Dynamic critical behaviour of superfluid phase transition

Iurii Molotkov

JINR, BLTP

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Critical Slowing Down and stochastic dynamics

Critical slowing down is considered to be an important indicator for predicting critical transitions in dynamical systems:

$$t_{relax} \sim \xi^z \sim |T - T_c|^{-\nu z} \to \infty \tag{1}$$

Phenomenological stochastic fields theory are often based on the form of Langevin's equations:

$$\partial_t \varphi_a = (\alpha_{ab} + \beta_{ab}) \frac{\delta S^{st}(\varphi)}{\delta \varphi_b} + \eta_a, \langle \eta_a(x) \eta_b(x') \rangle = 2\alpha_{ab} \delta(x - x'), \tag{2}$$

where α_{ab} - Onsager's coefficients, β_{ab} – intermode coupling coefficients

Time-dependent Green's function at finite temperature

Grand-canonical expectation value of the time-ordered product of Heisenberg field operators is the definition used here for the time-dependent Green functions at finite temperature:

$$G_n(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathrm{Tr}\,\{\hat{\rho}\, T\,[\hat{\varphi}_H(\mathbf{x}_1)\hat{\varphi}_H(\mathbf{x}_2)\ldots\hat{\varphi}_H(\mathbf{x}_n)]\}, \tag{3}$$

$$\mathbf{x} = (x, t)$$

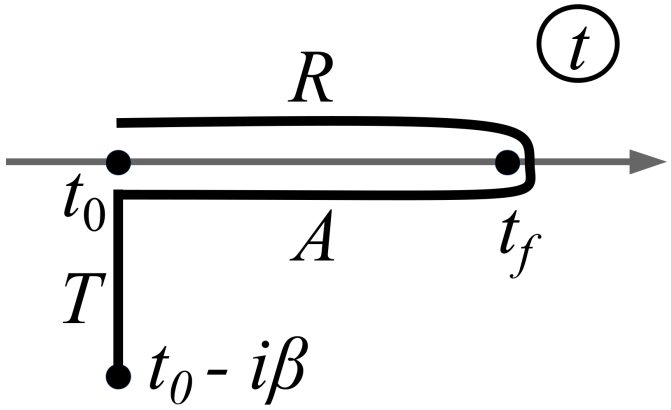
$$G_n(x_1, x_2, \ldots, x_n) = \frac{Z_0}{Z_G}\Bigg\{\exp\Bigg[\frac{1}{2}\sum_{l=1}^{4}\frac{\delta}{\delta\varphi_l}\Delta_{ll}\frac{\delta}{\delta\varphi_l} + \sum_{k<l}\frac{\delta}{\delta\varphi_k}n_{kl}\frac{\delta}{\delta\varphi_l}$$

$$+\frac{1}{2}\sum_{k,l=1}^{4}\frac{\delta}{\delta\varphi_k}d_{kl}\frac{\delta}{\delta\varphi_l}\Bigg]\varphi_3(x_1)\cdots\varphi_3(x_n)\,\exp\Bigg[-\frac{i}{\hbar}\int_{t_i}^{t_f}V_n(\varphi_3)\,dt + \frac{i}{\hbar}\int_{t_i}^{t_0}V_n(\varphi_2)\,dt$$

$$-\frac{1}{\hbar}\int_0^{\beta\hbar}V_n(\varphi_1)\,dt + \frac{i}{\hbar}\int_{t_0}^{t_f}V_n(\varphi_4)dt\Bigg]\Bigg\}\Bigg|_{\varphi_i=0}, \tag{4}$$

J. Honkonen, M. V. Komarova, Y. G. Molotkov, and M. Y. Nalimov, Nucl. Phys. B 939, 105 (2019)

Path integral formalism



The generating functional of the Green function can be written as

$$\int D\psi^+ D\psi e^{iS+A\psi^++A^+\psi}, \tag{5}$$

where the action:

$$S = \int\limits_C dt \int dx \left(\psi^+ \partial\psi - H\right) \tag{6}$$

Y. A. Zhavoronkov, M. V. Komarova, Y. G. Molotkov, M. Y. Nalimov, and J. Honkonen, Theor. Math. Phys. 200, 1237–1251 (2019).

Hamiltonian and propagators

We are investigating a non-relativistic gas of scalar bosons with a local repulsive density-density interaction near the critical point of condensation. Thus, the Hamilton operator is chosen in the form:

$$\hat{H} = \int d\mathrm{x} \left[\hat{\psi}^{+}(t,\mathrm{x}) \left(-\frac{\hbar^2 \nabla^2}{2m} - \mu \right) \hat{\psi}(t,\mathrm{x}) + \frac{g}{4} \hat{\psi}^{+2}(t,\mathrm{x}) \hat{\psi}^{2}(t,\mathrm{x}) \right] \tag{7}$$

where ψ, ψ^{+} - annihilation and creation field operators consisting of operator of creation and annihilation operators with usual commutators, μ - chemical potential, m - mass of a particle, g - coupling constant. The propagators have the following form:

$$G_{RR} = e^{-i\varepsilon(t-t')}(\Theta(t-t') + n_\varepsilon), \qquad G_{AA} = e^{-i\varepsilon(t-t')}(\Theta(t'-t) + n_\varepsilon),$$

$$G_{TT} = e^{-i\varepsilon(\tau-\tau')}(\Theta(\tau-\tau') + n_\varepsilon),$$

$$G_{RA} = e^{-i\varepsilon(t-t')} n_\varepsilon, \qquad G_{AR} = e^{-i\varepsilon(t-t')}(n_\varepsilon + 1),$$

$$G_{RT} = G_{AT} = e^{-i\varepsilon(t-t_0+i\tau')} n_\varepsilon, \qquad G_{TR} = G_{TA} = e^{-i\varepsilon(t_0-i\tau-t')}(n_\varepsilon + 1),$$

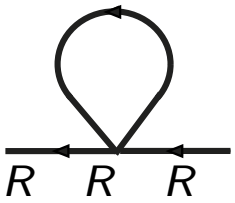
where $\tau \equiv -\mathrm{Im}\, t$, $n_\varepsilon = (e^{\beta\varepsilon} - 1)^{-1}$, $\varepsilon \equiv k^2/(2m_0) - \mu$, k – momentum, Θ – Heaviside step function.

Divergences and their regularization

Let us consider two-point Green's function

$$G_2(x_1, x_2) = \mathrm{Tr}\left\{\hat{\rho}_G\, T\left[\hat{\psi}_H(x_1)\hat{\psi}_H^+(x_2)\right]\right\} \tag{8}$$

First order correction to the Green's function:



$$= -ig\int d\mathrm{p}\,[\theta(0) + n_p]\int_{t_i}^{t_f} dt\,\exp\left[-i\epsilon(\mathrm{k})(t_1 - t)\right][\theta(t_1 - t) + n_k]$$

$$\times \exp\left[-i\epsilon(t - t_2)\right][\theta(t - t_2) + n_k] = -ig\int d\mathrm{p}\, n_p\,\exp\left[-i\epsilon(t_1 - t_2)\right]$$

$$\times\left[(t_1 - t_2) + (t_1 - t_i + t_f - t_2)n_k + (t_f - t_i)n_k^2\right].$$

The linear growth in time $t_1 - t_2$ obtained here is not a divergence, but nevertheless it is unusual behaviour of the loop correction. Oscillating time dependence occurs in other graphs as well with the subsequent generation of apparent divergences. It is utterly annoying to collect all the relevant terms prior passing to the limit of whole time axis. Therefore, one can find it convenient to regularize these IR divergences so that each separate has a finite limit, when $t_f \to \infty$, $t_i \to -\infty$.

Divergences and their regularization

It is convenient to use a different set of Green function. To this end, let us introduce a new set of fields:

$$\begin{pmatrix} \eta \\ \eta^+ \\ \xi \\ \xi^+ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_R - \psi_A \\ \psi_R^+ - \psi_A^+ \\ \psi_R + \psi_A \\ \psi_R^+ + \psi_A^+ \end{pmatrix} . \tag{9}$$

Regularized propagators:

$$\langle \xi\, \xi^+ \rangle_{\mathbf{0}} = e^{-i\varepsilon(t-t')-\gamma|t-t'|}\,(2n_\varepsilon + 1)\,, \qquad \langle \eta\, \eta^+ \rangle_{\mathbf{0}} = 0,$$

$$\langle \psi_T\, \psi_T^+ \rangle_{\mathbf{0}} = e^{-\varepsilon(\tau-\tau')}(\Theta(\tau - \tau') + n_\varepsilon),$$

$$\langle \xi\, \eta^+ \rangle_{\mathbf{0}} = \Theta(t - t')e^{-i\varepsilon(t-t')-\gamma|t-t'|}\,, \qquad \langle \eta\, \xi^+ \rangle_{\mathbf{0}} = -\Theta(t' - t)e^{-i\varepsilon(t-t')-\gamma|t-t'|}\,,$$

$$\langle \psi_T\, \xi^+ \rangle_{\mathbf{0}} = \sqrt{2}e^{-i\varepsilon(t_0-i\tau-t')-\gamma|t_0-t'|}(n_\varepsilon + 1), \qquad \langle \xi\, \psi_T^+ \rangle_{\mathbf{0}} = \sqrt{2}e^{-i\varepsilon(t-t_0+i\tau)-\gamma|t-t_0|}\,n_\varepsilon\,.$$

If we put t_0 (or choose ant other finite value), them the vertices of the density operator field remain connected to others and we are left with the full perturbation theory. For problems with time scales of the order of relaxation time this is appropriate choice. Here, we are interested in the situation at times much larger that the relaxation time and in this case it is reasonable to send the reference time t_0 to $-\infty$

IR effective field theory

This leads to propagators of the effective IR theory in dimensionless variables in the form:

$$
\begin{aligned}
&\langle \xi(t,\mathrm{k})\xi^{+}(t',-\mathrm{k})\rangle_0 = \Delta^{K}_{\mathrm{IR\,reg}}(t,\mathrm{k}) = \frac{2}{k^2}e^{-i\alpha u k^{\mathbf{2}}(t-t')-\alpha k^{\mathbf{2}}|t-t'|},\\
&\langle \eta(t,\mathrm{k})\eta^{+}(t',-\mathrm{k})\rangle_0 = 0,\\
&\langle \xi(t,\mathrm{k})\eta^{+}(t',-\mathrm{k})\rangle_0 = \Delta_{reg}(t,\mathrm{k}) = \theta(t-t')e^{-i\alpha u k^{\mathbf{2}}(t-t')-\alpha k^{\mathbf{2}}|t-t'|},\\
&\langle \eta(t,\mathrm{k})\xi^{+}(t',-\mathrm{k})\rangle_0 = -\widetilde{\Delta}_{reg}(t,\mathrm{k}) = -\theta(t'-t)e^{-i\alpha u k^{\mathbf{2}}(t-t')-\alpha k^{\mathbf{2}}|t-t'|},
\end{aligned}
\tag{10}
$$

Bare action:

$$
\begin{aligned}
S = {} & 4\eta\alpha\eta^{+} + \eta^{+}(\partial_t - i\alpha u\Delta - \alpha\Delta)\xi + \xi^{+}(\partial_t - i\alpha u\Delta + \alpha\Delta)\eta + \\
& + \frac{ig_1\alpha}{2}\eta^{+}\xi^{+}\xi\xi + \frac{ig_2\alpha}{2}\eta\xi\xi^{+}\xi^{+}
\end{aligned}
\tag{11}
$$

Canonical dimension of fields and parameters of the effective IR model

	ξ, ξ^+	η, η^+	α	g
d_Q^ω	$-\frac{1}{2}$	$\frac{1}{2}$	0	2
d_Q^k	$\frac{d}{2}$	$\frac{d}{2}$	0	$-d$
d_Q	$\frac{d}{2}-1$	$\frac{d}{2}+1$	0	$4-d$

The model is logarithmic in the four-dimensional space.

Two loops calculation

	s.c.	Z_i	value
	$\frac{1}{2}$	Z_5	$-\frac{\alpha g_1^2}{(iu+1)8\pi^2\varepsilon}$
	1	Z_5	$-\frac{\alpha g_1^2}{8\pi^2\varepsilon}$
	1	Z_5	$\frac{\alpha g_1 g_1^*}{8\pi^2\varepsilon}$
	$\frac{1}{2}$	Z_0	$-\frac{2\alpha g_1 g_1^*}{(8\,\pi^2)^2\,\varepsilon\,(u^2+1)}\left\{\pi\,u+2\,u\,\arctan\left(\frac{u}{4}-\frac{3}{4\,u}\right)+\log\frac{4096^8}{(u^2+1)\,(u^2+9)^3}\right\}$
	1	Z_1	$\frac{g_1^2}{2(8\,\pi^2)^2\varepsilon(u^2+1)^2}\left\{4u\arctan\frac{u}{3}-\left(u^2-1\right)\log\frac{16^2}{u^2+9}-2i\left[\left(u^2-1\right)\arctan\frac{u}{3}+u\log\frac{16^2}{u^2+9}\right]\right\}$
	1	Z_2	$\frac{\alpha g_1^2}{(8\,\pi^2)^2\varepsilon}\left[\frac{(u^2+6-iu)}{u^2+9}\right]$
	$\frac{1}{2}$	Z_1	$\frac{g_1 g_1^*}{2(8\,\pi^2)^2\varepsilon\,(u^2+1)^2}\left(2u\left[-\pi+2\arctan\left(\frac{u}{2}+\frac{3}{2u}\right)\right]+\left(u^2-1\right)\log\frac{16^4}{(u^2+1)\,(u^2+9)}-i\left\{\left(u^2-1\right)\left[\pi-2\arctan\left(\frac{u}{2}+\frac{3}{2u}\right)\right]+2u\log\frac{16}{(u^2+1)\,(u^2+9)}\right\}\right)$
	$\frac{1}{2}$	Z_2	$-\frac{\alpha g_1 g_1^*}{(8\,\pi^2)^2\epsilon}\left[\frac{2(u^2+3-2ui)}{u^2+9}\right]$

Two loops calculation

The coefficients at the counterterms Z_i were calculated in the MS scheme in the leading order perturbation theory:

$$Z_0 = 4\alpha + \frac{g_1 g_2 \alpha M_1}{4^3 \pi^4 \epsilon}, \qquad Z_1 = 1 + \frac{g_1^2 M_3}{2^7 \pi^4 \epsilon} + \frac{g_1 g_2 M_2}{2^8 \pi^4 \epsilon},$$
$$Z_2 = (1+iu)\alpha + \frac{\alpha(u+2i)g_1^2}{2^8 \pi^4 \epsilon (u+3i)} - \frac{\alpha g_1 g_2 (u+i)}{2^8 (u+3i)\pi^4 \epsilon}, \tag{12}$$
$$Z_5 = \frac{i g_1 \alpha}{2} - \frac{g_1^2 \alpha}{16(1+iu)\pi^2 \epsilon} - \frac{g_1^2 \alpha}{8\pi^2 \epsilon} + \frac{g_1 g_2 \alpha}{8\pi^2 \epsilon},$$

where

$$M_1 = \frac{A + 3B - \pi u - 2u \arctan C_-}{u^2+1}, \quad M_2 = \frac{(u-i)^2 (2i \arctan(2C_+) - \pi i - A - B)}{(u^2+1)^2},$$
$$M_3 = \frac{(u+i)^2 (B - 2i \arctan(u/3))}{(u^2+1)^2}, \quad A = \ln(u^2+1), \; B = \ln\left(\frac{u^2+9}{16}\right), \; C_\pm = \frac{u^2 \pm 3}{4u}.$$

β and γ functions:

$$\beta_{gn} = -\epsilon g_n + \epsilon g_n \sum_{m=r,i} g_m \frac{\partial [Z_{gm}]}{\partial g_m}, \quad \beta_u = \epsilon u \sum_{n=r,i} g_n \frac{\partial [Z_u]}{\partial g_n}, \quad \gamma_\zeta = -\epsilon \sum_{n=r,i} g_n \frac{\partial [Z_\zeta]}{\partial g_n},$$

Fixed points

- The trivial fixed point $g_{1*} = g_{2*} = 0$ with arbitrary u_*. Eigenvalues of the Jacobi matrix $\omega_{ij} = \partial_i \beta_j$ are $(-\epsilon, -\epsilon, 0)$ confirming the IR instability of the fixed point
- $g_{2*} = -2\pi^2\epsilon, \omega_* = \frac{1}{u} = 0$. The infinite value u_* is taken into account due to the nonperturbative nature of the charge u. This fixed point leads to eigenvalues of the matrix ω_{ij} equal to $(\epsilon, -\frac{\epsilon}{16}, 0)$. Thus, this is an unstable saddle point. It corresponds to quantum-mechanical behaviour of the system described by oscillating propagator without attenuation.
- $g_{1*} = 0$,$g_{2*} = -\frac{8\pi\epsilon}{5}$, $u_* = 0$. Eigenvalues of the ω matrix are $(\epsilon, \frac{\epsilon}{5}, 2\frac{\epsilon^2}{25}\log\frac{4}{3})$. The positive sign of all eigenvalues at $\epsilon > 0$ means that this fixed point is IR stable.

Fixed point and critical exponent z

The only one IR stable point is $g_{r*} = 0, g_{i*} = -8\pi^2\epsilon/5, u_* = 0$. Its matrix ω: (ϵ, $\epsilon/5$, $2\log(4/3)\,\epsilon^2/25$)

$$\gamma_\xi^* = 0.01\epsilon^2, \qquad \gamma_\eta^* = 0.0245218\epsilon^2, \qquad \gamma_\alpha^* = -0.0145218\epsilon^2. \tag{13}$$

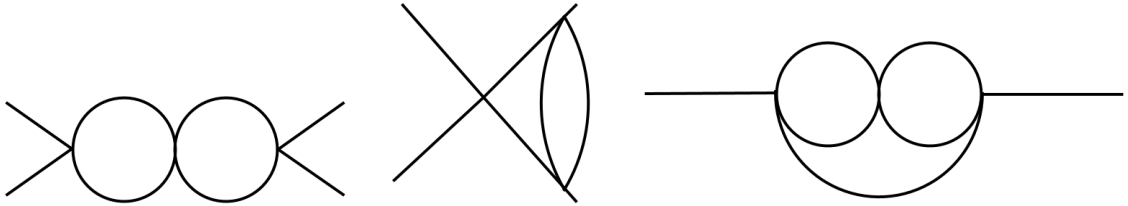
$$z = 2 + 0.0145218\epsilon^2 + O(\epsilon^3) \tag{14}$$

At the fixed point, the action of the model is differ from known 2-component stochastic model A only by change of the notation and a trivial rescaling parameters and fields. $\eta^+ = \phi_1' + \phi_2'$, $\eta = i\phi_2' - \phi_1'$, $\xi^+ = \phi_1 + i\phi_2$, $\xi = \phi_1 - i\phi_2$

$$S_A = -\alpha\phi'^2 + \phi'(\partial_t\phi - \alpha\Delta\phi + \frac{\alpha g}{6}\phi^3) \tag{15}$$

3-loop Feynman diagrams

At the third order of the perturbation theory there are three topological types of the diagrams, depicted below:



It is easily seen that the first type of these diagrams is a product of two diagrams of the first order. Z_i can be express as:

$$Z_0(g_1, g_2, u) = 4\alpha - \frac{\alpha g_1 g_2 M_0(u)}{2 \cdot 16^2 \pi^4 \varepsilon}$$
$$+ \frac{(-i) g_1 g_2^2 Q_{10}(u)}{16^3 \pi^6 \varepsilon} + \frac{(-i) g_1^2 g_2 Q_{11}(u)}{16^3 \pi^6 \varepsilon},$$

$$Z_1(g_1, g_2, u) = 1 + \frac{g_1^2 M_3(u)}{16^2 \pi^4 \varepsilon} - \frac{g_1 g_2 M_2(u)}{2 \cdot 16^2 \pi^4 \varepsilon}$$
$$+ \frac{(-i) g_1^3 Q_4(u)}{16^3 \pi^6 \varepsilon} + \frac{(-i) g_1^2 g_2 Q_5(u)}{16^3 \pi^6 \varepsilon} + \frac{(-i) g_1 g_2^2 Q_6(u)}{16^3 \pi^6 \varepsilon},$$

$$Z_2(g_1, g_2, u) = (1 + iu)\alpha + \frac{g_1^2 \alpha (2 - iu)}{16^2 \pi^4 (3 - iu)} -$$
$$- \frac{g_1 g_2 \alpha (1 - iu)}{16^2 (3 - iu) \pi^4} + \frac{(-i) g_1^3 Q_7(u)}{16^3 \pi^6 \varepsilon} + \frac{(-i) g_1^2 g_2 Q_8(u)}{16^3 \pi^6 \varepsilon} +$$
$$+ \frac{(-i) g_1 g_2^2 Q_9(u)}{16^3 \pi^6 \varepsilon}$$

$$Z_5(g_1, g_2, u) = \frac{i g_1 \alpha}{2} - \frac{g_1^2 \alpha}{16(1 + iu)\pi^2 \varepsilon} - \frac{g_1^2 \alpha}{8\pi^2 \varepsilon} + \frac{g_1 g_2 \alpha}{8\pi^2 \varepsilon}$$
$$- \frac{i g_1^3}{64\pi^4 \varepsilon} Q_1(u) - \frac{i g_1^2 g_2}{128\pi^4 \varepsilon} Q_2(u) - \frac{i g_1 g_2^2}{128\pi^4 \varepsilon} Q_3(u),$$

Stability of the fixed point

Next to leading calculation of ω matrix yields the following eigenvalues:

$$\begin{aligned} &\epsilon + 0.68\epsilon^2 \\ &0.2\epsilon + 0.245739\epsilon^2 \\ &0.0230146\epsilon^2 - 0.0258336\epsilon^3 \end{aligned} \tag{16}$$

At $\epsilon = 1$ it seems that the fixed point loses stability, however it is well known that such series are essentially asymptotic:

$$\begin{aligned} \omega_1 &= 0.2116697531 \\ \omega_2 &= 1.078902843 \\ \omega_3 &= 0.003120033385 \end{aligned} \tag{17}$$

Equivalence of F and A models of stochastic dynamics

Static action of model F:

$$S_{st} = \partial\psi^+\partial\psi + \frac{g_1(\psi^+\psi)^2}{6} - g_2 m\psi^+\psi + \frac{m^2}{2} + \frac{w\rho v^2}{2}, \tag{18}$$

$$\begin{aligned}
-S_{dyn} = {} & b_1\psi^{+\prime}\psi' - b_2 m'\partial^2 m' + b_3\partial_i v_j'\partial_i v_j' + b_4\partial_j v_j'\partial_i v_i' + a_1\psi^{+\prime}\partial^2\psi + a_2\psi'\partial^2\psi^+ \\
& + a_3 m'\partial_i v_i + a_4 m'\partial^2 m + a_5 v_i'\partial^2 v_i + a_6 v_i'\partial_i\partial_j v_j + a_7 v_i'\partial_i m - \psi^{+\prime}\partial_t\psi - \psi'\partial_t\psi^+ \\
& - m'\partial_t m - v'\partial_t v + \psi^{+\prime}\big[-v_i\partial_i\psi + a_8\psi^+\psi\psi + a_9 m\psi + a_{10}\psi v^2\big] \\
& + \psi'\big[-v_i\partial_i\psi^+ + a_{11}(\psi^+)^2\psi + a_{12} m\psi^+ + a_{13}\psi^+ v^2\big] \\
& + m'\big[-m\partial_i v_i + a_{14}\partial^2(\psi^+\psi) + a_{15}\partial^2 v^2 + a_{16}\psi^+\partial^2\psi + a_{17}\psi\partial^2\psi^+\big] \\
& + v_i'\big[-\partial_j(v_i v_j) + a_{18}(\partial_i\psi^+)\partial^2\psi + a_{19}\psi^+\psi^2\partial_i\psi^+ + a_{20} m\psi\partial_i\psi^+ \\
& + a_{21}(\partial_i\psi)\partial^2\psi^+ + a_{22}(\psi^+)^2\psi\partial_i\psi + a_{23} m\psi^+\partial_i\psi + a_{24}\partial_i(\psi^+\psi) + a_{25}\partial_i v^2\big]
\end{aligned} \tag{19}$$

Power counting shows that a_3 is the most IR-relevant theory parameter. Because a_3 is the coefficient of the action term $m\prime\partial_i u_i$ is quadratic in the fields, and its influence on the perturbation theory diagrams is therefore not initially obvious.

$$\begin{aligned}
-S_{dyn,eff} = {} & b_1\psi^{+\prime}\psi' + a_1\psi^{+\prime}\partial^2\psi + a_2\psi'\partial^2\psi^+ - \psi^{+\prime}\partial_t\psi - \psi'\partial_t\psi^+ + \psi^{+\prime}[a_8\psi^+\psi^2 \\
& - a_9 a_{24}(\psi^+\psi^2)/a_7] + \psi'[a_{11}(\psi^+)^2\psi - a_{12}a_{24}(\psi^+)^2\psi/a_7].
\end{aligned} \tag{20}$$

Conclusion

- We derived microscopic model suitable for superfluid transition.
- Then we performed construction of IR effective field theory. Proved renormalizability of the theory. Calculated z critical exponent at two-loops order
- Finally we investigated stability of the only one fixed point at 3-loop order and showed why stochastic model F can be reduced to model A.