Models in Quantum Field Theory (MQFT) 2022

Convergent perturbation theory in field models of statistical physics

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Outline

Introduction

Divergent Perturbation Theory

Divergent series in QFT = infinite single terms + divergence of the total sum.

Individual infinite diagrams:

1. Ultraviolet problem
   Standard solution: regularization and some renormalization scheme.
2. Infrared problem
   Standard solution: renormalization group (RG)

Infinite series sum: power series of perturbation theory are asymptotic.

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Asymptotic nature of perturbation theory

Asymptotic series in QFT:

$$G = \sum_N G^{(N)} g^N, \quad G^{(N)} \sim N!(-A)^N N^B C, \quad N \to \infty, \quad A, B, C < \infty.$$

Standard solution: various resummation procedures.

How is perturbation theory constructed in QFT?

Typical object: $\int \mathcal{D}[\Phi]\, e^{-S_0 - gS_{\text{int}}} = \int \mathcal{D}[\Phi] e^{-S_0} \sum_{N=0}^{\infty} \frac{(-g)^N}{N!} (S_{\text{int}})^N \Rightarrow \sum_{N=0}^{\infty} \int \mathcal{D}[\Phi] \ldots$

What is the reason for the divergence?

In all realistic field theories we have $S_0 \geqslant 0$ and $|S_{\text{int}}| \gg |S_0|$ (Dyson's argument).

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Convergent perturbation theory

When might perturbation theory converge?

A sufficient condition for the analyticity of the correlator in the vicinity of $g = 0$ is

$$|S_0| \gg |S_{\text{int}}| \qquad \text{(A. G. Ushveridze, 1982)} \tag{1}$$

General idea of the method

Choose in some way new $\mathcal{S}_0$ and $\mathcal{S}_{\text{int}}$ and also a new expansion parameter ζ in such a way that

1. Condition (1) is fulfilled,
2. New action $\mathcal{S}_0 + \zeta\, \mathcal{S}_{\text{int}}$ coincides with $S_0 + g S_{\text{int}}$ at some point $\zeta = \zeta_0$.
3. Convergence radius of the new perturbation theory is larger than $|\zeta_0|$.

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General remarks

- By proper choice of $\mathcal{S}_0$ and $\mathcal{S}_{\text{int}}$ it is possible to achieve a perturbation theory without any spurious divergences (so-called superconvergent) (A. G. Ushveridze, 1984).
- In a large class of models, it is possible to choose $\mathcal{S}_0$ and $\mathcal{S}_{\text{int}}$ in such a way as to represent the perturbation theory with respect to ζ by diagrams of the old (divergent) series (see e.g. A. G. Ushveridze, 1983). In this case, the proposed method works like some sort of resummation procedure (A. G. Ushveridze, 1982).

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Main results

General requirements for the field model

The most general class of models where it is possible to construct a convergent perturbation theory based on ordinary (divergent) diagrams can be described by the following action

$$S = S_0[\Phi] + S_{\text{int}}[\Phi, \{g_\sigma\}],$$

where

- $\Phi \equiv \{\phi, \psi, \chi, \ldots\} = \{\Phi^{(\alpha)}\}_{\alpha=1}^{n}$. All fields from the set $\{\Phi^{(\alpha)}\}$ are bosonic.
- $\{g_\sigma\}$ is a set of coupling constants
- MS-scheme, $D - \varepsilon$ regularization ($d^D k \to \mu^\varepsilon d^{D-\varepsilon} k$)
- $S_0[\Phi]$ is a massless free action, $S_{\text{int}}[\Phi, \{g_\sigma\}]$ is a λ-homogeneous functional, i.e.

$$S_{\text{int}}[\{z^{\lambda_\alpha}\Phi^{(\alpha)}\}, \{g_\sigma\}] = z S_{\text{int}}[\{\Phi^{(\alpha)}\}, \{g_\sigma\}], \qquad z \in \mathbb{C}.$$

- $\exists$ a nontrivial solution $\{\Phi_{st}^{(\alpha)}, g_{st}\}$ of the following system

$$\begin{cases} \delta S / \delta \Phi^{(\alpha)} = 0, \quad \alpha = 1, \ldots, n, \\ S_{\text{int}}[\{\Phi^{(\alpha)}\}, \{g_\sigma(g_{st})\}] = -g_{st}^2/2. \end{cases}$$

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Which physical models belong to the class described above?

The main requirement is the existence of an instanton solution of the equations of motion in the model under consideration.

- Euclidean field theories ϕ^n
  See e.g. (B. Shaverdyan et at., 1983)
- Vector model ϕ^4 with cubic symmetry (two-couplings)
  See e.g. (M. Nalimov et al., 2020)
- Standard ϕ^4-based models from A to H (dynamics and statics)
  See e.g. (J. Honkonen, et al., 2005, two papers)
- Kraichnan model of passive scalar advection in turbulent flow
  See e.g. (J. Andreanov, et al., 2006)

Construction of convergent perturbation theory

Replacement $\Phi^{(\alpha)} \longrightarrow z^{\lambda_\alpha}\Phi^{(\alpha)}$ selects in $S_0[\Phi]$ two terms $S_0[\Phi] = S_{01}[\Phi] + S_{02}[\Phi]$, where

$$S_{01}[\{z^{\lambda_\alpha}\Phi^{(\alpha)}\}] \equiv \sum_{k=1}^{m} z^{\Delta_k} S_{01}^{(k)}[\{\Phi^{(\alpha)}\}], \ \Delta_k < 1, \quad \forall k,$$

$$S_{02}[\{z^{\lambda_\alpha}\Phi^{(\alpha)}\}] = zS_{02}[\{\Phi^{(\alpha)}\}]$$

Proposed action restructuring scheme

$$\mathcal{S}(\zeta) = S_{01}[\Phi] + S_{02}[\Phi] + \zeta S_{\text{int}}[\Phi, \{g_\sigma\}] + a(1-\zeta)\sum_{k=1}^{m}\left(S_{01}^{(k)}[\Phi]\right)^{1/\Delta_k}, \quad a > 0.$$

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Main points of the proof of convergence

The proof of the convergence of the series with respect to ζ is based on the instanton analysis of the asymptotics of the higher order expansion coefficients.

Nth term: $$\frac{1}{2\pi i}\oint\limits_{\gamma}\frac{du}{u}\int\mathcal{D}[\Phi]\ \ldots\ e^{-S_0[\Phi]-S_{\text{int}}[\Phi,\{g_\sigma\}]}e^{-N\ln u} \quad\Longrightarrow\quad \text{saddle-point method.}$$

At the fixed point g_σ are proportional to some powers of $\varepsilon \Longrightarrow g_\sigma = u^{q(\sigma)}\bar{g}_\sigma,\ q(\sigma)\in\mathbb{N}$.

Standard perturbation theory (by variable u)

After changing of variables $\Phi^{(\alpha)} \longrightarrow \Phi^{(\alpha)}/u$ and $u \longrightarrow u/\sqrt{N}$, the Nth term becomes proportional to $(N!)^{1/2}$ and the stationarity equations take the form

$$\begin{cases}\dfrac{\delta}{\delta\Phi^{(\alpha)}}\Big(S_0+S_{\text{int}}\Big)=0,\\ S_0+S_{\text{int}}=-u^2/2.\end{cases}$$

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Replacement $\Phi^{(\alpha)} \longrightarrow N^{\lambda_\alpha}\Phi^{(\alpha)}/u$ in the action $\mathcal{S}(\zeta)$ leads to the new stationarity equations

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Stationarity equations for $\mathcal{S}(\zeta)$ in detailed notation

$$\{\Phi_{st}^{(\alpha)}, u_{st}\} \Longrightarrow \begin{cases} \dfrac{\delta}{\delta\Phi^{(\alpha)}}\left(\sum_k \left(u^{-2} S_{01}^{(k)}[\Phi]\right)^{1/\Delta_k} + S_{02}[\Phi] + \dfrac{\zeta}{u^2}\dfrac{S_{\text{int}}[\Phi, \{\bar{g}_\sigma\}]}{a(1-\zeta)}\right) = 0, & (\heartsuit) \\ S_{\text{int}}[\Phi, \{\bar{g}_\sigma\}] - a\sum_k \left(S_{01}^{(k)}[\Phi]\right)^{1/\Delta_k} = -u^2/\zeta. & (\diamondsuit) \end{cases}$$

Equation ($\heartsuit$) is invariant under replacement $\Phi_{st}^{(\alpha)} \longrightarrow z^{\lambda_\alpha}\Phi_{st}^{(\alpha)}$. In turn, the left-hand side of ($\diamondsuit$) is multiplied by z with such a replacement, i.e. we can always satisfy ($\diamondsuit$).

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Then, the equation (♡) can be satisfied by an appropriate choice of ζ

$$\zeta_{st}=\frac{1}{1+B/aA(u_{st})}$$

The largest contribution to the asymptotics of $G^{(N)}$, $N\to\infty$ is given by $\boxed{\sim e^{N\ln\zeta_{st}}=\zeta_{st}^N}$

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Convenient representation for correlators

The representation of the $2s$-point correlator using the known diagrams of the standard (divergent) perturbation theory is based on the following chain of identities

$$\prod_{k=1}^{m} e^{-a(1-\zeta)\left(S_{01}^{(k)}\right)^{1/\Delta_k}} \theta\left(S_{01}^{(k)}\right) = \prod_{k=1}^{m} \int_{-\infty}^{\infty} \mathrm{d}y_k \delta(y_k - S_{01}^{(k)}) \theta(y_k) e^{-a(1-\zeta) y_k^{1/\Delta_k}} =$$

$$\prod_{k=1}^{m} \int_{0}^{\infty} \mathrm{d}y_k \int_{-\infty}^{\infty} \mathrm{d}y'_k e^{-a(1-\zeta) y_k^{1/\Delta_k} + i y'_k \left(y_k - S_{01}^{(k)}\right)}$$

$$\Downarrow$$

$$\boxed{G_s(\{\bar{g}_\sigma\}, \zeta, a) = \prod_{k=1}^{m} \int_{-\infty}^{\infty} \mathrm{d}y'_k \int_{0}^{\infty} \mathrm{d}y_k \frac{e^{-a(1-\zeta) {y_k}^{1/\Delta_k} + i y'_k y_k}}{\left(1 + i y'_k\right)^{r(s,k)/2}} G_s\left(\left\{\frac{\zeta \bar{g}_\sigma}{\left(1 + i y'_k\right)^{\Delta_k/2}}\right\}\right)}$$

UV and IR behavior in convergent approach

- UV problem

$$Z_e = 1 + \frac{Z_e^1}{\varepsilon} + O\left(\frac{1}{\varepsilon^2}\right), \; Z_e \to \mathscr{Z}_e = Z_e\left(\left\{\frac{\zeta \bar{g}_\sigma}{\left(1 + i y_k'\right)^{\Delta_k/2}}\right\}\right), \; e = (\{\Phi^{(\alpha)}\}, \{\bar{g}_\sigma\}, \text{etc.})$$

- IR problem
  RG equation in our approach

$$\left[\mu\partial_\mu + \sum_{n=1}^{\infty} \beta_n^\zeta \partial_g^n - \gamma_\tau^\zeta \tau \partial_\tau + \gamma_s^\zeta\right] G_s^R(\zeta) = 0.$$

$$G_s^R(\zeta = 1) = G_s^R \quad \Longrightarrow \quad \text{at} \quad \zeta = 1 \quad \beta_n^\zeta = 0 \quad \text{for all} \quad k > 1.$$

UV and IR behavior in convergent approach

- UV problem

$$Z_e = 1 + \frac{Z_e^1}{\varepsilon} + O\left(\frac{1}{\varepsilon^2}\right), \; Z_e \to \mathscr{Z}_e = Z_e\left(\left\{\frac{\zeta \bar{g}_\sigma}{\left(1 + i y_k'\right)^{\Delta_k/2}}\right\}\right), \; e = (\{\Phi^{(\alpha)}\}, \{\bar{g}_\sigma\}, \text{etc.})$$

- IR problem

  RG equation in our approach

$$\left[\mu\partial_\mu + \sum_{n=1}^{\infty} \beta_n^\zeta \partial_g^n - \gamma_\tau^\zeta \tau \partial_\tau + \gamma_s^\zeta\right] G_s^R(\zeta) = 0.$$

$$G_s^R(\zeta = 1) = G_s^R \quad \Longrightarrow \quad \text{at} \quad \zeta = 1 \quad \beta_n^\zeta = 0 \quad \text{for all} \quad k > 1.$$

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$$\Downarrow$$

$$\left[\mu\partial_\mu + \beta_g^\zeta \partial_g - \gamma_\tau^\zeta \tau \partial_\tau + \gamma_k^\zeta\right] G_{2k}^{S_R(\zeta)}(\zeta) = 0, \quad \beta_g^\zeta \equiv \beta_1$$

Thanks for your attention

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