Anti-de Sitter-Beltrami spacetime in nonrelativistic limit

Sergei Manida

St. Petersburg State University

It is known that, similarly to the space-time of Minkowski, the space-time of constant non-zero curvature has the maximum symmetry. Such spaces are the de Sitter and Anti-de Sitter spaces, respectively. For an Anti-de Sitter (but not de Sitter) metric in Beltrami coordinates, there is a limit $c \to \infty$ leading to an *R*-space metric.

Possible Kinematics H.Bacry, J.-M.Lévy-Leblon, J.Math.Phys., 9, 1605 (1968)

$$[J_i, J_j] = \varepsilon_{ijk} J_k;$$

$$[J_i, K_j] = \varepsilon_{ijk} K_k; \quad [J_i, P_j] = \varepsilon_{ijk} P_k; \quad [J_i, H] = 0;$$

$$[P_i, K_j] = \rho \delta_{ij} H;$$

$$[H, P_i] = \alpha K_i; \quad [H, K_i] = \lambda P_i;$$

$$[P_i, P_j] = \beta \varepsilon_{ijk} J_k, \quad [K_i, K_j] = \mu \varepsilon_{ijk} J_k.$$

æ

イロト イポト イヨト イヨト

$$[J_i, J_j] = \varepsilon_{ijk} J_k;$$

$$[J_i, K_j] = \varepsilon_{ijk} K_k; \quad [J_i, P_j] = \varepsilon_{ijk} P_k; \quad [J_i, H] = 0;$$

$$[P_i, K_j] = \rho \delta_{ij} H;$$

$$[H, P_i] = \alpha K_i; \quad [H, K_i] = \lambda P_i;$$

$$[P_i, P_j] = \beta \varepsilon_{ijk} J_k, \quad [K_i, K_j] = \mu \varepsilon_{ijk} J_k.$$

$$ho
eq \mathsf{0}; \qquad lpha =
ho eta; \qquad \lambda = -
ho \mu;$$

・ロト・(部・・ヨト・ヨー・)への

$$[P_i, K_j] = \delta_{ij}H;$$

de Sitter

$$\begin{split} [H,P_i] &= +\frac{1}{R^2} K_i; \qquad [H,K_i] = \frac{1}{c^2} P_i; \\ [P_i,P_j] &= +\frac{1}{R^2} \varepsilon_{ijk} J_k, \qquad [K_i,K_j] = -\frac{1}{c^2} \varepsilon_{ijk} J_k. \\ & \text{Anti-de Sitter} \\ [H,P_i] &= -\frac{1}{R^2} K_i; \qquad [H,K_i] = \frac{1}{c^2} P_i; \\ [P_i,P_j] &= -\frac{1}{R^2} \varepsilon_{ijk} J_k, \qquad [K_i,K_j] = -\frac{1}{c^2} \varepsilon_{ijk} J_k. \end{split}$$

< />
▲ □ > < □ > < □ > □ □

The linear element of the metric AdS is induced on the surface of the hyperboloid

$$z_{-1}^2 + z_0^2 - \vec{z}^2 = R^2$$

in the ambient five-dimensional space with a metric

$$ds^2 = dz_{-1}^2 + dz_0^2 - d\vec{z}^2.$$

Let us set $z_0 = c au$ and write the metric as

$$ds^2 = c^2 d\tau^2 - d\vec{z}^2 + rac{(c^2 au d au - \vec{z} d\vec{z})^2}{R^2 + c^2 au^2 - \vec{z}^2}$$

In the limit $c\tau \gg R$, we get

$$ds^2=-rac{1}{ au^2}(R^2d au^2+(ec{z}d au- au dec{z})^2)$$

The transition to Beltrami coordinates $\{t, \vec{r}\}$

$$ct = R \frac{c\tau}{z_{-1}}, \quad \vec{r} = R \frac{\vec{z}}{z_{-1}}$$

does not commute with the limit $c\tau \gg R$.

The linear element of the AdSB metric is:

$$ds^{2} = rac{\eta_{\mu
u}dx^{\mu}dx^{
u}}{h^{2}} - rac{(\eta_{\mu
u}x^{\mu}dx^{
u})^{2}}{R^{2}h^{4}} = g_{\mu
u}dx^{\mu}dx^{
u},$$

The linear element of the *AdSB* metric is: where $\eta_{\mu\nu} = \text{diag}\{+1, -1, -1, -1\}$, indices μ, ν take the values 0,1,2,3, the summation is carried out over repeated indices, and the metric tensor

$$g_{\mu
u}=rac{1}{h^2}\left(\eta_{\mu
u}-rac{\eta_{\mueta}x^eta\eta_{
ulpha}x^lpha}{R^2h^2}
ight),$$
 где $h^2=1+rac{\eta_{\mu
u}x^\mu x^
u}{R^2}.$

Inverse tensor:

$$\mathsf{g}^{\mu
u}=\mathsf{h}^2\left(\eta^{\mu
u}+rac{\mathsf{x}^\mu\mathsf{x}^
u}{\mathsf{R}^2}
ight).$$

Determinant of the metric tensor $g = -h^{-10}$.

《曰》 《圖》 《臣》 《臣》

Let us write an expression for the linear element of the metric AdSB in the domain $ct \gg R$. To do this, we represent the time and space components in an explicit form:

$$ds^{2} = R^{2} \frac{c^{2} dt^{2} - d\vec{r}^{2}}{R^{2} + c^{2} t^{2} - \vec{r}^{2}} - R^{2} \frac{(c^{2} t dt - r dr)^{2}}{(R^{2} + c^{2} t^{2} - \vec{r}^{2})^{2}}$$

and expand in a series in the small parameter $R^2/(ct)^2$

$$ds^{2} = rac{R^{2}}{c^{2}t^{2}}\left\{rac{R^{2}}{t^{2}}dt^{2} - \left(dr - rac{r}{t}dt
ight)^{2} - r^{2}d\Omega^{2} + O(R^{2}/(ct)^{2})
ight\}$$

Consider this expression in a small neighborhood of some time $t = T + \tau$, $\tau \ll T$ and introduce the notation $R/T = c_0$:

$$ds^{2} = \frac{c_{0}^{2}}{c^{2}} \left\{ c_{0}^{2} dt^{2} - \left(dr - \frac{r}{R} c_{0} dt \right)^{2} - r^{2} d\Omega^{2} + O(c_{0}^{2}/c^{2}) + O(\tau/T) \right\}$$

Sergei Manida (SPbSU)

Laplace–Beltrami operator $\Box = rac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu
u} \partial_
u)$ представляется в виде

$$\Box = \left(1 + \frac{\eta_{\mu\nu} x^{\mu} x^{\nu}}{R^2}\right) \left(\left(\eta^{\mu\nu} + \frac{x^{\mu} x^{\nu}}{R^2}\right) \partial_{\mu} \partial_{\nu} + \frac{2}{R^2} x^{\mu} \partial_{\mu}\right).$$

Klein–Fock equation for a scalar function Φ :

$$\left(1+\frac{\eta_{\mu\nu}x^{\mu}x^{\nu}}{R^{2}}\right)\left(\left(\eta^{\mu\nu}+\frac{x^{\mu}x^{\nu}}{R^{2}}\right)\partial_{\mu}\partial_{\nu}+\frac{2}{R^{2}}x^{\mu}\partial_{\mu}\right)\Phi+\frac{m^{2}c^{2}}{\hbar^{2}}\Phi=0.$$
(1)

In the equation (1), we single out terms with certain powers of c:

$$\left(\frac{c^2t^2}{R^2} + \frac{R^2 - r^2}{R^2}\right) \left(\frac{1}{c^2}\partial_t^2 - \Delta + \frac{x^{\mu}x^{\nu}}{R^2}\partial_{\mu}\partial_{\nu} + \frac{2}{R^2}x^{\mu}\partial_{\mu}\right)\Phi + \frac{m^2c^2}{\hbar^2}\Phi = 0.$$
(2)

<ロ> (四) (四) (三) (三) (三) (三)

We will consider the equation (2) in some neighborhood of the point t = T, such that cT >> R. Multiply this equation by R^2/c^2 and get:

$$t^{2}\left(-\Delta+\frac{x^{\mu}x^{\nu}}{R^{2}}\partial_{\mu}\partial_{\nu}+\frac{2}{R^{2}}x^{\mu}\partial_{\mu}\right)\Phi+\frac{m^{2}R^{2}}{\hbar^{2}}\Phi+O(R^{2}/(cT)^{2})=0.$$

$$t^{2}\left(-\Delta + \frac{t^{2}}{R^{2}}\partial_{t}^{2} + \frac{r^{2}}{R^{2}}\partial_{r}^{2} + \frac{2rt}{R^{2}}\partial_{t}\partial_{r} + \frac{2t}{R^{2}}\partial_{t} + \frac{2r}{R^{2}}\partial_{r}\right)\Phi =$$
$$= -\frac{m^{2}R^{2}}{\hbar^{2}}\Phi + O(R^{2}/(cT)^{2}). \quad (3)$$

The equation (3) can be considered as the "non-relativistic" limit of the equation (2), discarding $O(R^2/(cT)^2)$, since formally (3) is obtained from (2) when $c \to \infty$.

The equation (3) allows variable separation. Indeed, setting $\Phi(t, \vec{r}) = F(\vec{r}/t)f(t)$, we get

$$-rac{t^2}{F(ec{r}/t)}\Delta F(ec{r}/t)+rac{t^2}{R^2f(t)}(t\partial_t+1)(t\partial_t)f(t)+rac{m^2R^2}{\hbar^2}=0.$$

The solution to this equation can be represented as a superposition of "plane" waves:

$$\exp\left\{\frac{i}{t\hbar}(a+\vec{\kappa}\vec{r})\right\}$$

with the condition $a^2 - \vec{\kappa}^2 R^2 = m^2 R^4$.

For fields localized in space-time interval $r \ll R$, $|t - T| \ll T$, it is possible to reduce the expression (3) to the form

$$\left(\Delta-rac{1}{c_0^2}\partial_t^2
ight)\Phi=rac{m^2c_0^2}{\hbar^2}\Phi+O(r/R)+O(|t-T|/T)+O(c_0^2/c^2),$$

coinciding with the Klein–Fock equation in the Minkowski space up to small order $O(c_0^2/c^2)$, O(r/R), O(|t - T|/T).

The Dirac equation in the Anti-de Sitter space in Beltrami coordinates can be represented as:

$$\left[i\gamma^{a}e^{\mu}_{a}\left(\partial_{\mu}-i\frac{1}{4}\omega^{ab}_{\mu}\sigma_{ab}\right)-\frac{mc}{\hbar}\right]\Psi=0,$$

where

$$\sigma_{ab} = \frac{i}{2} [\gamma_a, \gamma_b], \qquad \{\gamma_a, \gamma_b\} = 2\eta_{ab}, \qquad \eta_{ab} e^a_\mu e^b_\nu = g_{\mu\nu}, \qquad e^a_\mu e^\mu_b = \delta^a_b,$$
$$\omega_{ab\mu} = \frac{1}{2} e^c_\mu (\gamma_{cab} - \gamma_{abc} - \gamma_{bca}), \qquad \gamma^c_{ab} = (e^\mu_a e^\nu_b - e^\mu_b e^\nu_a) \partial_\nu e^c_\mu. \tag{4}$$

Tetrads e^a_μ and their inverses e^μ_a for the metric (7) are easy to calculate

$$e^{a}_{\mu} = \frac{1}{h} \left(\delta^{a}_{\mu} - \frac{\delta^{a}_{\beta} x^{\beta} \eta_{\mu\nu} x^{\nu}}{R^{2} h(h+1)} \right), \qquad e^{\mu}_{a} = h \left(\delta^{\mu}_{a} + \frac{\eta_{ab} \delta^{b}_{\nu} x^{\nu} x^{\mu}}{R^{2}(h+1)} \right).$$
(5)

Sergei Manida (SPbSU)

(日) (同) (日) (日) (日)

Substituting expressions (5) into formulas (4), we obtain

$$\omega_{\mu}^{ab} = rac{x^{
u} (\delta_{\mu}^{a} \delta_{\nu}^{b} - \delta_{\nu}^{a} \delta_{\mu}^{b})}{R^{2} h (1+h)},$$

and the Dirac equation takes the form

$$\left\{ih\gamma^{\mu}\left[\partial_{\mu}+\frac{\eta_{\mu\nu}x^{\nu}x^{\alpha}\partial_{\alpha}}{R^{2}(h+1)}-\frac{i}{4}\frac{x^{\nu}(\delta^{d}_{\nu}\delta^{b}_{\mu}-\delta^{d}_{\mu}\delta^{b}_{\nu})}{R^{2}h(h+1)}\sigma_{bd}\right]-\frac{mc}{\hbar}\right\}\Psi=0.$$
 (6)

Obviously, the equation (6) in the domain $r \ll R$ coincides with the Dirac equation in the Minkowski space. In this regard, the equation (6) is used to approximate corrections of the order of $O(r^2/R^2)$ to solutions of the ordinary Dirac equation.

We consider this equation in the vicinity of the auxiliary moment of time t = T, such that $cT \gg R$. Multiply the equation (6) by R/cT, expand all expressions into series with respect to this small parameter and discard terms of order O(R/cT). The resulting expression looks like

$$i\left[\frac{t^2}{TR}\gamma^0\partial_t + \frac{tr}{TR}\gamma^0\partial_r + \frac{t}{T}\gamma^i\partial_i\right]\Psi - \frac{mR}{\hbar T}\Psi = 0.$$
 (7)

In a small neighborhood of the world point t = T, r = 0 in the equation (7) one can neglect small quantities of the order of O(r/R), O(|t - T|/T) and obtain an equation $(R/T \equiv c_0)$

$$i\left[\frac{1}{c_0}\gamma^0\partial_t+\gamma^i\partial_i\right]\Psi-\frac{mc_0}{\hbar}\Psi=0,$$

that formally coincides with the Dirac equation in the Minkowski space.

イロト イポト イヨト イヨト

We use the representation of gamma matrices, in which

$$\gamma^{0} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^{i} = \begin{pmatrix} 0 & \sigma^{i} \\ -\sigma^{i} & 0 \end{pmatrix},$$

and the bispinor Ψ can be represented as

$$\Psi = \begin{pmatrix} \phi \\ \theta \end{pmatrix}.$$

The equation (7) for the spinors ϕ and θ can be represented as

$$D_0\phi + D\theta - \frac{mR}{\hbar}\phi = 0,$$
(8)

$$D_0\theta + D\phi + \frac{mn}{\hbar}\theta = 0,$$
 (9)

where the notation

$$D_0 = i \frac{t}{R} (t \partial_t + r \partial_r), \qquad D = i t \sigma^i \partial_i$$
(10)

It is easy to check that $[D_0, D] = 0$, and the operator $D^2 - D_0^2$ coincides with the Laplace–Beltrami operator on the left side of the equality (10):

$$t^2\left(-\Delta+rac{x^\mu x^
u}{R^2}\partial_\mu\partial_
u+rac{2}{R^2}x^\mu\partial_\mu
ight).$$

Thus, as a solution to the equations (8), (9), we can take two solutions to the equation (3) related by the relation (8) or (9).

The Anti-de Sitter-Beltrami metric is covariant under the following transformations:

Spatial translations:

$$x' = \frac{x - \rho}{1 - \frac{\rho x}{R^2}},$$
(11)
$$\{t', y', z'\} = \frac{\{t, y, z\}\sqrt{1 - \frac{\rho^2}{R^2}}}{1 - \frac{\rho x}{R^2}}.$$
(12)

Time translation:

$$t' = \frac{t - \alpha R^2 / c^2}{1 + \alpha t},$$
(13)

$$\vec{r'} = \frac{\vec{r} \sqrt{1 - \frac{\alpha^2 R^2}{c^2}}}{1 + \alpha t}.$$
(14)

Lorentz transformations:

$$x' = \gamma(x - ut),$$

$$t' = \gamma(t - \frac{u}{c^2}x),$$

Sergei Manida (SPbSU)

17 / 25

э

In the non-relativistic limit, spatial translations do not change, Lorentz transformations turn into ordinary Galilean transformations

$$x' = x - ut, \tag{15}$$

$$t'=t$$

and time translations take the form:

$$t' = \frac{t}{1 + \alpha t},$$
(16)

$$\vec{r}' = \frac{\vec{r}}{1 + \alpha t}.$$
(17)

A characteristic feature of this transformation is the invariance of the hyperplane t = 0. How did we get the relativistic Klein-Fock and Dirac equations? The fact is that time in all the above transformation formulas is measured not from an arbitrarily chosen initial moment, but from this invariant hyperplane. To obtain an expression relating the coordinates of two inertial reference frames whose origins coincide at some time T, it is necessary to perform the Galilean transformation (15) with the parameter u, then shift the spatial coordinates (11), (12) with the parameter $\rho = uT$, while the time t = T will go to $t' = T\sqrt{1 - u^2T^2/R^2}$. Next, we need to make a transformation (16), (17) with the parameter

$$\alpha = T^{-1} \left(\sqrt{1 - u^2 T^2 / R^2} - 1 \right).$$

As a result of these transformations, the point t = T, r = 0 remains unchanged, and the coordinates of an arbitrary space-time point are transformed as

$$t' = \frac{t}{\gamma - (\gamma - 1)t/T + \gamma uxT/R^2)},$$

$$x' = \frac{\gamma(x - u(t - T))}{\gamma - (\gamma - 1)t/T + \gamma uxT/R^2)},$$

$$\{y', z'\} = \frac{\{y, z\}}{\gamma - (\gamma - 1)t/T + \gamma uxT/R^2)},$$

where $\gamma = \frac{1}{\sqrt{1 - \frac{u^2 T^2}{R^2}}}$. And this is nothing but fractional-linear Lorentz-Fock transformations. Introducing the notation $c_0 = R/T$ and passing to limit $R \gg x$, $T \gg |t - T|$ for a fixed c_0 , we obtain the usual Lorentz transformations for the quantities $\{t - T, x, y, z\}$.

《曰》 《聞》 《臣》 《臣》 三臣

The Schrödinger equation can be considered as a non-relativistic analogue of the wave equation. The group of coordinate transformations considered above consists of Galilean transformations and fractional linear space-time translations. In the region $r \ll R$, linear-fractional spatial translations become linear, while transformations

$$t' = \frac{t}{1 + \alpha t},$$
$$\vec{r}' = \frac{\vec{r}}{1 + \alpha t},$$

that have the meaning of reverse-time translations remain unchanged. The usual free Schrödinger equation

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2m}\Delta\Psi \tag{18}$$

is invariant under time translations.

Let's try to modify the equation (18) in such a way that it becomes covariant with respect to translations of the reverse time:

$$i\hbar\left(rac{t^2}{T^2}\partial_t+rac{tr}{T^2}\partial_r
ight)\Psi=-rac{\hbar^2}{2m}rac{t^2}{T^2}\Delta\Psi.$$

This equation has solutions similar to the solutions of the Klein–Fock equation:

$$\Psi(t,\vec{r}) = \exp\left\{\frac{i}{\hbar t}(a+\vec{k}\vec{r})\right\},$$

but with another condition:

$$a = rac{k^2}{2m}.$$

To describe unstable states, the right side of the equation should include Γ_0 – the probability of decay per unit time, measured at the moment T:

$$i\hbar\left(\frac{t^2}{T^2}\partial_t + \frac{tr}{T^2}\partial_r\right)\Psi = -\frac{\hbar^2}{2m}\frac{t^2}{T^2}\Delta\Psi + i\hbar\frac{\Gamma_0}{2}\Psi.$$

The solution of this equation differs from the original one by the factor $\exp(\Gamma_0 T \frac{T-t}{2t})$, which leads to the probability density in the form

$$P(t) = P(T) \exp\left(\Gamma_0 T \frac{T-t}{t}\right) = P(T) \exp\left(-\Gamma_0 \frac{\tau}{1+\tau/T}\right)$$

where $(\tau \equiv t - T)$

Let us finally turn to the Schwarzschild metric in the Anti-de Sitter-Beltrami space *SAdSB*.

$$ds^{2} = \left(\frac{f(r,t)}{h_{0}^{2}} - \frac{r^{2}t^{2}c^{2}}{R^{4}h^{6}f(r,t)}\right)c^{2}dt^{2} + 2\frac{h_{0}^{2}c^{2}trdtdr}{R^{2}h^{6}f(r,t)} - \frac{h_{0}^{4}dr^{2}}{h^{6}f(r,t)} - \frac{r^{2}d\Omega^{2}}{h^{2}}$$

Here

$$R^{2}h^{2} = R^{2} + c^{2}t^{2} - r^{2},$$

$$R^{2}h_{0}^{2} = R^{2} + c^{2}t^{2},$$

$$f(r, t) = 1 + \frac{r^{2}}{R^{2}h^{2}} - \frac{2MGh}{c^{2}r}.$$

э

∃ ► < ∃ ►

Passing to the region $ct \gg R$ and expanding in a series in a small parameter, we obtain

$$R^{2}h^{2} = c^{2}t^{2}(1 + O(R^{2}/(ct)^{2})),$$

$$R^{2}h_{0}^{2} = c^{2}t^{2}(1 + O(R^{2}/(ct)^{2})),$$

$$f(r, t) = 1 - \frac{2Mgt}{Rr}, \quad g \equiv G/c$$

$$ds^{2} = \frac{R^{2}}{c^{2}t^{2}} \left\{ \frac{f(r, t)R^{2}dt^{2}}{t^{2}} - \frac{(rdt - tdr)^{2}}{t^{2}f(r, t)} - r^{2}d\Omega^{2} + O(R^{2}/(ct)^{2}) \right\}.$$

This expression for $f \rightarrow 1$ coincides with (8). all the thermodynamic formulas obtained in the previous report for the AdSB space coincide in our "nonrelativistic"limit with similar formulas in the Minkowski space.