Phenomenological description of particle production: Riemannian geometry + scalar field

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Motivation

- The importance of investigation of particle production by strong fields and the back reaction of these processes on a space-time metric which includes not only the influence of the created particles but also the contribution due to the vacuum polarization.

- The main obstacle in accounting for the back reaction is that the rigorous solution of the quantum problem requires the knowledge of the boundary conditions, while the latter can be imposed only after solving the field equations. To avoid this difficulty an attempt has been made to describe the particle creation process phenomenologically on the classical level.

- Using the example of the action for an ideal fluid with a variable number of particles, it can be shown that spherically symmetric vacuum solutions of the black hole type in GR (and also for some cases of QG) cannot describe the "pregnant vacuum", i.e. a state when particles can be produced, but they are not still present.

Action for the perfect fluid in Eulerian variables

$$S_m = \int \{ -E(n, X) + \mu_0 (u_a u^a - 1) + \mu_1 \nabla_a (n u^a) + \\ + \mu_2 \partial_a X u^a \} \sqrt{|g|} d^4 x, \quad (1)$$

$\mu_0(x)$, $\mu_1(x)$ and $\mu_2(x)$ are the Lagrange multipliers.

J.R.Ray J.Math.Phys. 13 (1972)

The dynamical variables are: the number density $n(x)$, the four velocity vector of fluid's flow u^a, some auxiliary field $X(x)$ for enumeration of the world-lines and the invariant energy density of the fluid:

$$E(n, X) = n \left(m(X) + \Pi(n)\right),$$

where $\Pi(n)$ is the potential energy describing the (self)interaction between the constituent particles, and $m(X)$ is their mass distribution.

The hydrodynamic pressure is: $p = n^2 \frac{d\Pi}{dn} = -E + n \frac{dE}{dn}$.

Here signature $(+, -, -, -)$ and geometric units in which $c = G = 1$ are used.

Motion equations and constraints

Variation of the matter action with respect to dynamical variables and Lagrange multipliers gives the following system of motion equations and constraints:

$$\delta n : -\frac{\partial E}{\partial n} - \partial_a \mu_1 \, u^a = 0 \,, \tag{2}$$

$$\delta u^a : \; \mu_2 \, \partial_a X + 2\mu_0 \, u_a - \partial_a \mu_1 \, n = 0 \,, \tag{3}$$

$$\delta X : \; -\frac{\partial E}{\partial X} - \nabla_a \left(\mu_2 \, u^a \right) = 0 \,, \tag{4}$$

$$\delta \mu_0 : \; u_a \, u^a = 1 \,, \tag{5}$$

$$\delta \mu_1 : \; \nabla_a \left(n \, u^a \right) = 0, \tag{6}$$

$$\delta \mu_2 : \; \partial_a X \, u^a = 0 \,, \tag{7}$$

$$\delta g_{ab} : \; T^{ab} = (p + E) \, u^a u^b - p \, g^{ab} \,. \tag{8}$$

The phenomenological description of particle creation

In order to move on to a model with a variable number of particles it suffices to replace the term in the original action, which, when varied with respect to μ_1, gives the continuity equation, by the law of particle creation:

$$\mu_1 \nabla_a (n\, u^a) \to \mu_1 \, \{\nabla_a (n\, u^a) - \Phi\}$$

$\Phi(inv)$ is some function of the invariants characterizing the field(s) that causes the particle creation

V.A.Berezin Int.J.Mod.Phys. A 2 (1987)

The continuity equation is replaced by the law of particle creation:

$$\nabla_a (n\, u^a) = 0 \to \nabla_a (n\, u^a) = \Phi \tag{9}$$

Energy-momentum tensor:

$$T^{ab} = (p + E)\, u^a u^b - p\, g^{ab} + \frac{2}{\sqrt{|g|}} \frac{\delta \left(\mu_1 \Phi \sqrt{|g|}\right)}{\delta g_{ab}}$$

Conformal invariance of the rate of particle production

Under a local conformal transformation of the metric: $g_{ab} = e^{2\omega}\,\widetilde{g}_{ab}$, the number density, the four-velocity, and the determinant of the metric change as follows:

$$n = \frac{\widetilde{n}}{e^{3\omega}}, \quad u^a = \frac{\widetilde{u}^a}{e^{\omega}}, \quad \sqrt{-g} = \sqrt{-\widetilde{g}}\, e^{4\omega}\,,$$

therefore:

$$\nabla_a \left(n\, u^a\right)\, \sqrt{-g} = \partial_a \left(n\, u^a\, \sqrt{-g}\right) = \partial_a \left(\widetilde{n}\, \widetilde{u}^a\, \sqrt{-\widetilde{g}}\right) = \\ = \widetilde{\nabla}_a \left(\widetilde{n}\, \widetilde{u}^a\right)\, \sqrt{-\widetilde{g}}$$

Then the relation (9) also implies the conformal invariance of $\Phi\, \sqrt{-g}$.

The particle production law in the absence of external fields

In the Riemannian geometry the combination $C^2\sqrt{-g} = C_{abcd}C^{abcd}\sqrt{-g}$ is conformally invariant, where C_{abcd} is the Weyl tensor defined as:

$$C_{abcd} = R_{abcd} + \frac{1}{2}\left(R_{ad}\,g_{bc} + R_{bc}\,g_{ad} - R_{ac}\,g_{bd} - R_{bd}\,g_{ac}\right) + \\ + \frac{1}{6}R\left(g_{ac}\,g_{bd} - g_{ad}\,g_{bc}\right) . \quad (10)$$

Without external classical fields the particle are created solely by the vacuum fluctuations due to the gravitational field thus Φ should depend on the geometric invariants. Restricting ourselves to combinations of geometric invariants that are at most quadratic in curvature, we get: $\Phi = \beta C^2$.

The same result was obtained in: *Ya. B. Zel'dovich A.A., Starobinskii JETP. Lett. 26 (1977)* for the particle creation by the vacuum fluctuations of the massless scalar field on the background metric of the homogeneous and slightly anisotropic cosmological spacetime. Now it becomes fundamental for any Riemannian geometry, irrespective of the form of the gravitational Lagrangian and also the back reaction is taken into account.

Pregnant vacuum

Of particular physical interest is the situation when the possibility of particle production exists but is not realized. This is the so-called "pregnant vacuum" (*arXiv:2203.04257 [gr-qc]*), which is an example of a physical vacuum. It corresponds to the solution of motion equations for which $n = \Phi = 0$ although the term responsible for the particle creation is still present in the action. In this case u^a is an arbitrary non-zero vector normalized by the condition (5) (the exception is cosmology), so for the equation (2):

$$\partial_a \mu_1 \, u^a = -\frac{p+E}{n}, \tag{11}$$

two options are possible:

1. matter that can potentially be born has non-zero pressure $p > 0$ and $\lim\limits_{n \to 0} \dfrac{E+p}{n} = 0$, therefore $\partial_a \mu_1 \, u^a = 0$. The arbitrariness of u^a implies $\mu_1 = const$.
2. dust is born, i.e. $p = 0$, $E = m_0 \, n$, then $\partial_a \mu_1 \, u^a = -m_0$. Similarly we obtain $m_0 = 0$ and $\mu_1 = const$.

It means that "the pregnant vacuum" cannot produce dust.

Let's choose quadratic gravity as gravity action:

$$S_q = -\frac{1}{16\pi}\int \sqrt{-g}\left(\alpha_1 R_{abcd}R^{abcd} + \alpha_2 R_{ab}R^{ab} + \alpha_3 R^2 + \alpha_4 R + \alpha_5 \Lambda\right)\, d^4x$$

Field equations:

$$2\left(4\alpha_1 + \alpha_2\right) B_{ab} + \frac{1}{3}\left(3\alpha_3 + \alpha_1 + \alpha_2\right) D_{ab} + \\ + \alpha_4 G_{ab} - \frac{\alpha_5}{2} g_{ab}\Lambda = 8\pi T_{ab}, \quad (12)$$

where
$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R, \quad D_{ab} = \left(2R_{ab} - \frac{1}{2}g_{ab}R + 2\,g_{ab}\Box - 2\,\nabla_b\nabla_a\right) R,$
$B_{ab} = \left(\nabla^c\nabla^d + \frac{1}{2}R^{cd}\right) C_{acbd}$ - Bach tensor.

Energy-momentum tensor for $\Phi = \beta\, C^2$:

$$T^{ab} = (p + E)\, u^a u^b - p\, g^{ab} - 8\beta\left(\nabla_c\nabla_d + \frac{1}{2}R_{cd}\right)\left(\mu_1\, C^{acbd}\right).$$

For the "pregnant vacuum" $\mu_1 = const,\ n = 0$ therefore:
$T^{ab} = -8\beta\,\mu_1\, B^{ab}$.

Spherical symmetry

Let's consider spherically symmetric geometries:

$$ds^2 = r^2(x)\left(\widetilde{\gamma}_{\alpha\beta}\, dx^\alpha dx^\beta - d\Omega^2\right), \quad \alpha, \beta = 0, 1. \qquad (13)$$

Invariants:
$\Delta = g^{ab}\partial_a r \partial_b r$, $\widetilde{R}$ - scalar curvature of $\widetilde{\gamma}_{\alpha\beta}$, $\sigma = \Box\, r - \frac{2}{r}\,\Delta$,
connection with four-dimensional scalar curvature:
$R = \frac{1}{r^2}\left(\widetilde{R} - 2 - 6\, r\, \sigma\right)$.

For the spherically symmetric metric: $C^2 = \frac{(\widetilde{R}-2)^2}{3r^4}$, therefore "the pregnant vacuum" corresponds to $\widetilde{R} = 2$.

The Bach tensor of the spherically symmetric metric with $\widetilde{R} = 2$ is zero (*V. A. Berezin et al J. Cosmol. Astropart. Phys. JCAP01(2016)019*), so the condition $T^{ab} = 0$ is automatically satisfied.

Physical vacuum in quadratic gravity

In GR($\alpha_1 = \alpha_2 = \alpha_3 = 0$) the general case of spherically symmetric vacuum is the Schwarzschild-de Sitter(anti-de Sitter) metric:

$$ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 - r^2 d\Omega^2, \quad f(r) = 1 - \frac{2M}{r} - \frac{r^2}{6}\frac{\alpha_5}{\alpha_4}\Lambda,$$

for which $\widetilde{R} = 2 - \frac{12M}{r}$.

Therefore for GR with the considered matter action only pure de Sitter(anti-de Sitter) spacetime corresponds to the "pregnant vacuum".

The same is true for the GR+C^2($\alpha_2 = -2\alpha_1$, $\alpha_3 = \frac{1}{3}\alpha_1$) since the Bach tensor is zero for $\widetilde{R} = 2$.

As for the general case of quadratic gravity the QG version of trace no-hair theorem implies that for static spacetimes with R sufficiently quickly approaching a constant, $R = -2\frac{\alpha_5}{\alpha_4}\Lambda$ throughout the spacetime (*Pravda V. et al Phys. Rev. D 103 (2021); Lu H. et al Phys. Rev. D 92(2015); Nelson W. Phys. Rev. D 82 (2010)*). For a spherically symmetric geometry with $\widetilde{R} = 2$ this condition again leads to the fact that the de Sitter metric is the only possible solution of (12), taking into account all the restrictions.

Let's consider function Φ of a more complex form, when some external scalar field φ also contributes to the process of particle creation. Since the combination $\sqrt{-g}\,\Phi$ is conformally invariant, one of the easiest ways to introduce a scalar field is:
$\Phi = \beta\, C^2 + \alpha\left(\varphi\,\Box\varphi - \frac{1}{6}\,\varphi^2\,R + \Lambda_0\,\varphi^4\right).$

Variation of the matter action with respect to φ gives an additional equation of motion:

$$\delta\varphi: \quad \mu_1\Box\varphi + \Box\left(\mu_1\varphi\right) + 4\mu_1\,\Lambda_0\varphi^3 - \frac{1}{3}\mu_1\,\varphi\,R = 0\,. \tag{14}$$

The energy-momentum tensor for this model is:

$$T^{ab} = (p+E)\,u^a u^b - p\,g^{ab} - 8\beta\left(\nabla_c\,\nabla_d + \frac{1}{2}\,R_{cd}\right)\left(\mu_1\,C^{acbd}\right) + \\ + \alpha\,\mu_1 g^{ab}\Lambda_0\,\varphi^4 - \alpha\,g^{ab}\,\partial_c\left(\mu_1\varphi\right)\partial^c\varphi + \alpha\,\partial^a\left(\mu_1\varphi\right)\partial^b\varphi + \\ + \alpha\,\partial^b\left(\mu_1\varphi\right)\partial^a\varphi + \frac{\alpha}{3}\left\{\mu_1\varphi^2\,G^{ab} - \nabla^a\nabla^b\left(\mu_1\varphi^2\right) + g^{ab}\Box\left(\mu_1\varphi^2\right)\right\} \tag{15}$$

Pregnant vacuum for the model with scalar field

Let's consider the "pregnant vacuum" for the model with the scalar field. Unlike previous case, the energy-momentum tensor (15) is not necessarily zero:

$$\frac{1}{\mu_1\alpha}\,T^{ab} = -\frac{8\beta}{\alpha}\,B^{ab} + g^{ab}\left(\Lambda_0\,\varphi^4 - \partial_c\varphi\,\partial^c\varphi\right) + \\ + 2\partial^a\varphi\,\partial^b\varphi + \frac{1}{3}\left\{\varphi^2\,G^{ab} - \nabla^a\nabla^b\left(\varphi^2\right) + g^{ab}\,\Box\left(\varphi^2\right)\right\}, \quad (16)$$

The condition $\Phi = 0$ together with the motion equation (14) form the following system:

$$\Phi = \beta\,C^2 + \alpha\left(\varphi\,\Box\varphi - \frac{1}{6}\,\varphi^2\,R + \Lambda_0\,\varphi^4\right) = 0\,, \quad (17)$$

$$2\Box\varphi + 4\Lambda_0\,\varphi^3 - \frac{1}{3}\,\varphi\,R = 0\,, \quad (18)$$

expressing $\Box\,\varphi$ from the first equation and substituting it into the second, we get: $\varphi^4 = \frac{\beta}{\alpha\Lambda_0}\,C^2$.

Pregnant vacuum with scalar field. Spherical symmetry

For spherically symmetric geometries this formula has the form:

$$\varphi^4 = \frac{\beta}{\alpha\Lambda_0}\,\frac{(\widetilde{R}-2)^2}{3r^4}\,. \tag{19}$$

If we consider the Schwarzschild-de Sitter(anti-de Sitter) spacetime then from (19) it follows that $\varphi \propto r^{-\frac{3}{2}}$ in this case:

$$\Box\varphi = -\frac{1}{r^2}\,\partial_r\left(r^2 f(r)\,\partial_r\varphi\right) \propto \left(-\frac{1}{2}r^{-\frac{7}{2}} + \frac{3M}{r} - \frac{\Lambda\,\alpha_5}{4\,\alpha_4}\,r^{-\frac{3}{2}}\right)\,,$$

but this result contradicts the equation of motion (18).

Thus, in GR the physical vacuum for the considered model with a scalar field in the spherically symmetric case cannot be the Schwarzschild de Sitter(anti-de Sitter) spacetime. An exception is pure de Sitter(anti-de Sitter), but then $\varphi = 0$.

Scalar field. Induced gravity

Let the energy density also depend on the scalar field: $E(n, X, \varphi)$. Therefore, the equation of motion for φ changes:

$$\mu_1 \Box\varphi + \Box(\mu_1\varphi) + 4\mu_1\,\Lambda_0\varphi^3 - \frac{1}{3}\mu_1\,\varphi\,R = -\frac{1}{\alpha}\frac{\partial E}{\partial\varphi}\,.$$

Trace of the energy-momentum tensor for this model is:

$$T = E - 3p + 4\alpha\,\mu_1\,\Lambda_0\,\varphi^4 - \frac{\alpha}{3}\,\mu_1\,\varphi^2\,R + \alpha\,\varphi\,\Box(\mu_1\varphi) + \\ + \alpha\,\mu_1\varphi\,\Box\varphi = E - 3p - \varphi\,\frac{\partial E}{\partial\varphi}\,. \quad (20)$$

Let's consider a situation where gravity action is conformally invariant. In this case $T = 0$: $E - 3p = \varphi\,\frac{\partial E}{\partial\varphi}$.

1. For dust $p = 0$, so $E \propto n\,\varphi$.
2. For radiation $E = 3p$, therefore, two options are possible: either there is no scalar field $\varphi = 0$, or $\frac{\partial E}{\partial\varphi} = 0$, that is, the energy density does not depend on φ or φ as a solution of the equations of motion corresponds to the extrema of the E function.

Conclusions

Let's summarize the results obtained:

- The conformal invariance of the term in the action responsible for the law of particle production leads to restrictions on form of the function Φ which depends on the invariants of external fields responsible for the creation processes.

- In the absence of external fields, when the only cause of creation is gravity, the square of the Weyl tensor is the most basic version for Riemannian geometry.

- When an external scalar field is introduced into the creation law, the following combination is chosen: $\varphi \Box \varphi - \frac{1}{6} \varphi^2 R + \Lambda_0 \varphi^4$ since it gives a nontrivial equation of motion and is conformally invariant when multiplied by $\sqrt{|g|}$.

- It is demonstrated that "the pregnant vacuum" cannot produce dust but it can give birth to matter with non-zero pressure, for example, thermal radiation.

Conclusions

- In the absence of external fields in GR there are no spherically symmetric vacuum solutions of the black hole type corresponding to "the pregnant vacuum". The same is true for quadratic gravity if we restrict ourselves to static spacetimes with R sufficiently quickly approaching a constant.

- In GR "the pregnant vacuum" for the model with the external scalar field in the spherically symmetric case cannot be described by vacuum solutions of the black hole type.

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