

Modified geodesic deviation equation in the Kerr metric

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Aims of this research

- Investigation of the modified geodesic deviation equation in the Kerr metric;
- Obtaining the dependence of tidal accelerations on distance in the Kerr metric for the modified deviation equation;
- Analysis of the nature of tidal acceleration for two particles falling radially.

Deviation of geodetic lines

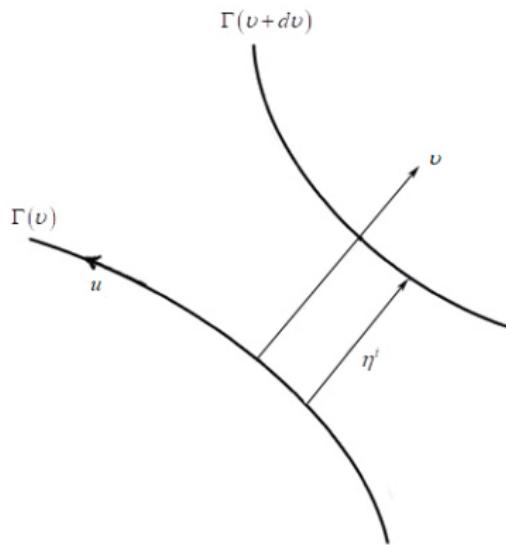


Fig.1: The deviation of curve $\Gamma(\nu)$ from curve $\Gamma(\nu + d\nu)$

Deviation of geodetic lines¹

- 1 $\Gamma(\nu)$ и $\Gamma(\nu + d\nu)$: $\frac{Du_1^i}{d\tau_1} = 0$, $\frac{Du_2^i}{d\tau_2} = 0$

where u^i is the tangent vector to the geodesic, τ_1, τ_2 are affine parameters.

- 2 η^i – infinitesimal vector deviation.

If $d\tau_1$ is an infinitesimal arc on geodesic 1 and $d\tau_2$ the arc on geodesic 2 corresponding to the connecting vectors $\eta^i(\tau)$ и $\eta^i(\tau + d\tau)$, we have:

$$\frac{d\tau_2}{d\tau_1} = 1 + \lambda, \text{ where } \frac{d\lambda}{d\tau_1} = 0$$

- 3 The geodesics are infinitesimally close in a neighborhood U , where the relative change in the curvature is small
- 4 The difference between the tangent vectors to the two geodesics is infinitesimally small in the neighborhood U :

$$\frac{du^i}{u^i} \ll 1 \tag{1}$$

where $du^i = u_2^i(\tau_2) - u_1^i(\tau_1)$.

- 5 When deriving the geodesic deviation equation, only first-order terms are taken into account.

¹Synge J.L., 1934, 1960

Geodesic deviation equation

$$\frac{D^2\eta^\alpha}{d\tau^2} = R_{\beta\mu\nu}^\alpha u^\beta \eta^\mu u^\nu \quad (2)$$

where $R_{\beta\mu\nu}^\alpha = g^{\alpha\rho} R_{\rho\beta\mu\nu}$ is the Riemann curvature tensor, u^ν is 4-velocity vectors. Let u_1^α , u_2^α is not an infinitesimal quantity ²:

$$\begin{aligned} \frac{D^2\eta^\alpha}{ds^2} &= R_{\beta\mu\nu}^\alpha u_1^\beta \eta^\mu u_1^\nu - \Gamma_{\mu\nu,\rho}^\alpha \eta^\rho \Delta u^\mu \Delta u^\nu - \\ &2\Gamma_{\mu\nu,\rho}^\alpha \eta^\rho u_1^\mu \Delta u^\nu - \Gamma_{\mu\nu}^\alpha \Delta u^\mu \Delta u^\nu \end{aligned} \quad (3)$$

where $\Gamma_{\mu\nu}^\alpha$ is Christoffel symbols, $\Delta u^\alpha \equiv u_2^\alpha - u_1^\alpha$.

² Ciufolini I., 1986; Hodgkinson D.E., 1987

Kerr metric

$$ds^2 = \left(1 - \frac{2Mr}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{2Mr^2 a^2 \sin^2\theta}{\rho^2}\right) \sin^2\theta dt^2 + \frac{2Mr^2 a}{\rho^2} \sin^2\theta dt d\phi \quad (4)$$

where M is mass black hole, a is angular momentum, $\rho^2 = r^2 + a^2 \cos^2\theta$,
 $\Delta = r^2 - 2Mr + a^2$, $r_{\pm} = M \pm \sqrt{M^2 - a^2}$.

Christoffel symbols

$$\begin{aligned}
\Gamma_{01}^0 &= \frac{M(r^2 + a^2)(r^2 - a^2 \cos^2 \theta)}{\Delta \rho^2}, \quad \Gamma_{02}^0 = -\frac{M r a^2 \sin 2\theta}{\rho^2}, \\
\Gamma_{13}^0 &= \frac{am ((a^4 - a^a r^2) \cos^2 \theta - a^2 r^2 - 3r^4) \sin^2 \theta}{\Delta \rho^2}, \quad \Gamma_{23}^0 = \Gamma_{32}^0 = \frac{M r a^3 \sin 2\theta \sin^2 \theta}{\rho^2}, \\
\Gamma_{00}^1 &= \frac{\Delta M (r^2 - a^2 \cos^2 \theta)}{\rho^3}, \quad \Gamma_{03}^1 = -\frac{\Delta M a \sin^2 \theta (r^2 - a^2 \cos^2 \theta)}{\rho^3}, \\
\Gamma_{11}^1 &= \frac{a^2 (m - r) \cos^2 \theta + r a^2 - m r^2}{\Delta \rho^2}, \quad \Gamma_{12}^1 = \Gamma_{30}^1 = -\frac{a^2 \sin 2\theta}{2\rho}, \quad \Gamma_{21}^1 = -\frac{\Delta r}{\rho}, \quad (5) \\
\Gamma_{33}^1 &= -\frac{a^4 (m - r) \cos^4 \theta - a^2 (m \rho + 2r^3 \cos^2 \theta + a^2 m r^2 - r^5) \sin^2 \theta \Delta}{\rho^3}, \\
\Gamma_{00}^2 &= -\frac{M r a^2 \sin 2\theta}{\rho^3}, \quad \Gamma_{03}^2 = \frac{M r a \sin 2\theta (r^2 + a^2)}{\rho^3}, \quad \Gamma_{11}^2 = \frac{a^2 \sin 2\theta}{2\Delta \rho}, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{r}{\rho}, \\
\Gamma_{22}^2 &= -\frac{a^2 \sin 2\theta}{2\rho}, \quad \Gamma_{30}^2 = \frac{M r a \sin 2\theta}{\rho^3}, \\
\Gamma_{33}^2 &= -\frac{a^4 \Delta \cos^4 \theta + 2a^2 r^2 \Delta \cos^2 \theta + 2a^4 M r + (4M r^3 + r^4) a^2 + r^6}{2\rho^3} \sin 2\theta ...
\end{aligned}$$

The nonzero components of the curvature tensor in the Kerr metric

$$R_{1023} = -\frac{1}{\rho^6} \left(aM\cos\theta \left(3r^2 - a^2\cos^2\theta \right) \right),$$

$$R_{1230} = -\frac{aM\cos\theta}{\rho^6} \left(3r^2 - a^2\cos^2\theta \right) \Sigma^{-2} \left[(r^2 + a^2)^2 + 2a^2\Delta\sin^2\theta \right],$$

$$R_{1302}^0 = \frac{aM\cos\theta}{\rho^6} \left(3r^2 - a^2\cos^2\theta \right) \Sigma^{-2} \left[2(r^2 + a^2)^2 + a^2\Delta\sin^2\theta \right],$$

$$-R_{3002} = R_{1213} = -\frac{aM\cos\theta}{\rho^6} \left(3r^2 - a^2\cos^2\theta \right) \left(\frac{3a\Delta^{0.5}}{\Sigma^2} \right) (r^2 + a^2) \sin\theta,$$

$$-R_{1220} = R_{1330} = -\frac{rM}{\rho^6} \left(3r^2 - a^2\cos^2\theta \right) \left(\frac{3a\Delta^{0.5}}{\Sigma^2} \right) (r^2 + a^2) \sin\theta,$$

$$-R_{1010} = R_{2323} = -\frac{rM}{\rho^6} \left(3r^2 - a^2\cos^2\theta \right) = R_{0202} + R_{0303}$$

$$-R_{1220} = R_{1330} = -\frac{rM}{\rho^6} \left(3r^2 - a^2\cos^2\theta \right) \Sigma^{-2} \left[(r^2 + a^2)^2 + 2a^2\Delta\sin^2\theta \right]$$

$$-R_{1212} = R_{0303} = -\frac{rM}{\rho^6} \left(3r^2 - a^2\cos^2\theta \right) \Sigma^{-2} \left[(r^2 + a^2)^2 + 2a^2\Delta\sin^2\theta \right]$$

where $\Sigma^2 = (r^2 + a^2)^2 - a^2\Delta$

Geodesic motion in Kerr spacetime

$$u_1^i = (u_1^0, u_1^1, 0, 0), \quad u_2^i = (u_2^0, u_2^1, 0, 0) \quad (6)$$

The normalization condition:

$$g_{ik} u_1^i u_1^k = 1, \quad g_{ik} u_2^i u_2^k = 1, \quad (7)$$

The geodesic equation for Kerr spacetime $\theta = \frac{\pi}{2}$:

$$\frac{dt}{ds} = \frac{1}{\Delta} \left[\left(r^2 + a^2 - \frac{2a^2 M}{r} \right) E + \frac{2aML}{r} \right],$$

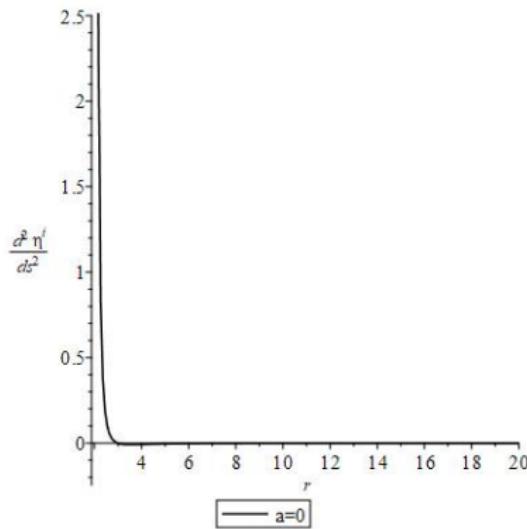
$$r^2 \left(\frac{dr}{ds} \right)^2 = -\Delta + r^2 E^2 + \frac{2M}{r} (L - aE)^2 - (L^2 - a^2 E^2) \quad (8)$$

$$\frac{d\phi}{ds} = \frac{1}{\Delta} \left[\left(1 - \frac{2M}{r} \right) L + \frac{2aME}{r} \right],$$

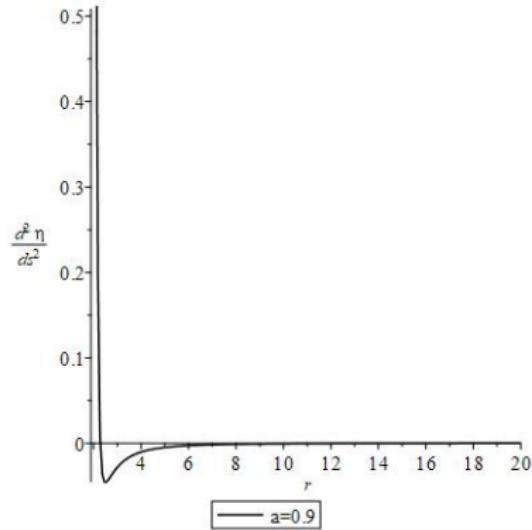
Generalized geodesic deviation equation

$$\begin{aligned}\frac{D^2\eta^1}{ds^2} &= R_{001}^1 u_1^0 \eta^0 u_1^0 + R_{010}^1 u_1^0 \eta^1 u_1^0 - \Gamma_{00}^1 (\Delta u^0)^2 - 2\Gamma_{11}^1 (\Delta u^1)^2 - \\ &2(\Gamma_{00,i}^1 \eta^i) \left[(\Delta u^0)^2 - 2u_1^0 \Delta u^0 \right] - 2(\Gamma_{11,i}^1 \eta^i) \left[(\Delta u^1)^2 - 2u_1^1 \Delta u^1 \right] \\ \frac{D^2\eta^2}{ds^2} &= R_{001}^2 u_1^0 \eta^0 u_1^0 + R_{010}^2 u_1^0 \eta^1 u_1^0 - 2\Gamma_{00}^2 (\Delta u^0)^2 - 2\Gamma_{11}^2 (\Delta u^1)^2 - \\ &2(\Gamma_{00,i}^2 \eta^i) \left[(\Delta u^0)^2 + 2u_1^0 \Delta u^1 \right] - 2(\Gamma_{11,i}^2 \eta^i) \left[(\Delta u^1)^2 + 2u_1^1 \Delta u^1 \right] \quad (9) \\ \frac{D^2\eta^3}{ds^2} &= R_{101}^3 u_1^1 \eta^0 u_1^1 + R_{110}^3 u_1^1 \eta^1 u_1^0 - 4\Gamma_{01}^3 \Delta u^0 \Delta u^1 - \\ &2(\Gamma_{01,i}^3 \eta^i) \left[2\Delta u^0 \Delta u^1 + u_1^0 \Delta u^1 + u_1^1 \Delta u^0 \right]\end{aligned}$$

Tidal acceleration $\left(\frac{D^2 \eta^r}{ds^2} \right)$



a)



b)

Fig.2(a,b): The dependence of the components of the tidal acceleration vector on the distance. The mass of the black hole is $M = 1$, the minimum distance to the horizon is $10^{-4} r_h$, the angular momentum of a radially incident particle is $L_1 = L_2 = 0.3$, , the energy of the particles is $E_1 = 1$, $E_2 = \frac{2\sqrt{2}}{3}$.

Tidal acceleration $\left(\frac{D^2 \eta^\theta}{ds^2} \right)$

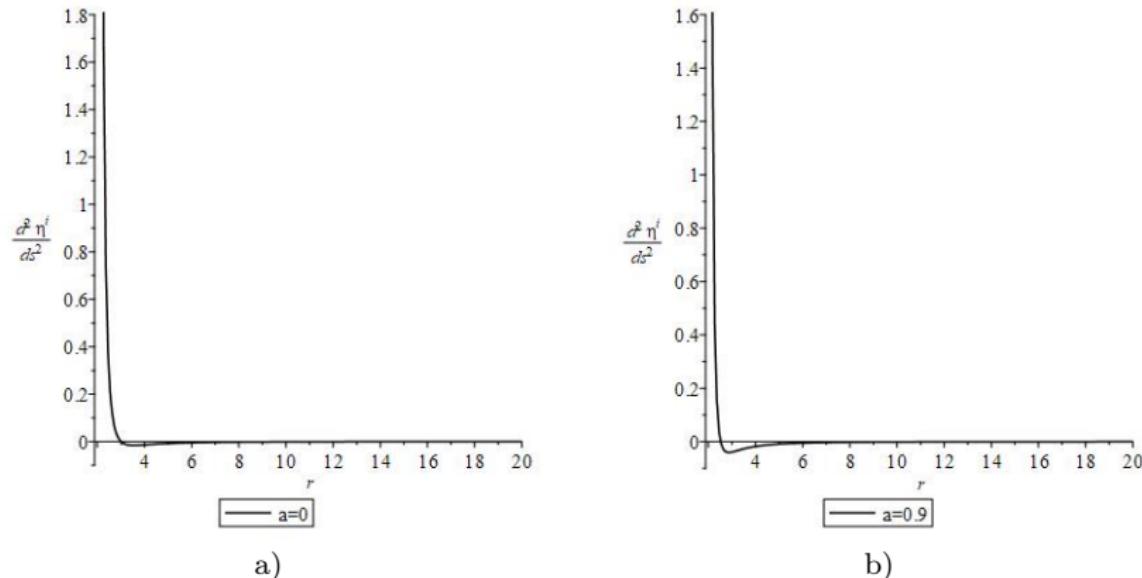


Fig.3 (a,b): The dependence of the components of the tidal acceleration vector on the distance. The mass of the black hole is $M = 1$, the minimum distance to the horizon is $10^{-4}r_h$, the angular momentum of a radially incident particle is $L_1 = L_2 = 0.3$, , the energy of the particles is $E_1 = 1$, $E_2 = \frac{2\sqrt{2}}{3}$.

Tidal acceleration $\left(\frac{D^2 \eta^\phi}{ds^2} \right)$

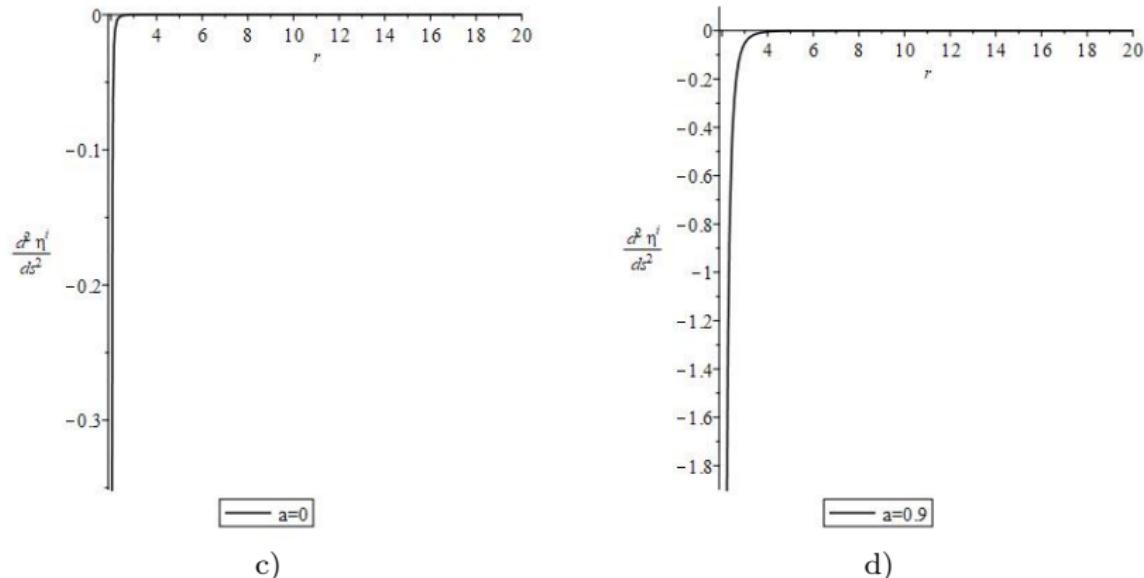


Fig.3 (c,d): The dependence of the components of the tidal acceleration vector on the distance. The mass of the black hole is $M = 1$, the minimum distance to the horizon is $10^{-4}r_h$, the angular momentum of a radially incident particle is $L_1 = L_2 = 0.3$, , the energy of the particles is $E_1 = 1$, $E_2 = \frac{2\sqrt{2}}{3}$.

Conclusion

- The paper presents an analytical form of the modified geodesic deviation equation for two test particles falling radially in the equatorial plane.
- We have shown that for reverse orbits, there is a change in the sign of the tidal acceleration components.
- An increase in the energy and angular momentum of the particles does not significantly affect the nature of the change in the components of tidal acceleration.

Thank you for your attention!

Литература

1. Synge J.L. On the deviation of geodesics and null-geodesics, particularly in relation to the properties of spaces of constant curvature and indefinite line element // Ann. Math. – 1934. – Vol.35, №4. – P.705–713
2. Synge J.L. Relativity: The General Theory. Amsterdam: North-Holland Publishing company, 1960 – 432 p.
3. Ciufolini I. Generalized geodesic deviation equation // Physical Review D. 1986. vol.34. N34. pp. 1014-1017.
4. Schutz B.F. On Generalised Equations of Geodesic Deviation // Galaxies, Axisymmetric Systems and Relativity, 1985 pp. 237-246.