Noether Charge, Thermodynamics and Phase Transition of the Schwarzschild Black Hole in Anti-de Sitter-Beltrami spacetime

T. Angsachon and K. Ruenearom

Division of Physics, Faculty of Science and Technolgy, Thammasat University

VII International Conference
Models in Quantum Field Theory
October 10-14, 2022

Noether Charge, Thermodynamics and Phase Transition of the Schwarzschild Black Hole in Anti-de Sitter-Beltrami spacetime

- arXiv:2208.07209
- Part I:Schwarzschild-Anti-de Sitter-Beltrami metric
- Part II:Entropy of the Schwarzschild Anti-de Sitter-Beltrami Black Hole
- Part III:Thermal Phase Transition of the Schwarzschild-Anti de Sitter-Beltrami Black Hole

Schwarzschild-Anti-de Sitter-Beltrami metric

- Schwarzschild Anti-de Sitter metric from the gravitating mass M and AdS radius L ; (Using unit $G = \hbar = c = k_B = 1$)

$$ds^2 = \left(1 + \frac{r^2}{L^2} - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{\left(1 + \frac{r^2}{L^2} - \frac{2M}{r}\right)} - r^2 d\Omega^2, \quad (1)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$.

- Beltrami coordinates transformations

$$t \Rightarrow L \arcsin \frac{t}{L\sqrt{1 + \frac{t^2}{L^2}}}, \quad r \Rightarrow \frac{r}{\sqrt{1 + \frac{t^2}{L^2} - \frac{r^2}{L^2}}}. \quad (2)$$

- The line element of the Schwarzschild-Anti-de Sitter-Beltrami(SAdSB) metric is [T.Angsachon and S.N.Manida(arXiv:1301.4198)]

$$ds^2 = \left(\frac{1}{h_0^4} f(r,t) - \frac{r^2 t^2}{L^4 h^6 f(r,t)}\right) dt^2 + 2\frac{h_0^2 tr}{L^2 h^6 f(r,t)} dtdr - \frac{h_0^4}{h^6 f(r,t)} dr^2 - \frac{r^2}{h^2} d\Omega^2, \quad (3)$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2, \quad (4)$$

$$f(r,t) = 1 + \frac{r^2}{L^2 h^2} - \frac{2Mh}{r}, \quad (5)$$

$$h(t,r) = \sqrt{1 + \frac{t^2 - r^2}{L^2}}, \quad (6)$$

$$h_0(t) = \sqrt{1 + \frac{t^2}{L^2}}. \quad (7)$$

- The Killing vector of the SAdSB metric [S.N.Manida(arxiv:1111.3676), H.Y. Gao and etc.(hep-th/0311156)]

$$\xi^{\mu}=\left(h_0^2,\frac{rt}{L^2},0,0\right), \tag{8}$$

is the time-translation generator.

- It obeys to the Killing equation

$$\nabla^{\mu}\xi^{\nu}+\nabla^{\nu}\xi^{\mu}=2\nabla^{[\mu}\xi^{\nu]}=0 \tag{9}$$

- We define the event horizon radius the SAdSB black hole as r_+ and see that from the condition of the zero-norm of the Killing vector (8)

$$\xi^2=f(r,t)=1+\frac{r^2}{L^2h^2}-\frac{2Mh}{r}=0. \tag{10}$$

Schwarzschild-Anti-de Sitter-Beltrami metric

- The event horizon radius of the SAdSB black hole is determined by

$$x_+ = \frac{r_+}{h} = \frac{2}{\sqrt{3}} L \sinh\left(\frac{1}{3}\sinh^{-1}\frac{3\sqrt{3}M}{L}\right), \tag{11}$$

$$r_+ = \frac{\frac{2}{\sqrt{3}} L \sinh\left(\frac{1}{3}\sinh^{-1}\left(\frac{3\sqrt{3}M}{L}\right)\right)}{\sqrt{1 + \frac{4}{3}\frac{r_b^2}{L^2}\sinh^2\left(\frac{1}{3}\sinh^{-1}\left(\frac{3\sqrt{3}M}{L}\right)\right)}}, \tag{12}$$

where $r_b = \sqrt{L^2 + t^2}$ is a bounded radius of the SAdSB metric.

- The relation between the mass and the event horizon radius can be also obtained as

$$M = \frac{x_+(L^2 + x_+^2)}{2L^2} = \frac{r_+ r_b^2 L}{2(r_b^2 - r_+^2)^{3/2}}, \tag{13}$$

Entropy of the Schwarzschild Anti-de Sitter-Beltrami Black Hole

- We propose to determine the entropy of the SAdSB black hole by the Noether charge method. The Noether charge and the Noether current density from the Einstein-Hilbert's action with the negative constant curvature can be represented as [PhysRevD.50.846,PhysRevD.49.6587,PhysRevD.74.044007]

$$J^{\mu} = \frac{1}{8\pi}\nabla_{\nu} Q^{\mu\nu}, \tag{14}$$

where

$$Q^{\mu\nu} = \frac{1}{8\pi}\nabla^{[\mu}\xi^{\nu]} = \frac{1}{16\pi}(\nabla^{\mu}\xi^{\nu} - \nabla^{\nu}\xi^{\mu}) \tag{15}$$

- The non-zero element of the Noether charge density for the SAdSB metric is

$$Q^{01} = -Q^{10} = \frac{1}{16\pi}(\nabla^{0}\xi^{1} - \nabla^{1}\xi^{0}) = \frac{h^2}{8\pi L^2}\left(r - Mh + \frac{ML^2 hh_0^2}{r^2}\right), \tag{16}$$

Entropy of the Schwarzschild Anti-de Sitter-Beltrami Black Hole

- The non-zero component of the two-dimensional integral element is

$$d\sigma_{01} = -d\sigma_{10} = \frac{r^2}{2h^5} \sin\theta d\theta d\phi. \tag{17}$$

- The entropy of the SAdSB black hole can be formulated as the integral

$$S_{\mathrm{BH}} = \frac{2\pi}{\kappa_{\mathrm{H}}} Q = \frac{2\pi}{\kappa_{\mathrm{H}}} \int_{\partial\Sigma} Q^{\mu\nu} d\sigma_{\mu\nu} = \frac{\pi r_+^2 L^2}{(r_b^2 - r_+^2)}, \tag{18}$$

- where κ_{H} is the surface gravity of the SAdSB metric is evaluated as

$$\kappa_H = \sqrt{-\frac{1}{2}\nabla^\nu \xi^\mu \nabla_\nu \xi_\mu} = \frac{1}{2}\frac{df(x)}{dx}\bigg|_{x_+} = \frac{L^2 + 3x_+^2}{2L^2 x_+}. \tag{19}$$

Entropy of the Schwarzschild Anti-de Sitter-Beltrami Black Hole

- The horizon area of the SAdSB black hole is calculated by the surface integration

$$\mathcal{A}_H = \int_0^{\pi} \int_0^{2\pi} \sqrt{g_{\theta\theta} g_{\phi\phi}}\, d\theta d\phi = 4\pi x_+^2 = 4\pi \frac{r_+^2 L^2}{r_b^2 - r_+^2}. \qquad (20)$$

- The entropy of SAdSB can be written in the usual form of the Bekenstein-Hawking area law as

$$S_{\rm BH} = \frac{\mathcal{A}_H}{4}. \qquad (21)$$

- The Smarr formula in term of black hole mechanics of the SAdSB black hole is

$$\frac{M}{2} = \frac{\kappa_{\rm H}}{8\pi} \mathcal{A}_H - \frac{x_+^3}{2L^2} \qquad (22)$$

Thermal Phase Transition of the Schwarzschild-Anti de Sitter-Beltrami Black Hole

- The thermodymic quantities and thermodynamic relation are determined for the SAdSB black hole .
- It is khown that the curvature radius of the AdS spacetime L can be represented as the thermodynamic pressure as [D.Kastor, S.Ray J,Traschen-0904.2765, D.Kuzbinak,R.Mann M.Teo-1608.06147, B.P.Dolan-1209.1272]

$$P = \frac{3}{8\pi L^2} \tag{23}$$

- The mass of the SAdSB black hole (13) can be interpreted as the enthalpy $H(S_{\mathrm{BH}}, P)$

$$M = H(S_{\mathrm{BH}}, P) = \sqrt{\frac{S_{\mathrm{BH}}}{4\pi}}\left(1 + \frac{8 S_{\mathrm{BH}} P}{3}\right) \tag{24}$$

- Using the Maxwell's thermodynamic relation to determine the Hawking temperature and the thermodynamic volume of the SAdSB black hole as

$$T_H = \frac{\partial H}{\partial S}\bigg|_P = \sqrt{\frac{1}{16\pi S_{\mathrm{BH}}}}(1 + 8PS_{\mathrm{BH}}) = \tag{25}$$

$$= \frac{\kappa_{\mathrm{H}}}{2\pi} = \frac{1}{4\pi r_+ L}\left(\frac{r_b^2 + 2r_+^2}{\sqrt{r_b^2 - r_+^2}}\right),$$

$$V = \frac{\partial H}{\partial P}\bigg|_{S_{\mathrm{BH}}} = \frac{4}{3}\frac{S_{\mathrm{BH}}^{3/2}}{\sqrt{\pi}} = \frac{4}{3}\frac{\pi r_+^3 L^3}{(r_b^2 - r_+^2)^{3/2}} \tag{26}$$

Thermal Phase Transition of the Schwarzschild-Anti de Sitter-Beltrami Black Hole

- The first law of black hole mechanics is determined by the variation of mass, event horizon area and pressure as

$$dM = \frac{\kappa_{\mathrm{H}}}{8\pi} d\mathcal{A}_H + VdP. \tag{27}$$

- The first law of black hole thermodynamics in the Anti-de Sitter-Beltrami spacetime is formulated as

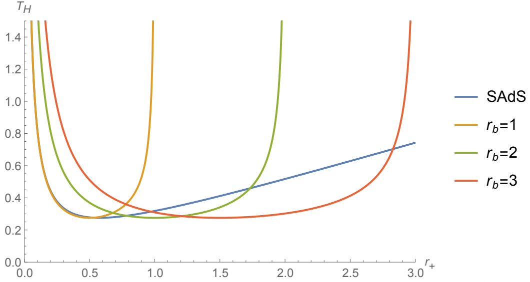
$$dM = dH(S_{\mathrm{BH}}, P) = T_{\mathrm{H}} dS_{\mathrm{BH}} + VdP \tag{28}$$

- The heat capacity of the SAdSB black hole is given from the relation

$$C = \frac{\partial M}{\partial T_H} = \frac{2\pi r_+^2 L^2 (r_b^2 + 2r_+^2)}{(r_b^2 - r_+^2)(4r_+^2 - r_b^2)}. \tag{29}$$

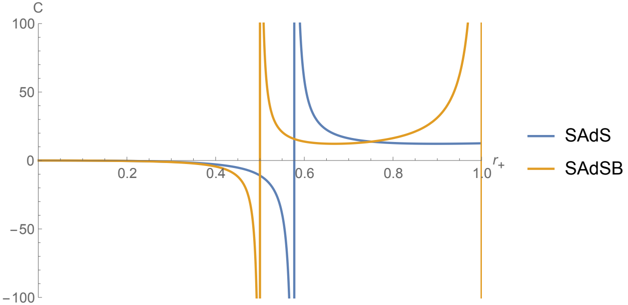
Thermal Phase Transition of the Schwarzschild-Anti de Sitter-Beltrami Black Hole

- Now we describe the thermodynamic phenomena of the SAdSB black hole.
- There is the phase transition between small and large black holes.
- The graph of the Hawking temperature T_H versus the event horizon radius r_+ for a fixed value of AdS radius $L = 1$ is plotted with $r_b = 1$ (orange), $r_b = 2$ (green) and $r_b = 3$ (red), compare with the case of SAdS black holes (blue).



Thermal Phase Transition of the Schwarzschild-Anti de Sitter-Beltrami Black Hole

- The phase transition is taken place under the discontinuity condition of the heat capacity at the horizon radius $r_c = \frac{r_b}{2}$ and minimal temperature $T_{min} = \frac{\sqrt{3}}{2\pi L}$.
- The graph of the heat capacity of the SAdS black holes (blue) and SAdSB black holes with $r_b = 1$ (orange).

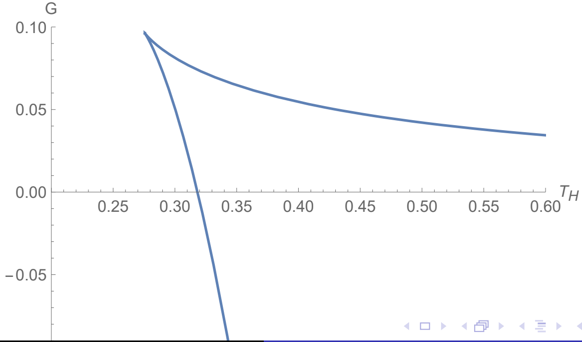


Thermal Phase Transition of the Schwarzschild-Anti de Sitter-Beltrami Black Hole

- The Gibbs free energy of the SAdSB is determined to consider the phase transition

$$G = M - T_H S_{BH} = \frac{r_+ L(r_b^2 - 2r_+^2)}{4(r_b^2 - r_+^2)^{3/2}} \tag{30}$$

- The graph of the Gibbs free energy and the Hawking temperature is drawn as



Thermal Phase Transition of the Schwarzschild-Anti de Sitter-Beltrami Black Hole

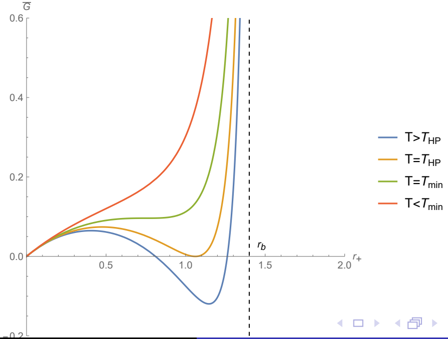
- There is discontinuity of the slope which characterizes the phase transition of the SAdSB Black hole.
- The discontinuous point of the free energy graph is a value of the Gibbs free energy at the critical minimal temperature T_{min}. This graph also shows that there are small and large states for the SAdSB black hole
- The first order phase transition occurs from thermal AdS spacetime to a large black hole in Beltrami coordinatesthe is characterized by the Hawking-Page temperature (T_{HP}) determined at the intercept $G = 0$ of the graph between free energy and temperature of the SAdSB black hole.

$$T_{HP} = \frac{1}{\pi L} \tag{31}$$

- Finally, we consider the canonical ensemble of the SAdSB in the contexts of the off-shell Gibbs free energy expressed as

$$\bar{G} = M - TS = \frac{Lr_+ r_b^2}{2(r_b - r_+)^{3/2}} - \frac{\pi T L^2 r_+^2}{r_b^2 - r_+^2}. \tag{32}$$

- The graph of $\bar{G}$ as the function of r_+ is plotted in the below figure



Thermal Phase Transition of the Schwarzschild-Anti de Sitter-Beltrami Black Hole

- We see that $\bar{G}$ of the SAdSB black hole is bounded within the r_b region in the same way as the case of the asymptotically flat Schwarzschild black hole in a cavity [PhysRevD.102.024006].
- There is no black hole phases at $T < T_{min}$ and only the thermal radiation appears in an AdSB spacetime.
- The black hole configuration appears at the inflection point after increasing the temperature to $T = T_{\mathrm{min}}$.
- At $T_{\mathrm{min}} < T < T_{\mathrm{HP}}$ two black hole configurations appear, namely small and large black hole phases, which correspond to the local maximum and minimum of $\bar{G}$, respectively.

Thermal Phase Transition of the Schwarzschild-Anti de Sitter-Beltrami Black Hole

- The Hawking-Page phase transition characterizing discontinuous changing from the thermal radiation to large black hole, occurs at the temperature $T = T_{\mathrm{HP}}$, the off-shell Gibbs free energy of radiation phase and large black hole phase are equal, namely $\bar{G}_{\mathrm{rad}} = \bar{G}_{\mathrm{LBH}} = 0$.
- The large black hole phase is the most thermodynamically preferred state in the free energy landscape of the system at the temperature range $T > T_{\mathrm{HP}}$.

Conclusion

- The solution of the gravitating mass in the inertial(Beltrami) frame of the Anti-de Sitter spacetime has been realized.
- The entropy of the SAdSB black hole can be determined by the Iyer-Wald's method of Noether charge integral. It satisfies the Bekenstein-Hawking area law.
- The Hawking temperature and Gibbs free energy depend on the bounded radius and their character is analogous to a flat black hole in a cavity. But the boundary of a cavity of the SAdSB metric varies in time.
- The given phase-transition temperature T_{HP} remains as the case of the SAdS black hole, whereas the transition radius r_{HP} of the SAdSB black hole decreases when it is compared with the black hole in non-inertial coordinates.

Thank you for attention.