The possibility of constructing stable solutions in Horndeski theory avoiding the no-go theorem

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» Modified gravity: why?

General relativity is a very successful theory of gravity, but we have some reasons to explore modified theories.

* First and probably the most major reason in recent years arises from the discovery of the accelerated expansion of the present universe. This may be caused by the extremely fine-tuned cosmological constant, but currently it would be better to have other possibilities at hand and a long distance modification of general relativity is one of such possible alternatives.
* Secondly, in order to test gravity, we need to know predictions of theories other than general relativity. This motivation is becoming increasingly important after the first detection of gravitational waves
* Thirdly, aside from phenomenology, pursuing consistent modifications of gravity helps us to learn more deeply about general relativity and gravity. For example, by trying to develop massive gravity one can gain a deeper understanding of general relativity and see how special a massless graviton is.

» Beyond Horndeski lagrangian

We are considering the theory with following Lagrangian:

$$S = \int \mathrm{d}^4x\sqrt{-g}\,(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_{\mathcal{BH}})\,, \tag{1a}$$

$$\mathcal{L}_2 = F(\pi, X), \tag{1b}$$

$$\mathcal{L}_3 = K(\pi, X)\Box\pi, \tag{1c}$$

$$\mathcal{L}_4 = -G_4(\pi, X)R + 2G_{4X}(\pi, X)\left[(\Box\pi)^2 - \pi_{;\mu\nu}\pi^{;\mu\nu}\right], \tag{1d}$$

$$\mathcal{L}_5 = G_5(\pi, X)\,G^{\mu\nu}\pi_{;\mu\nu} + \frac{1}{3}G_{5X}\left[(\Box\pi)^3 - 3\Box\pi\pi_{;\mu\nu}\pi^{;\mu\nu} + 2\pi_{;\mu\nu}\pi^{;\mu\rho}\pi_{;\rho}{}^{\nu}\right], \tag{1e}$$

$$\mathcal{L}_{\mathcal{BH}} = F_4(\pi, X)\epsilon^{\mu\nu\rho}{}_{\sigma}\epsilon^{\mu'\nu'\rho'\sigma}\pi_{,\mu}\pi_{,\mu'}\pi_{;\nu\nu'}\pi_{;\rho\rho'} + \tag{1f}$$
$$+ F_5(\pi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\pi_{,\mu}\pi_{,\mu'}\pi_{;\nu\nu'}\pi_{;\rho\rho'}\pi_{;\sigma\sigma'},$$

where π is the scalar field, $X = g^{\mu\nu}\pi_{,\mu}\pi_{,\nu}$, $\pi_{,\mu} = \partial_\mu\pi$, $\pi_{;\mu\nu} = \nabla_\nu\nabla_\mu\pi$, $\Box\pi = g^{\mu\nu}\nabla_\nu\nabla_\mu\pi$, $G_{4X} = \partial G_4/\partial X$, etc. The Horndeski theory corresponds to $F_4(\pi, X) = F_5(\pi, X) = 0$.

» Scalar perturbation sector

$$ds^2 = (1+2\Phi)\,dt^2 - \partial_i\beta\ dtdx^i - a^2(1+2\Psi\delta_{ij}+2\partial_i\partial_j E)\,dx^i dx^j. \qquad (2)$$

The scalar field perturbation is denoted by $\delta\pi = \chi$.
Then the quadratic action for the scalar perturbations has the form

$$\begin{aligned}
S^{(2)} = \int \mathrm{d}t\,\mathrm{d}^3x\,a^3 \Bigg(& A_1\,\dot{\Psi}^2 + A_2\,\frac{(\overrightarrow{\nabla}\Psi)^2}{a^2} + A_3\,\Phi^2 + A_4\,\Phi\frac{\overrightarrow{\nabla}^2\beta}{a^2} + A_5\,\dot{\Psi}\frac{\overrightarrow{\nabla}^2\beta}{a^2} \\
& + A_6\,\Phi\dot{\Psi} + A_7\,\Phi\,\frac{\overrightarrow{\nabla}^2\Psi}{a^2} + A_8\,\Phi\frac{\overrightarrow{\nabla}^2\chi}{a^2} + A_9\,\dot{\chi}\frac{\overrightarrow{\nabla}^2\beta}{a^2} + A_{10}\,\chi\ddot{\Psi} + A_{11}\,\Phi\dot{\chi} \\
& + A_{12}\,\chi\frac{\overrightarrow{\nabla}^2\beta}{a^2} + A_{13}\,\chi\frac{\overrightarrow{\nabla}^2\Psi}{a^2} + A_{14}\,\dot{\chi}^2 + A_{15}\,\frac{(\overrightarrow{\nabla}\chi)^2}{a^2} + B_{16}\,\dot{\chi}\frac{\overrightarrow{\nabla}^2\Psi}{a^2} + A_{17}\,\Phi\chi \\
& + A_{18}\,\dot{\Psi}\chi + A_{19}\,\Psi\chi + A_{20}\,\chi^2 + A_{21}\,\chi\overrightarrow{\nabla}^2 E + A_{22}\,\overrightarrow{\nabla}^2 E(\ddot{\chi}) + A_{23}\,\overrightarrow{\nabla}^2 E(\dot{\chi}) \\
& + A_{24}\,\overrightarrow{\nabla}^2 E\left(\dot{\Phi}\right) + A_{25}\,\Phi\overrightarrow{\nabla}^2 E + A_{26}\,\overrightarrow{\nabla}^2 E\left(\ddot{\Psi}\right) + A_{27}\,\overrightarrow{\nabla}^2 E\left(\dot{\Psi}\right) \Bigg),
\end{aligned}$$

where an overdot stands for derivative with respect to cosmic time t, and coefficients A_i are expressed in terms of the Lagrangian functions and their derivatives.

» No-go theorem

From unitary gauge analysis when $E = \chi = 0$, we know, that we can, after integrating the constraints, obtain the following action:

$$S^{(2)} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3 \left(\mathcal{G}_S \left(\dot{\Psi} \right)^2 - \mathcal{F}_S \frac{\left(\overrightarrow{\nabla} \Psi \right)^2}{a^2} \right) \tag{3}$$

Where

$$\mathcal{G}_S = \frac{4}{9} \frac{A_3 {A_1}^2}{{A_4}^2} - A_1, \tag{4a}$$

$$\mathcal{F}_S = \frac{1}{a} \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{a A_5 \cdot A_7}{2 A_4} \right] - A_2 \tag{4b}$$

The quadratic action for tensor perturbations is

$$S^{(2)}_{tensor} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3 \left[\frac{A_5}{2} \left(\dot{h}^T_{ik} \right)^2 - A_2 \frac{\left(\overrightarrow{\nabla} h^T_{ik} \right)^2}{a^2} \right], \tag{5}$$

where h^T_{ik} denotes transverse traceless tensor perturbation.

» No-go theorem

To avoid ghost and gradient instabilities one requires $A_5 > 0$, $\mathcal{G}_S > 0$ and $A_2 > 0$, $\mathcal{F}_S > 0$.
From the definition of $\mathcal{F}_S$:

$$\frac{d}{dt}\left[\frac{aA_5 \cdot A_7}{2A_4}\right] = a \cdot (\mathcal{F}_S + A_2) > 0, \tag{6}$$

The point is that

$$\xi = \frac{aA_5 \cdot A_7}{2A_4}, \quad A_5 + A_7 = 4F_4\dot{\pi}^4 + 12HF_5\dot{\pi}^5 \tag{7}$$

is, therefore, a monotonously growing function. Hence, ξ necessarily crosses zero somewhere during the evolution. This is the case if $A_4 \to \infty$. Since A_4 is dependent on the background fields, this means that there can be no non-singular solutions in General Horndeski theory. It were shown that healthy bounce and genesis solutions could be achieved in case of beyond Horndeski theory, and in the case of General Horndeski theory, attempts to construct such a solution meet the problem of strong coupling in the tensor sector.

» Gauge invariant variables

This action is invariant with respect to small coordinate transformations:

$$x^\mu \to x^\mu - \xi^\mu,$$

where $\xi^\mu = \left(\xi_0, \xi_T^i + \delta^{ij}\partial_j \xi_S\right)^{\mathrm{T}}$. In which the fields change as:

$$\Phi \to \Phi + \dot{\xi}_0, \quad \beta \to \beta - \xi_0 + a^2 \dot{\xi}_S, \quad \chi \to \chi + \xi_0 \dot{\pi}, \quad \Psi \to \Psi + \xi_0 H, \quad E \to E - \xi_S.$$

The action can be rewritten in explicitly gauge-invariant form by introducing new variables (Bardeen variables):

$$X = \chi + \dot{\pi}\left(\frac{\beta}{a^2} + \dot{E}\right), \tag{8a}$$

$$Y = \Psi + H\left(\frac{\beta}{a^2} + \dot{E}\right), \tag{8b}$$

$$Z = \Phi + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\beta}{a^2} + \dot{E}\right]. \tag{8c}$$

» Three variables action

In terms of these variables, the action will take the form

$$S^{(2)} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3 \Bigg(A_1 \left(\dot{Y}\right)^2 + A_2\, \frac{(\vec{\nabla}\, Y)^2}{a^2} + A_3\, Z^2 + A_6\, Z\dot{Y} + A_7\, Z \frac{\vec{\nabla}^2\, Y}{a^2}$$
$$+ A_8\, Z \frac{\vec{\nabla}^2 X}{a^2} + A_{10}\, X\ddot{Y} + A_{11}\, Z\dot{X} + A_{13}\, X \frac{\vec{\nabla}^2\, Y}{a^2} + A_{14} \left(\dot{X}\right)^2$$
$$+ A_{15}\, \frac{(\vec{\nabla} X)^2}{a^2} + A_{16}\, \dot{X} \frac{\vec{\nabla}^2\, Y}{a^2} + A_{17}\, ZX + A_{18}\, X\dot{Y} + A_{20} X^2 \Bigg) \tag{9}$$

At this point it is clearly seen that the field Z is non-dynamic and we can derive a Z–constraint which has the following form:

$$Z = \frac{1}{2A_3} \left(-A_7 \frac{\vec{\nabla}^2\, Y}{a^2} - A_8 \frac{\vec{\nabla}^2 X}{a^2} + 3A_4\, \dot{Y} - A_{11}\, \dot{X} - A_{17} X \right) \tag{10}$$

We used that $A_6 = -3A_4$.

» $A_4 = 0$

We can consider this case if we just put $A_4 = 0$ in our three variables action. So, after substituting the Z–constraint into it and and make a substitution

$$\zeta = Y + \eta X, \quad \eta = \eta(A_i), \tag{11}$$

we get the following action:

$$S^{(2)} = \int \mathrm{d}t\,\mathrm{d}^3x\,a^3 \left(A_1 \left(\dot{\zeta}\right)^2 + A_2 \frac{\left(\overrightarrow{\nabla}\zeta\right)^2}{a^2} - \frac{1}{9}\frac{{A_1}^2}{A_3}\frac{\left(\overrightarrow{\nabla}^2\zeta\right)^2}{a^4} + \frac{1}{3}\frac{A_1 A_{11}}{A_3}\frac{\left(\overrightarrow{\nabla}^2 X\right)}{a^2}\dot{\zeta} \right) \tag{12}$$

Which leads to constrain $\dot{\zeta} = 0$, which means the absence of dynamics of the field ζ.
This result might be expected, since when the limit of the speed of sound is equal to zero:

$$\lim_{A_4 \to 0, \dot{A}_4 \to 0} c_S = 0. \tag{13}$$

» $A_3 = 0$ situation

From the view of the Z–constraint, we can also distinguish the case $A_3 = 0$ as a singular point. By reason of the following ratios on the coefficients

$$A_3 = \frac{3}{2} A_4 H - \frac{1}{2} A_{11} \dot{\pi}, \tag{14a}$$

$$A_{17} = 3 \frac{\dot{H}}{\dot{\pi}} A_4 - \frac{\ddot{\pi}}{\dot{\pi}} A_{11} \tag{14b}$$

we have two options: $A_4 = 0, A_{11} = 0$ and $A_4 = 0, \dot{\pi} = 0$.

» $A_4 = 0, A_{11} = 0$

In this case, the Z–constraint gives us the condition:

$$X = -\frac{A_7}{A_8} Y$$

Which brings the action into the following form:

$$S^{(2)} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3\, mY^2 \tag{15}$$

where

$$m = \text{(Some VERY big expression)} \tag{16}$$

» $A_4 = 0, \dot{\pi} = 0$

In this case, the condition $A_4 = 0$ takes the form of:

$$G_4 H = 0 \tag{17}$$

For $A_4 = 0$ it is also necessary to impose the condition $H = 0$. And the action takes the form:

$$S^{(2)} = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3 \left(\mathcal{G}_S \left(\dot{Y} \right)^2 + m Y^2 - \mathcal{F}_S \frac{\left(\vec{\nabla} Y \right)^2}{a^2} \right) \tag{18}$$

where

$$\mathcal{G}_S = \mathcal{F}_S = \frac{2 G_4}{G_{4\pi}^2} \left(6 G_{4\pi}^2 + 4 F_X G_4 - 4 K_\pi G_4 \right) \tag{19a}$$

$$m = 2 F_{\pi\pi} \frac{G_4^2}{G_{4\pi}^2} \tag{19b}$$

And the corresponding equation of motion:

$$\ddot{Y} + \frac{G_4 F_{\pi\pi}}{2 G_4 K_\pi - 2 G_4 F_X - 3 {G_{4\pi}}^2} Y - \frac{\left(\vec{\nabla} Y \right)^2}{a^2} = 0 \tag{20}$$

The case of the Minkowski space in GR ($G_4 = \frac{1}{2}$) is a special case of this solution.

» Reconstruction of Lagrangian functions

Without loss of generality we choose the following form of the scalar field

$$\pi(t) = t, \tag{21}$$

so that $X = 1$. To reconstruct the theory which corresponds some solution we use the following ansatz for the Lagrangian functions

$$F(\pi, X) = f_0(\pi) + f_1(\pi) \cdot X + f_2(\pi) \cdot X^2, \tag{22a}$$

$$K(\pi, X) = k_0(\pi) + k_1(\pi) \cdot X + k_2(\pi) \cdot X^2, \tag{22b}$$

$$G_4(\pi, X) = \frac{1}{2}, \tag{22c}$$

$$G_5(\pi, X) = F_4(\pi, X) = F_5(\pi, X) = 0. \tag{22d}$$

We are interested to consider the case $G_4 = \text{const}$, which corresponds to GR.

» Reconstruction of Lagrangian functions

Only the equations of motion and the condition $A_4 = 0$ remain as possible constraints:

$$f_2 = f_0 + \dot{H} \tag{23a}$$

$$f_1 = -2f_2 - 3H^2 + \dot{k_0} - \dot{k_2} \tag{23b}$$

$$k_1 = H - 2k_2 \tag{23c}$$

It is easy to see that we can freely put $f_2 = k_0 = k_2 = 0$, and then the remaining functions can be directly expressed through the Hubble parameter and its derivative:

$$f_0 = -\dot{H}, \tag{24a}$$

$$f_1 = -3H^2, \tag{24b}$$

$$k_1 = H. \tag{24c}$$

Thank you for your attention!