
*Eur. Phys. J. C*, 82:390, 2022.
arXiv 2112.01043
[5] V. Demchik and V. Skalozub. Spontaneous creation of chromomagnetic field and A(0)-condensate at high temperature on a lattice.
*J. Phys.*, A41:164051, 2008

# In the present talk,

I suggest a new solution to the problem with the imaginary part. It consists in the application of the Higgs mechanism to the unstable (tachyonic) mode. I define the unstable mode by its quantum numbers $p_0$, $p_3$, and $n = 0$, $s = -1$. This way, it represents a complex scalar field, $\psi(x_0, x_3)$, with negative mass square and a quartic self-interaction, living in the remaining two dimensions $(x_0, x_3)$. The corresponding Lagrangian reads

$$\mathcal{L} = \frac{1}{2}\psi^* \left(\partial_0^2 - \partial_3^2 - m^2\right) \psi - \lambda(\psi^*\psi)^2.$$

The color magnetic background field enters through the mass square and through the vertex factor

$$m^2 = gB, \qquad \lambda = 8g^2 \frac{gB}{2\pi}.$$

To realize this program, I consider a quantum field theory with the tachyonic mode.

# Field theory with the tachyonic mode

I consider the action of SU(2) dropping all modes except for the tachyonic one. This should be understood as a first approximation in the assumption that the tachyonic mode captures the essential basics and that the remaining modes can be handled as perturbations. The background is $B_\mu^a = \delta^{a3}\frac{B}{2}(-x_2, x_1, 0, 0)_\mu$.
The initial formulas read
$\mathcal{L}_{QCD} = -\frac{1}{4}\left(F_{\mu\nu}^a\right)^2, \quad S = \int d^4x\ \mathcal{L}_{QCD}, \quad Z = \int DA_\mu^a\ e^S$
We turn the gauge potential $A_\mu^a$ into the so-called charged basis,

$$W_\mu(x) = \frac{1}{\sqrt{2}}\left(A_\mu^1 + iA_\mu^2\right), \qquad A_\mu = A_\mu^3,$$
$$W_\mu^*(x) = \frac{1}{\sqrt{2}}\left(A_\mu^1 - iA_\mu^2\right).$$

The field $A_\mu$ is the color neutral component and $W_\mu$ is color charged. It is a complex field whereas $A_\mu$ remains real. This way the theory has a neutral and a charged vector fields (dropping the word 'color' from now on).
The covariant derivative for the charged field is $D_\mu = \partial_\mu - iB_\mu$.

# The action

$S = S_c + S_2 + S_3 + S_4,$
$S_c = -\frac{B^2}{2}$ – classical part
$S_2 =$
$\int dx \ \left[\frac{1}{2}A_\mu(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A_\nu + W^*_\mu\left(g_{\mu\nu}D^2 - D_\mu D_\nu + 2iF_{\mu\nu}\right)W_\nu\right]$
– quadratic part
$S_4 = g^2 \int dx \left(W^*_\mu W_\nu W^*_\mu W_\nu - W^*_\mu W_\mu W^*_\nu W_\nu\right)$ – quartic part
The tachyonic contribution is in the charged field and it reads

$$W^{ta}_\mu(x) = \int \frac{dk_\alpha}{(2\pi)^2}\ e^{ik_\alpha x^\alpha} u_0\left(x_\perp\right)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\\ 0\\ 0\end{pmatrix}_\mu \tilde{\psi}(k_\alpha), \quad (\alpha = 3, 4),$$

where

$$x_\perp = \sqrt{x_1^2 + x_2^2}, \qquad u_0(x_\perp) = \left(\frac{B}{2\pi}\right)^{1/2} \exp\left(-\frac{B}{4}x_\perp^2\right),$$

and $u_0(x_\perp)$ is the ground state wave function of a harmonic oscillator belonging to the lowest Landau level.

# The action with only the tachyonic mode

The tachyonic mode is a complex scalar field, $\tilde{\psi}(k_\alpha)$, in two dimensions. We introduce its Fourier transform back into configuration space,

$$\psi(x_\alpha) = \int \frac{dk_\alpha}{(2\pi)^2}\; e^{ik_\alpha x^\alpha}\, \tilde{\psi}(k_\alpha), \;\; W^{ta}_\mu(x) = u_0(x_\perp)\frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ i \\ 0 \\ 0 \end{array} \right)_\mu \psi(x_\alpha).$$

Now, dropping all other modes, the remaining action is

$$S_2 = \int dx_3 dx_4 \;\; \psi^*(x_\alpha)\left(\partial_\alpha^2 + gB\right)\psi(x_\alpha).$$

$$S_4 = g^2 \frac{B}{2\pi} \int dx_3 dx_4 \;\; \left(\psi^*(x_\alpha)\psi(x_\alpha)\right)^2 ,$$

where we used

$$\int dx_1 dx_2 \;\; u_0(x_\alpha)^4 = \frac{B}{2\pi}.$$

# Applying the second Legendre transform (or CJT – formalism) to the tachyonic action

The complex field $\psi(x)$ is equivalent to two real scalar fields (Higgs and Goldstone fields), $\psi(x_\alpha) = \frac{1}{\sqrt{2}}(\eta(x_\alpha) + i\phi(x_\alpha))$. So we have an O(2) model in 2 dimensions, and consider finite temperature. The action turns into

$$S = \int dx_1 dx_2 \Big[\frac{1}{2}\eta(x_\alpha)(\partial_\alpha^2 + m^2)\eta(x_\alpha) + \phi(x_\alpha)(\partial_\alpha^2 + m^2)\phi(x_\alpha) - \frac{\lambda}{8}\left(\eta(x_\alpha)^2 + \phi(x_\alpha)^2\right)^2\Big]$$

with the notations $m^2 = gB, \quad \lambda = 8g^2\frac{gB}{2\pi}$

Like in the Higgs model, one must make a shift,

$$\eta(x_\alpha) \to \eta(x_\alpha) + v,$$

which realizes the symmetry breaking, and quantize the field after this shift.

$v$ is the condensate, $\langle\bar{\psi}\psi\rangle = \frac{1}{2}v^2$.

# Applying the second Legendre transform (or CJT – formalism) to the tachyonic action

The shift results in new masses,

$$\mu_\eta^2 = -m^2 + \frac{3}{2}\lambda v^2, \qquad \mu_\phi^2 = -m^2 + \frac{1}{2}\lambda v^2.$$

We follow [6]. The propagator is considered as an arbitrary argument and the Legendre transform is with respect to the propagator. The result is a representation of the free energy in terms of 2PI (two particle irreducible) diagrams.

$$\begin{aligned} W = \frac{m^2}{2}v^2 - \frac{\lambda}{8}v^4 + \frac{1}{2}\mathrm{tr}\ln\beta_\eta + \frac{1}{2}\mathrm{tr}\ln\beta_\phi \\ - \frac{1}{2}\mathrm{tr}\Delta_\eta^{-1}\beta_\eta - \frac{1}{2}\mathrm{tr}\Delta_\pi^{-1}\beta_\pi + W^{2\mathrm{PI}}[\beta_\eta, \beta_\phi] \end{aligned}$$

The inverse free propagators read (in momentum representation) $\Delta_\eta^{-1} = k_\alpha^2 + \mu_\eta^2$, $\Delta_\phi^{-1} = k_\alpha^2 + \mu_\phi^2$,

[6] M. Bordag and V. Skalozub. Temperature phase transition and an effective expansion parameter in the O(N)-model.
*Phys. Rev. D*, 65:085025, 2002

# The gap equations and the Hartree approximation

The above representation is equivalent to a resummation of all *daisy*-diagrams (mass insertions, also *foam*-diagrams). The price for that are the Schwinger-Dyson equations,

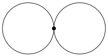
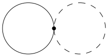
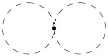
$$\beta_\eta^{-1} = \Delta_\eta^{-1} - \Sigma_\eta(k), \qquad \beta_\phi^{-1} = \Delta_\phi^{-1} - \Sigma_\phi(k),$$

where

$$\Sigma_\eta(k) = 2\frac{\delta W_2[\beta_\eta, \beta_\phi]}{\delta \beta_\eta}, \qquad \Sigma_\phi(k) = 2\frac{\delta W_2[\beta_\eta, \beta_\phi]}{\delta \beta_\phi},$$

are functional derivatives from the 2PI graphs with respect to the propagators.

In lowest order (Hartree approximation), the 2PI-graphs are

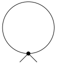
$W_2^{Hartree} = \frac{1}{8}$  $+ \frac{1}{4}$  $+ \frac{1}{4}$  ,

$$= -\frac{3\lambda}{8}\left(\Sigma_\eta^{(0)}\right)^2 - \frac{\lambda}{4}\Sigma_\eta^{(0)}\Sigma_\phi^{(0)} - \frac{3\lambda}{8}\left(\Sigma_\phi^{(0)}\right)^2,$$

where the solid line represents $\beta_\eta$ and the dashed line is $\beta_\phi$.

# The gap equations

In this approximation, the mass insertions

$$\Sigma_\eta^{(0)} \equiv \mathrm{tr}\beta_\eta =$$  $$, \quad \Sigma_\phi^{(0)} \equiv \mathrm{tr}\beta_\phi =$$  ,

are constants.
We make the ansatz $\beta_\eta(k) = \frac{1}{k_\beta^2 + M_\eta^2}, \beta_\phi(k) = \frac{1}{k_\beta^2 + M_\phi^2}$,
and we are left with gap equations

$$M_\eta^2 = -m^2 + \frac{3\lambda}{2}v^2 + \frac{3\lambda}{2}\Sigma_\eta^{(0)} + \frac{\lambda}{2}\Sigma_\phi^{(0)},$$
$$M_\phi^2 = -m^2 + \frac{\lambda}{2}v^2 + \frac{\lambda}{2}\Sigma_\eta^{(0)} + \frac{3\lambda}{2}\Sigma_\phi^{(0)}.$$

The one loop graphs are simple, for example
$\Sigma_\eta^{(0)} = \frac{-1}{4\pi}\ln\frac{M^2}{2\mu^2} + \frac{1}{\pi}S_2(M/T)$ ($\mu = 1$ in the following) with
$S_2(x) = \sum_{n=1}^{\infty} K_0(nx) = \int_x^\infty dy\, \frac{(y^2 - x^2)^{-1/2}}{e^y - 1}$

Inserting the above expressions, for the effective potential $W = -V_{eff}$ we arrive at

$$\begin{aligned} V_{eff} = & -\frac{m^2}{2}v^2 + \frac{\lambda}{8}v^4 + \frac{1}{2}V_1(M_\eta) + \frac{1}{2}V_1(M_\phi) \\ & - \frac{3\lambda}{8}\left(\Delta_0(M_\eta)\right)^2 - \frac{\lambda}{4}\Delta_0(M_\eta)\Delta_0(M_\phi) - \frac{3\lambda}{8}\left(\Delta_0(M_\phi)\right)^2 , \end{aligned}$$

Using the solutions of the gap equations, this effective potential becomes a function of the condensate $v$ (and the parameters $m$, $\lambda$ and $T$).

<u>Comment</u> The gap equations have always (for all values of $v$ and of the parameters) a real solution, $M_\eta$, $M_\phi$. We mention that this is a result of the resummations, and happens already at zero temperature.

# Phase transitions and symmetry restoration

With the above formulas, in our $O(2)$-model after the symmetry breaking by the condensate $v$, the phase transition may be investigated numerically at finite temperature. The known result is a first-order phase transition; at some $T_c$, the condensate disappears and the symmetry is restored.

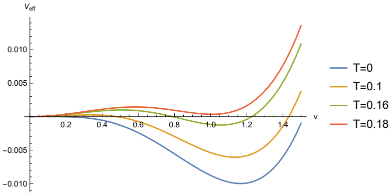


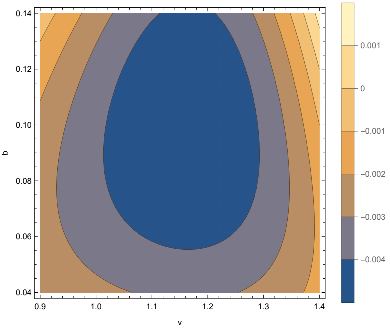

Figure: $V_{eff}$ as function of $v$ for $b = 0.1$ and $g = 1$.

<u>Comment</u> From general grounds one would expect a second order transition. That we see a first order is a known defect of the Hartree approximation. However, the symmetry breaking at low temperature, and its restoration at higher temperature are correct also in the given approximation.

# Minimum of the effective potential as function of both, condensate and magnetic field

The complete effective potential consists of the classical energy and the quantum correction, $\tilde{V}_{eff}(v, B) = \frac{B^2}{2} + V_{eff}$, and, in fact, we have to look for a minimum of <u>this</u> expression. This is, what comes in place of the old Heisenberg-Euler formula.

A numerical evaluation shows a minimum in the plane of the two parameters, $v$ and $B$. $T = 0$ (left panel), $T = 0.08$ (right panel)



# Symmetry restoration

As mentioned, the restoration of the symmetry at high temperature was a problem for the vacuum with the tachyonic mode (besides its instability). In the given approach this problem is solved. Raising the temperature, the minimum in the $(v, B)$-plane becomes shallower and disappears finally.

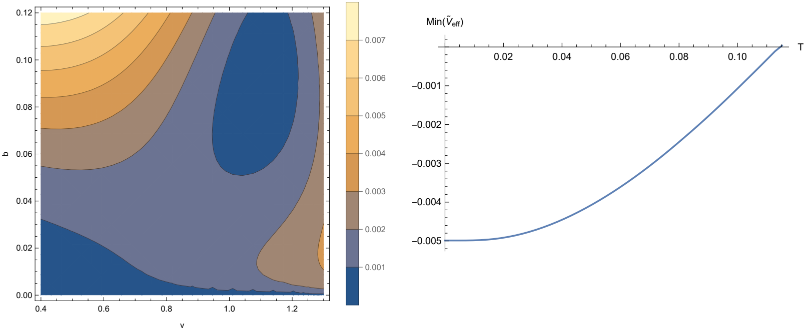



Figure: The vacuum energy $\tilde{V}_{eff}(v, B)$, for $T = 0.12$ (left panel) with two minima. The right panel shows the depth $Min(\tilde{V}_{eff})$ of the minimum as function of the temperature.

# Conclusions

In this talk, I showed a new model for the vacuum of QCD. It rests on the assumption that the unstable, tachyonic mode in a chromomagnetic background field forms a condensate. This means that the tachyonic mode is quantized around the condensate and that the corresponding effective potential is lower than the perturbative one.
This way, the original symmetry will be spontaneously broken by two, the magnetic background and the tachyonic condensate. This situation is similar to the Higgs model and its 'Mexican hat' potential. In our case, the contribution $-gB$ to the spectrum plays the role of the negative mass square.

To realize the above proposition, I assume that the basic features can be investigated in an approximation where all gluon modes, except for the tachyonic one, are neglected. The resulting theory is an O(2)-model in two dimensions with a mass and a coupling which depend on the background field.

# Conclusions (cont.)

Raising the temperature, the mentioned minimum becomes shallower, and a second minimum at the origin appears, which becomes the deeper one when further raising the temperature. This way, the symmetry will be restored.

I hope, the tachyon condensation observed will open a new window for the confinement problem.

Over the past decades, a much-discussed question was what happens with the unstable, tachyonic mode. The physical answer is that tachyons are created (from the instability) until these come into equilibrium with their repulsive self-interaction and that these tachyons form a stable condensate (relativistic BEC).

# Further development

The above investigation uses the simplest model.
Further steps should be:

1. account for the backreaction of the condensate on the background
2. consider non homogeneous background (for example, a center vortex)
3. account for the stable modes
4. ...

Thank you for attention