

Cosmological particle creation: Weyl geometry + scalar field

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A little bit of mathematics

Differential geometry

Metric tensor $g_{\mu\nu} \Rightarrow$ interval

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \quad (g_{\mu\nu}g^{\nu\lambda} = \delta_\mu^\lambda)$$

Connections $\Gamma_{\mu\nu}^\lambda \Rightarrow$ covariant derivative

$$\nabla_\mu l^\nu = l^\nu_{,\mu} + \Gamma_{\sigma\mu}^\nu l^\sigma, \dots$$

Curvature tensor $R^\mu_{\nu\lambda\sigma}$

$$R^\mu_{\nu\lambda\sigma} = \frac{\partial \Gamma^\mu_{\nu\sigma}}{\partial x^\lambda} - \frac{\partial \Gamma^\mu_{\nu\lambda}}{\partial x^\sigma} + \Gamma^\mu_{\nu\lambda}\Gamma^\kappa_{\kappa\sigma} - \Gamma^\mu_{\nu\sigma}\Gamma^\kappa_{\kappa\lambda}$$

Ricci tensor $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$

Curvature scalar $R = g^{\mu\lambda}R_{\mu\lambda}$

$$g_{\mu\nu}(x), \quad \Gamma_{\mu\nu}^\lambda(x)$$

A little bit of mathematics

Differential geometry II

Three tensors $g_{\mu\nu}(x)$, $S_{\mu\nu}^\lambda$, $Q_{\lambda\mu\nu}$

Torsion $S_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda - \Gamma_{\nu\mu}^\lambda$

Nonmetricity $Q_{\lambda\mu\nu} = \nabla_\lambda g_{\mu\nu}$

Connections $\Gamma_{\mu\nu}^\lambda = C_{\mu\nu}^\lambda + K_{\mu\nu}^\lambda + L_{\mu\nu}^\lambda$

$$C_{\mu\nu}^\lambda = \frac{1}{2}g^{\lambda\kappa}(g_{\kappa\mu,\nu} + g_{\kappa\nu,\mu} - g_{\mu\nu,\kappa})$$

$$Q_{\lambda\mu\nu} = \nabla_\lambda g_{\mu\nu} \Rightarrow Q_{\lambda\mu\nu} = Q_{\lambda\nu\mu}$$

$$K_{\mu\nu}^\lambda = \frac{1}{2}(S_{\mu\nu}^\lambda - S_\mu^\lambda{}_\nu - S_\nu^\lambda{}_\mu)$$

$$L_{\mu\nu}^\lambda = \frac{1}{2}(Q_{\mu\nu}^\lambda - Q_\mu^\lambda{}_\nu - Q_\nu^\lambda{}_\mu)$$

A little bit of mathematics

Differential geometry III

Riemann geometry

$$S_{\mu\nu}^\lambda = 0, \quad Q_{\lambda\mu\nu} = 0$$

$$S_{\mu\nu}^\lambda = 0, \quad Q_{\lambda\mu\nu} = 0, \quad \Rightarrow \quad \Gamma_{\mu\nu}^\lambda = C_{\mu\nu}^\lambda$$

Weyl geometry

$$S_{\mu\nu}^\lambda = 0 \quad \Rightarrow \quad \Gamma_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda$$

$$Q_{\lambda\mu\nu} = A_\lambda(x)g_{\mu\nu}(x)$$

$$\Gamma_{\mu\nu}^\lambda = C_{\mu\nu}^\lambda + W_{\mu\nu}^\lambda$$

$$W_{\mu\nu}^\lambda = -\frac{1}{2}(A_\mu\delta_\nu^\lambda + A_\nu\delta_\mu^\lambda - A^\lambda g_{\mu\nu})$$

$A_\lambda(x)$ — “Weyl vector”

Local conformal transformation

$$ds^2 = \Omega^2(x) d\hat{s}^2 = \Omega^2 \hat{g}_{\mu\nu}(x) dx^\mu dx^\nu$$

$$g_{\mu\nu} = \Omega^2 \hat{g}_{\mu\nu}, \quad g^{\mu\nu} = \frac{1}{\Omega^2} \hat{g}^{\mu\nu}$$

$$\sqrt{-g} = \Omega^4(x) \sqrt{-\hat{g}}$$

$$C_{\mu\nu}^\lambda = \hat{C}_{\mu\nu}^\lambda + \left(\frac{\Omega_{,\nu}}{\Omega} \delta_\mu^\lambda + \frac{\Omega_{,\mu}}{\Omega} \delta_\nu^\lambda - g^{\lambda\kappa} \frac{\Omega_{,\kappa}}{\Omega} \hat{g}_{\mu\nu} \right)$$

If A_μ = gauge field

$$A_\mu = \hat{A}_\mu + 2 \frac{\Omega_{,\mu}}{\Omega} \quad \Rightarrow$$

$$\Gamma_{\mu\nu}^\lambda = \hat{\Gamma}_{\mu\nu}^\lambda \quad \Rightarrow$$

$$R^\mu_{\nu\lambda\sigma} = \hat{R}^\mu_{\nu\lambda\sigma}, \quad R_{\mu\nu} = \hat{R}_{\mu\nu}$$

Strength tensor

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = A_{\nu,\mu} - A_{\mu,\nu}$$

Weyl gravity

$$\mathcal{L}_W = \alpha_1 R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R^2 + \alpha_4 F_{\mu\nu} F^{\mu\nu}$$

$$S_W = \int \mathcal{L}_W \sqrt{-g} d^4x, \quad \frac{\delta S_W}{\delta \Omega} = 0$$

Dynamical variables: $g_{\mu\nu}(x)$, $A_\mu x$

Matter fields $S_{\text{tot}} = S_W + S_m$, $S_m = \int \mathcal{L}_m \sqrt{-g} d^4x$

S_m is not necessary conformal invariant

δS_m does!

$$\begin{aligned} \delta S_m &\stackrel{\text{def}}{=} -\frac{1}{2} \int T^{\mu\nu}(\delta g_{\mu\nu}) \sqrt{-g} d^4x - \int G^\mu(\delta A_\mu) \sqrt{-g} d^4x \\ &\quad + \int \frac{\delta \mathcal{L}_m}{\delta \Psi}(\delta \Psi) \sqrt{-g} d^4x = 0 \end{aligned}$$

Ψ — collective dynamical variable for matter fields, $\delta \mathcal{L}_m / \delta \Psi = 0$
 $T^{\mu\nu}$ — energy-momentum tensor, G^μ — “Weyl current”

Weyl gravity II

$$\delta g_{\mu\nu} = \frac{2}{\Omega} g_{\mu\nu}(\delta\Omega) + \Omega^2(\delta\hat{g}_{\mu\nu})$$

$$\delta A_\mu = (\delta A_\mu) + 2(\delta(\log\Omega),_\mu)$$

$$\delta\hat{g}_{\mu\nu} : \frac{\delta S_m}{\delta\hat{g}_{\mu\nu}} \stackrel{\text{def}}{=} -\frac{1}{2}\hat{T}_{\mu\nu} \Rightarrow \hat{T}_{\mu\nu} = \Omega^2 T_{\mu\nu}$$

$$(\hat{T}_\mu^\nu = \Omega^4 T_\mu^\nu, \hat{T}^{\mu\nu} = \Omega^6 T_{\mu\nu})$$

$$\delta\hat{A}_\mu : \frac{\delta S_m}{\delta\hat{A}_\mu} \stackrel{\text{def}}{=} -\hat{G}^\mu \Rightarrow \hat{G}^\mu = \Omega^4 G^\mu$$

Self-consistency condition

$$\delta\Omega : 2G_{;\mu}^\mu = \text{Trace}[T^{\mu\nu}]$$

";" — covariant derivative with $C_{\mu\nu}^\lambda$

Perfect fluid, Riemann geometry I

J.R. Ray (1972):

$$S_m = - \int \varepsilon(X, n) \sqrt{-g} d^4x + \int \lambda_0(u_\mu u^\mu - 1) \sqrt{-g} d^4x \\ + \int \lambda_1(nu^\mu)_{;\mu} \sqrt{-g} d^4x + \int \lambda_2 X_{,\mu} u^\mu \sqrt{-g} d^4x$$

$\varepsilon(X, n)$ — energy density

$u^\mu(x)$ — four-velocity

$n(x)$ — particle number density,

$X(x)$ — auxiliary variable,

$\lambda_i(x)$ — Lagrange multipliers

Constraints

$u^\mu u_\mu = 1$ — normalization

$(nu^\mu)_{;\mu} = 0$ — particle number conservation

$X_{,\mu} u^\mu = 0$ — numbering of the trajectories

Perfect fluid, Riemann geometry II

Energy-momentum tensor

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - p g^{\mu\nu}$$

Hydrodynamical pressure

$$p = n \frac{\partial \varepsilon}{\partial n} - \varepsilon$$

V.A.Berezin (1987):

Phenomenological description of particle creation

$$(n u^\mu)_{;\mu} = 0 \quad \Rightarrow \quad (n u^\mu)_{;\mu} - \Phi(inv) = 0$$

Perfect fluid, Weyl geometry: G^μ — ? How to incorporate A_μ ?



New possibilities

Single particle u^μ

Riemann geometry

$$S_{\text{part}} = -m \int ds$$

Weyl geometry — new invariant $B = A_\mu u^\mu$

$$S_{\text{part}} = \int f_1(B) ds + \int f_2(B) d\tau = \int \{ f_1(B) \sqrt{g_{\mu\nu} u^\mu u^\nu} + f_2(B) \} d\tau$$

Equations of motion

$$\begin{aligned} f_1(B) u_{\lambda;\mu} u^\mu &= \left((f'_1(B) + f''_2(B)) A_\lambda - f'_1(B) u_\lambda \right) B_{,\mu} u^\mu \\ &+ (f'_1(B) + f'_2(B)) F_{\lambda\mu} u^\mu \end{aligned}$$

Since $F_{\lambda\mu} u^\lambda u^\mu \equiv 0$ and $u_{\lambda;\sigma} u^\lambda \equiv 0$

Either $(f'_1 + f''_2)B = f'_1$ or $B_{,\mu} u^\mu = 0$

The invariant $B = A_\mu u^\mu$ is closely tied to the number density n .
Hence,

$$\varepsilon(X, n) \Rightarrow \varepsilon(X, \varphi(B)n)$$

And how about the constraint $(nu^\mu)_{;\mu} = \Phi(inv)$?
Conformal transformation

$$(nu^\mu)_{;\mu} \sqrt{-g} = (nu^\mu \sqrt{-g})_{,\mu}$$

$$n = \frac{\hat{n}}{\Omega^3}, \quad u^\mu = \frac{\hat{u}^\mu}{\Omega}, \quad \sqrt{-g} = \Omega^4 \sqrt{-\hat{g}} \Rightarrow$$

$$(nu^\mu \sqrt{-g})_{,\mu} = (\hat{n}\hat{u}^\mu \sqrt{-\hat{g}})_{,\mu} \Rightarrow$$

$\Phi \sqrt{-g}$ — conformal invariant \Rightarrow

In the absence of the classical fields, i.e., in the case of the particle creation by the vacuum fluctuations

$$\Phi = \alpha'_1 R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + \alpha'_2 R_{\mu\nu} R^{\mu\nu} + \alpha'_3 R^2 + \alpha'_4 F_{\mu\nu} F^{\mu\nu}$$

Riemann geometry — the only possibility

$$\Phi \propto C^2$$

$C^2 = C_{\mu\nu\lambda\sigma} C^{\mu\nu\lambda\sigma}$ — square of the Weyl tensor.

Y.B.Zel'dovich and A.A.Starobinsky (1977):

Particle creation by the vacuum fluctuation of the massless scalar field on the homogeneous slightly anisotropic cosmological background.

Now it becomes fundamental!

Scalar field

1

$$\begin{aligned} n &\rightarrow f(B)n \rightarrow f(B)m(\varphi)n \Rightarrow \\ \varepsilon(X, n) &\rightarrow \varepsilon(X, z) \end{aligned}$$

$$z = f(B)m(\varphi)n$$

$$\varepsilon + p = +n \frac{\partial \varepsilon}{\partial n} = z \frac{\partial \varepsilon}{\partial z}$$

2

$$(nu^\mu)_{;\mu} = \Phi(\text{inv})$$

$$\Phi = \alpha'_1 R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + \alpha'_2 R_{\mu\nu} R^{\mu\nu} + \alpha'_3 R^2 + \alpha'_4 F_{\mu\nu} F^{\mu\nu} + \Theta(\varphi^2 R + \Lambda \varphi^4)$$

Induced gravity $= S_m$ and nothing more $\Rightarrow G^\mu = 0, T^{\mu\nu} = 0$

Self-consistency condition

$$\begin{aligned} S_m = & -\int \varepsilon \sqrt{-g} d^4x + \int \lambda_0 (u_\mu u^\mu - 1) \sqrt{-g} d^4x \\ & + \int \lambda_1 ((nu^\mu)_{;\mu} - \Phi) \sqrt{-g} d^4x + \int \lambda_2 X_{,\mu} u^\mu \sqrt{-g} d^4x \end{aligned}$$

Only the first integral is not explicitly conformal invariant

$$\varepsilon - 3p = 2 \left((\varepsilon + p) \frac{1}{f} \frac{df}{dB} u^\sigma \right)_{;\sigma} + (\varepsilon + p) \left(\frac{B}{f} \frac{df}{dB} + \frac{\varphi}{m} \frac{dm}{d\varphi} \right)$$

Equation for $\delta\varphi$

$$(\varepsilon + p) \frac{1}{m} \frac{dm}{\varphi} = 2\lambda_1 \Theta(\varphi R - 2\Lambda\varphi^3)$$

Cosmology I

Cosmology = homogeneity and isotropy
Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \right), \quad k = 0, \pm 1$$

$$A_\mu = (A_0(t), 0, 0, 0) \quad \Rightarrow$$

$$F_{\mu\nu} \equiv 0$$

$$T_\nu^\mu = (T_0^0(t), T_1^1(t) = T_2^2 = T_3^3)$$

Special gauge

$$A_0(t) = 0$$

Cosmology II

In cosmology Weyl tensor equals zero

$$\begin{aligned} C_{\mu\nu\lambda\sigma} &= R_{\mu\nu\lambda\sigma} - \frac{1}{2}(R_{\mu\lambda}g_{\nu\sigma} - R_{\nu\sigma}g_{\mu\lambda} + R_{\mu\sigma}g_{\nu\lambda} - R_{\nu\lambda}g_{\mu\sigma}) \\ &\quad + \frac{1}{6}R(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda}) \end{aligned}$$

$$C^2 = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$$

$$\begin{aligned} \lambda'_1 R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} + \lambda'_2 R_{\mu\nu}R^{\mu\nu} + \lambda'_3 R^2 &= \alpha C^2 + \beta GB + \gamma R^2 \\ &= \beta GB + \gamma R^2 \end{aligned}$$

Gauss-Bonnet:

$$GB = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 = -2R_{\mu\nu}R^{\mu\nu} + \frac{2}{3}R^2$$

Field equations I

$$\frac{\delta S_m}{\delta g^{\mu\nu}} = -\frac{1}{2} T^{\mu\nu}$$

$$\frac{\delta S_m}{\delta A_\mu} = -G^\mu = 0$$

$$G^\mu = G^\mu[\text{part}] + G^\mu[\text{cr}] + G^\mu[\varphi]$$

$$G^\mu[\text{part}] = -(\varepsilon + p) \frac{1}{f} \frac{df}{dB} u^\mu$$

$$G^\mu[\text{cr}] = 4\beta\lambda_{1;\sigma} R^{\mu\sigma} - 2(\beta + 3\gamma)\lambda_1{}^{;\mu} R - 6\gamma\lambda_1 R^{;\mu}$$

$$G^\mu[\varphi] = 3\Theta(\lambda_1\varphi^2)^{;\mu}$$

$$G^0[\text{part}] = -(\varepsilon + p) \frac{1}{f} \frac{df}{dB}$$

$$G^0[\text{cr}] = 2\beta\dot{\lambda}_1(2R_0^0 - R) - 6\gamma(\lambda_1 R)$$

$$G^0[\varphi] = 3\Theta(\lambda_1\varphi^2)$$

Field equations II

$$T^{\mu\nu} = T^{\mu\nu}[\text{part}] + T^{\mu\nu}[\text{cr}] + T^{\mu\nu}[\varphi]$$

$$T = \text{trace } T^{\mu\nu}, \quad T_0^0, \quad (T_1^1 = \frac{1}{3}(T - T_0^0))$$

$$T_0^0[\text{part}] = \varepsilon, \quad T[\text{part}] = \varepsilon - 3\rho$$

$$T_0^0[\text{cr}] = 8\beta\dot{\lambda}_1\frac{\dot{a}}{a}R_0^0 - (4\beta + 3\gamma)\dot{\lambda}_1\frac{\dot{a}}{a}R - \gamma\lambda_1 \left(12\frac{\dot{a}}{a}\dot{R} + R(4R_0^0 - R) \right)$$

$$\begin{aligned} T[\text{cr}] &= \ddot{\lambda}_1(8\beta R_0^0 - 4\beta R - 12\gamma R) - 4\dot{\lambda}_1 \left(\beta\frac{\dot{a}}{a}(R + 2R_0^0) + 6\gamma\dot{R} + 9\gamma\frac{\dot{a}}{a}R \right) \\ &\quad - 12\gamma\lambda_1 \left(\ddot{R} + 3\frac{\dot{a}}{a}\dot{R} \right) \end{aligned}$$

$$T_0^0[\varphi] = \Theta \left\{ \lambda_1\varphi^2(2R_0^0 - R) + \lambda_1\Lambda\varphi^4 + 6\frac{\dot{a}}{a}(\lambda_1\varphi^2)\dot{} \right\}$$

$$T[\varphi] = 2\Theta \left\{ -\lambda_1\varphi^2R + 2\lambda_1\Lambda\varphi^4 + \frac{3}{a^3}(\lambda_1\varphi^2a^3)\dot{} \right\}$$

Field equations III

$$R_0^0 = -3\frac{\ddot{a}}{a}$$

$$R = -6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} \right), \quad k = 0, \pm 1$$

Pregnant vacuum

$$n = \dot{n} = 0 \Rightarrow \Phi = 0 \Rightarrow$$

$$\frac{8}{3}\beta(R - 2R_0^0) - \gamma R^2 = \Theta(\Lambda\varphi^4 - \varphi^2 R)$$

$$G^0[\text{part}] = 0, \quad T_0^0[\text{part}] = 0, \quad T[\text{part}] = 0$$

$$\lambda_1\varphi(R - 2\Lambda\varphi^2) = 0$$

$$\frac{\varepsilon + p}{n} = -\dot{\lambda}_1$$

- ① $\lim_{n \rightarrow 0} = \infty \rightarrow \text{no vacuum solutions} \rightarrow \text{Birth? Dark energy?}$
- ② $\lim_{n \rightarrow \infty} = 0 \rightarrow \lambda_1 = \text{const} \rightarrow \text{non-dust pregnancy}$
- ③ $\lim_{n \rightarrow 0} \neq 0, \infty \rightarrow \text{dust pregnancy}, f(0)m(\varphi) = -\dot{\lambda}_1, B = 0$

Non-dust pregnancy I

$$\dot{\lambda}_1 = 0$$

$$G^0 = 0 \quad \Rightarrow \quad 6\lambda_1(-\gamma\dot{R} + \Theta\varphi\dot{\varphi}) = 0$$

$$T_0^0 = 0 \quad \Rightarrow \quad \lambda_1 \left\{ -\gamma \left((12\frac{\dot{a}}{a}\dot{R}) + R(4R_0^0 - R) \right) \right.$$

$$\left. + \Theta \left(\varphi(2R_0^0 - R) + \Lambda\varphi^4 + 12\frac{\dot{a}}{a}\varphi\dot{\varphi} \right) \right\} = 0$$

$$T = 0 \quad \Rightarrow \quad \lambda_1 \left\{ -12\gamma\frac{(a^3\dot{R})}{a^3} + 2\Theta \left(-\varphi^2R + 2\Lambda\varphi^4 + \frac{3}{a^3}(\varphi^2a^3)\dot{} \right) \right\}$$

$$\Phi = 0 \quad \Rightarrow \quad \frac{8}{3}\beta(R - 2R_0^0) - \gamma R^2 = \Theta(\Lambda\varphi^4 - \varphi^2R)$$

$$(\delta\varphi) \quad \Rightarrow \quad \lambda_1\varphi(R - 2\Lambda\varphi^2)$$

Non-dust pregnancy II

- ① $\lambda_1 = 0 \Rightarrow$ a lot of vacua with any scale factor $a(t)$
- ② $\lambda_1 = \text{const} \neq 0 \Rightarrow \dot{R} = 0, \dot{\varphi} = 0$ — there are solutions.
But!

Dust pregnancy

$$p = 0 \quad \Rightarrow \quad \dot{\lambda}_1 = -f(0)m(\varphi)$$

- ① Unusual $\dot{\lambda}_1 = 0 \quad \Rightarrow \quad m(\varphi) = 0 \quad \Rightarrow \quad \varphi = \varphi_0 = \text{const}$

$$G^0 = 0 \quad \Rightarrow \quad \lambda_1 \dot{R} = 0$$

$$T_0^0 = 0 \quad \Rightarrow \quad \lambda_1 \left\{ -\gamma R(4R_0^0 - R) + \Theta(\varphi_0(2R_0^0 - R) + \Lambda\varphi_0^4) \right\} = 0$$

$$T = 0 \quad \Rightarrow \quad \lambda_1 \left\{ -12\gamma \ddot{R} + 2\Theta(-\varphi_0^2 R + 2\Lambda\varphi_0^4) \right\}$$

$$\Phi = 0 \quad \Rightarrow \quad \frac{8}{3}\beta(R - 2R_0^0) - \gamma R^2 = \Theta(\Lambda\varphi_0^4 - \varphi_0^2 R)$$

$$(\delta\varphi) \quad \Rightarrow \quad \lambda_1 \Theta(\varphi_0 R - 2\Lambda\varphi_0^3) = 0$$

- ② $\lambda_1 = 0 \quad \Rightarrow \quad \text{only } \Phi = 0 \quad \text{— solution exists}$
- ③ $\lambda_1 = \text{const} \neq 0 \quad \Rightarrow \quad \dot{R} = 0 \quad \text{— Milne } (R = R_0^0 = 0) \text{ or}$
 $R = \text{const} \neq 0, R_0^0 = \text{const} \neq 0 \quad \text{— special parameters}$
- ④ $\dot{\lambda}_1 \neq 0 \quad \Rightarrow \quad \text{Milne } (R = R_0^0 = 0), \text{ De Sitter } (R = 4R_0^0) \quad \text{—}$
Otherwise: violent birth of dust particles (homogeneous and isotropic!). Dark matter?

The End