Why does expectation value of stress energy tensor blow up near the event horizons?

MQFT 2022

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13 October, 2022

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Introduction

What we want to calculate and why?

Black hole solution of the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_s}{r}} - r^2 \left(\sin^2\theta d\varphi^2 + \theta^2\right), \quad r_s = 2MG$$

Horizon at $r = r_s$

What we want to calculate and why?

But, in fact, quantum average of the stress-energy tensor of a matter field should be taken into account:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\langle\hat{T}_{\mu\nu}\rangle$$

From the one hand G is small constant, and we expect perturbated solution has the form of:

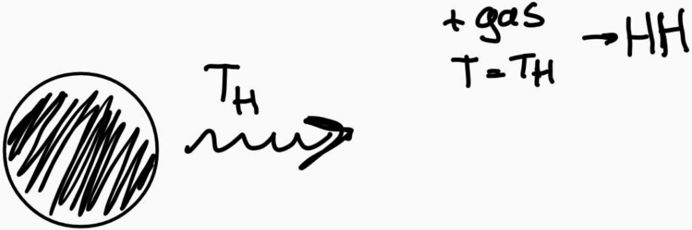
$$g_{\mu\nu} = g^0_{\mu\nu} + Gh_{\mu\nu}$$

From the other hand:

$$g_{00}(r = r_s) = 0$$

So, metric is sensitive to the perturbations $\rightarrow$ Hawking radiation ?

What we want to calculate and why?



What we want to calculate and why?

Calculating of the quantum average of the stress-energy tensor in the curved backround is a hard technical problem.
The key goal of the talk is to discuss properties of the quantum average of the stress-energy tensor in the spaces with horizons:

$$\langle \hat{T}_{\mu\nu} \rangle = ?$$

What we want to calculate and why?

The key goal of the talk is to discuss properties of the quantum average of the stress-energy tensor in the spaces with horizons:

$$\langle \hat{T}_{\mu\nu} \rangle = ?$$

We fix the background metric
and find a method of calculation such averages.

Setup

Difficulties

$$\langle \hat{T}_{\mu\nu} \rangle = ?$$

- Regularization?
- State (density matrix?
- Coordinate dependence?

Framework

Action of the matter field:

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left(\partial_\mu \varphi \partial_\mu \varphi - m^2 \varphi^2 \right),$$

SET expectation value from the Wightman function as:

$$T_{\mu\nu}(x)_\beta = \left(\frac{\partial}{\partial x_1^\mu} \frac{\partial}{\partial x_2^\nu} - \frac{1}{2} g_{\mu\nu} \left[g^{\alpha\beta} \frac{\partial}{\partial x_1^\alpha} \frac{\partial}{\partial x_2^\beta} - m^2 \right] \right) W_\beta(x_1|x_2) \Big|_{x_1 = x_2 = x}$$

which is defined as follows:

$$W_\beta(x_1|x_2) = \langle \hat{\varphi}(x_1) \hat{\varphi}(x_2) \rangle_\beta, \qquad \langle \hat{O} \rangle_\beta \equiv \frac{\mathrm{Tr} e^{-\beta \hat{H}} \hat{O}}{\mathrm{Tr} e^{-\beta \hat{H}}}$$

2D

$$ds^2 = \left(1 - \frac{r_g}{r}\right)\, dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} = ds^2 = \left[1 - \frac{r_g}{r(r^*)}\right]\left(dt^2 - dr^{*2}\right) \tag{1}$$

No Einstein equation, but simple calculations! The tortoise coordinate

$$r^* = r + r_g \log\left(\frac{r}{r_g} - 1\right)$$

near the horizon ($r^* \to -\infty$) the metric looks like the Rindler's one:

$$ds^2_{\mathrm{nh}} \approx e^{\frac{r^*}{r_g}}\left(dt^2 - dr^{*2}\right)$$

$$\partial_t^2\varphi - \partial_{r^*}^2\varphi + m^2 g_{00}\,\varphi = 0 \tag{2}$$

$$-\partial_{r^*}^2\varphi_\omega(r^*) + m^2 g_{00}(r^*)\,\varphi_\omega(r^*) = \omega^2\varphi_\omega(r^*)$$

Near the horizon:

$$\begin{aligned}\hat{\varphi}(t, r^*) = &\int_0^m \frac{d\omega}{\sqrt{2\omega}}\, e^{-i\omega t}\varphi_\omega(r^*)\hat{c}_\omega + \\ &+ \int_m^{+\infty} \frac{d\omega}{\sqrt{2\omega}} e^{-i\omega t}\left[R_\omega(r^*)\hat{a}_\omega + L_\omega(r^*)\hat{b}_\omega\right] + h.c.\end{aligned}$$

In the lightcone coordinates $V = t + r^*$, $U = t - r^*$ the metric (**??**) takes the form

$$ds^2 = C(U,V)dUdV, \quad C(U,V) = \frac{\mathcal{W}(e^{\frac{V-U}{4M}-1})}{1+\mathcal{W}(e^{\frac{V-U}{4M}-1})},$$

where $\mathcal{W}(r^*)$ is the Lambert function. Near the horizon

$$W((V^+,U^+),(V^-,U^-)) \approx \int \frac{d\omega}{4\pi\omega}\frac{1}{e^{\beta\omega}-1}\Big(e^{i\omega(V^+-U^-)+2i\delta_\omega}+e^{i\omega(V^+-V^-)}+ \\ + e^{i\omega(U^+-U^-)} + e^{i\omega(U^+-V^-)-2i\delta_\omega}\Big)$$

$$T_{\mu\nu} \approx \Theta_{\mu\nu} + \frac{R}{48\pi}g_{\mu\nu},$$

where

$$\Theta_{UU} = -\frac{1}{12\pi}C^{1/2}\partial_U^2 C^{-1/2} + \frac{\pi}{12\beta^2} = -\frac{\pi}{12\beta_H^2} + \frac{\pi}{12\beta^2}$$

$$\Theta_{VV} = -\frac{1}{12\pi}C^{1/2}\partial_V^2 C^{-1/2} + \frac{\pi}{12\beta^2} = -\frac{\pi}{12\beta_H^2} + \frac{\pi}{12\beta^2}$$

$$\Theta_{UV} = \Theta_{VU} = 0$$

4D

$$\Big[- \partial_{r*}^2 + V_l(r) \Big] L(r^*) = \omega^2 L(r^*),$$

$$V_l(r) = f(r) \Big(m^2 + \frac{1}{r^2} l(l+1) + \frac{f'_r(r)}{r} \Big).$$

$$W_\beta(x, x') \equiv \langle \hat{\varphi}(t, r, \theta, \phi) \varphi(t', r', \theta', \phi') \rangle_\beta =$$

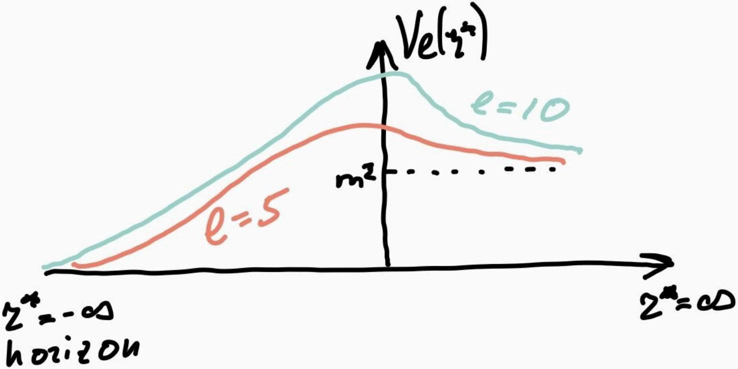
$$= \int_{-\infty}^{+\infty} d\omega \frac{1}{e^{\beta\omega} - 1} \frac{e^{-i\omega(t'-t)}}{rr' 4\pi\omega} \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l \big(\vec{x} \cdot \vec{x}' \big) \times$$

$$\times \Big[R^*_{\omega,l}(r') R_{\omega,l}(r) + L^*_{\omega,l}(r') L_{\omega,l}(r) \Big].$$

Equation of motion:

$$\Big[-\partial_{r^*}^2 + V_l(r)\Big]L(r^*) = \omega^2 L(r^*)$$

$$V_l(r) = f(r)\Big(m^2 + \frac{1}{r^2}l(l+1) + \frac{f_r'(r)}{r}\Big).$$



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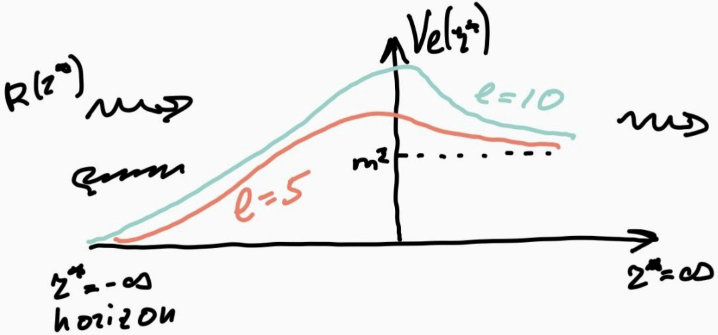

$$\Big[-\partial^2_{r*} + V_l(r)\Big]L(r^*) = \omega^2 L(r^*)$$

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$$W_\beta(x,x') \equiv \langle\hat{\varphi}(t,r,\theta,\phi)\varphi(t',r',\theta',\phi')\rangle_\beta =$$

$$= \int_{-\infty}^{+\infty} d\omega \frac{1}{e^{\beta\omega}-1}\frac{e^{-i\omega(t'-t)}}{rr'4\pi\omega}\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}P_l\big(\vec{x}\cdot\vec{x}'\big)\times$$

$$\times\Big[R^*_{\omega,l}(r')R_{\omega,l}(r) + L^*_{\omega,l}(r')L_{\omega,l}(r)\Big].$$

$$W_\beta(x,x') = \left\langle \hat{\varphi}(t,r,\theta,\phi)\varphi(t',r',\theta',\phi') \right\rangle_\beta =$$

$$= \int_{-\infty}^{+\infty} d\omega \frac{1}{e^{\beta\omega}-1} \frac{e^{-i\omega(t'-t)}}{rr'4\pi\omega} \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l\left(\vec{x}\cdot\vec{x}'\right) \left[R^*_{\omega,l}(r')R_{\omega,l}(r)\right]$$

$$\sum_{l=0}^{\infty}(2l+1)\left[\left|R_{\omega,l}(r)\right|^2\right] \approx \omega^2\left(1-\frac{1}{2r}\right)^{-1},$$

$$\sum_{l=0}^{\infty}(2l+1)l(l+1)\left[\left|R_{\omega,l}(r)\right|^2\right] \approx \frac{1}{6}\left(\omega^2+\omega^4\right)\left(1-\frac{1}{2r}\right)^{-2},$$

$$\sum_{l=0}^{\infty}(2l+1)\left[\left|\partial_{r*}R_{\omega,l}(r)\right|^2\right] \approx \frac{1}{3}\left(4\omega^2+\omega^4\right)\left(1-\frac{1}{2r}\right)^{-1}.$$

We define regularization as follows:

$$\langle : \hat{T}_{\mu\nu} : \rangle_\beta = \underbrace{\langle : \hat{T}_{\mu\nu} : \rangle_\beta - \langle : \hat{T}_{\mu\nu} : \rangle_{\beta_H}}_{\text{finite term(no regularization)}} + \langle : \hat{T}_{\mu\nu} : \rangle_{\beta_H} \approx \langle \hat{T}_{\mu\nu} \rangle_\beta - \langle \hat{T}_{\mu\nu} \rangle_{\beta_H}$$

Where β_H is inverse Hawking radiation. From point splitting or effective actions:

$$\langle : \hat{T}_{\mu\nu} : \rangle_{\beta_H} \sim g_{\mu\nu}$$

We consider

$$M = 1/2 \qquad \rightarrow \qquad \beta_H = 2\pi$$

$$\langle : \hat{T}^{\mu}_{\nu} : \rangle_{\beta=\frac{1}{T}} \approx \frac{1}{480\pi^2}\left(1-\frac{1}{2r}\right)^{-2}\left((2\pi T)^4-1\right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & -\frac{1}{3} \end{pmatrix} +$$

$$+ \frac{1}{48\pi^2}\left(1-\frac{1}{2r}\right)^{-2}\left((2\pi T)^2-1\right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{2}{3} \end{pmatrix}.$$

Divergent at the horizon! And mass independent!

$$\langle : \hat{T}^{\mu}_{\mu} : \rangle_{\beta=\frac{1}{T}} \approx \frac{1}{24\pi^2}\left(1-\frac{1}{2r}\right)^{-2}\left((2\pi T)^2 - 1\right)$$

Trace is mass independent

Other spaces:

$$ds^2 = e^{2\xi\alpha}\left(d\eta^2 - d\xi^2\right) - d\vec{z}_\perp^2,$$

$$ds^2 = f(r)\,dt^2 - \frac{dr^2}{f(r)} - r^2\big(\sin^2\theta d\varphi^2 + \theta^2\big), \quad f(r) = 1 - H^2\,r^2$$

The similar results!

Peculiarities of Wightman functions near the horizons

$$\sum_{l=0}^{+\infty}(2l+1)P_l\big(\cos\theta\big)\Big[|R_{\omega,l}(r^*)|^2+|L_{\omega,l}(r^*)|^2\Big]\approx$$

$$\approx\frac{8\omega}{\theta^2}\sin\left(2\omega\log\frac{e^{r^*}}{\theta}\right),\quad\text{if}\ \ e^{r^*}\ll\theta\ll1$$

Then the approximate form is as follows:

$$W_\beta\approx\int_{-\infty}^{+\infty}d\omega\frac{1}{e^{\beta\omega}-1}\frac{e^{-i\omega(t'-t)}}{4\pi^2\omega}\frac{8\omega}{\theta^2}\sin\left(2\omega\log\frac{e^{r^*}}{\theta}\right)\approx\frac{2}{\pi}\frac{1}{\beta}\frac{1}{\theta^2}$$

Note that in the limit in question, the geodesic distance:

$$L\approx r\sqrt{2(1-\cos\theta)}\approx\frac{\theta}{2}$$

$r=\frac{1}{2}$. Finally, for the Wightman function, we obtain:

$$W_\beta\approx\frac{2\pi}{\beta}\times\frac{1}{4\pi^2}\frac{1}{L^2}$$

$$\hat{H} = \int \sqrt{g} dx \hat{T}_0^0$$

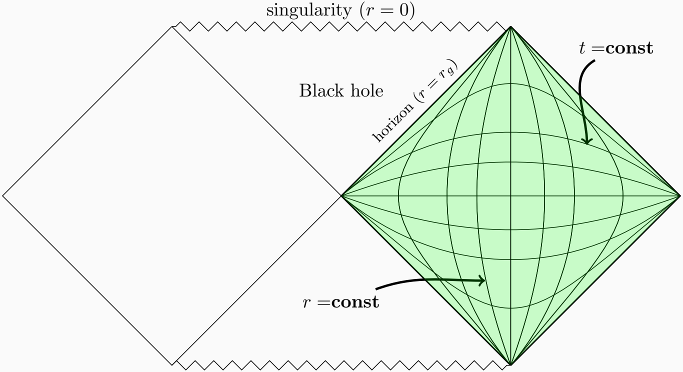
To obtain Hamiltonian we use the field operator:

$$\hat{\varphi}(t, r, \phi, \theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^{\infty} d\omega \frac{1}{r} \frac{1}{\sqrt{4\pi\omega}} \times \\ \times \left[Y_l^m(\phi, \theta) e^{i\omega t} \left(R_{\omega,l} \hat{a}^\dagger_{\omega,l} + L_{\omega,l} \hat{b}^\dagger_{\omega,l} \right) + h.c. \right]$$

Here *h.c.* is a Hermitian conjugated term. Straightforward calculation gives the following Hamiltonian in the Schwarzschild space time:

$$:\hat{H} := \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^{\infty} d\omega \omega \left(\hat{b}^\dagger_{\omega,l} \hat{b}_{\omega,l} + \hat{a}^\dagger_{\omega,l} \hat{a}_{\omega,l} \right)$$

$$g_{00}(r = r_s) = 0.$$



time-like trajectories became light-like on the horizon!

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Cite 4D:
Notes on peculiarities of quantum fields in space–times with horizons, K.V. Bazarov, e-Print: 2112.02188 [hep-th], Class.Quant.Grav. 39 (2022) 21, 217001

Cite 2D:
Heating up an environment around black holes and inside de Sitter space, E.T. Akhmedov , P.A. Anempodistov, K.V. Bazarov , D.V. Diakonov, U. Moschella e-Print: 2010.10877 [hep-th], Phys.Rev.D 103 (2021) 2, 025023

Thank you for your attention!