Renormalization group in a model of self-organized criticality: Anisotropic system in isotropic environment

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Contents

The original stochastic model

Phenomenon of SOC

- Systems with self-organized criticality (SOC) [1] don’t have a turning parameter and evolve to the critical state as a result of their intrinsic dynamics.
- Systems with SOC are ubiquitous in Nature [2]!
- Examples are provided by biological systems [3], including neural systems among them [4], social networks [5] and many others.

The Hwa-Kardar equation

- The Hwa-Kardar (HK) stochastic equation [6] is a semi-phenomenological continuous model for the SOC behaviour in a coarse-grained “running” sandpile.
- The system is manifestly anisotropic!
- The surface tilt is specified by a constant unit vector $\boldsymbol{n}$: $\boldsymbol{x} = \boldsymbol{x}_{\perp} + \boldsymbol{n} x_{||}$, $|\boldsymbol{n}| = 1$, $(\boldsymbol{x}_{\perp}\boldsymbol{n}) = 0$.

The HK equation:

$$\partial_t h = \nu_{\perp 0}\, \partial_{\perp}^2 h + \nu_{||0}\, \partial_{||}^2 h - \partial_{||} h^2/2 + f, \tag{1}$$

where $h = h(t, \boldsymbol{x})$ is a deviation of the height of the sandpile from its average value; $\nu_{\perp 0}$ and $\nu_{||0}$ are kinetic coefficients; the derivatives are

$$\partial_t = \frac{\partial}{\partial t}, \quad \boldsymbol{\partial}_{\perp}^2 = (\boldsymbol{\partial}_{\perp}\boldsymbol{\partial}_{\perp}) = \frac{\partial}{\partial x_{\perp i}}\frac{\partial}{\partial x_{\perp i}};$$

summation over repeated tensor indices is implied here and throughout the presentation; index i in $x_{\perp i}$ runs from 1 to $(d - 1)$ with d being the dimension.

The Hwa-Kardar equation

The random noise $f(x)$ has zero mean and prescribed Gaussian statistics:

$$\langle f(x) f(x') \rangle_f = C_0 \, \delta(t - t') \, \delta^{(d)}(\mathbf{x} - \mathbf{x}'), \tag{2}$$

where $C_0 > 0$ is a positive amplitude and brackets $\langle \ldots \rangle_f$ denote averaging over the Gaussian statistics of the random noise f.

Let's move on to the environment!

The Navier-Stokes equation

The environment motion is described by the Navier-Stokes stochastic differential equation for an isotropic incompressible viscous fluid with an external random stirring force [7]:

$$\nabla_t \boldsymbol{v}_i = \nu_0 \partial^2 \boldsymbol{v}_i - \partial_i \wp + \eta_i, \tag{3}$$

where

$$\nabla_t = \partial_t + (\boldsymbol{v}\boldsymbol{\partial}) \tag{4}$$

is the Lagrangian (Galilean covariant) derivative, $\wp$ is the pressure and ν_0 is the kinematic viscosity coefficient.

Due to incompressibility of the fluid, the velocity field is transverse: $(\boldsymbol{\partial}\boldsymbol{v}) = 0$, in the sense that $(\boldsymbol{k}\boldsymbol{v}) = 0$ in the momentum representation.

The Navier-Stokes equation

The random noise $\eta_i(x)$ has a Gaussian probability distribution with zero mean and the given correlation function

$$\langle \eta_i(t,\mathbf{x})\eta_j(t',\mathbf{x}')\rangle_\eta = \delta(t-t')\int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d} P_{ij}(\mathbf{k})\, d_v(k) \exp i(\mathbf{k}(\mathbf{x}-\mathbf{x}'))\,. \quad (5)$$

$$d_v(k) = D_1 + D_2 k^{4-d-y}.$$

Here the brackets $\langle \ldots \rangle_\eta$ stand for the averaging over the noise statistics, $k \equiv |\boldsymbol{k}|$ is the wave number, $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ is the transverse projector, and $D_1, D_2 > 0$ are positive amplitude factor.

The contribution with D_1 is called local [7] and one with D_2 − unlocal [8].

The coupling of the fields h and $\boldsymbol{v}$ is introduced by the “minimal” substitution [9]:

$$\partial_t h \to \nabla_t h \equiv \partial_t h + (\boldsymbol{v}\cdot\boldsymbol{\partial})h.$$

Field theoretic formulation of the model

The action functional

The full stochastic problem (1)–(5) is equivalent (see, e.g., Sec. 5.3 in [10]) to the field theoretic model with the doubled set of fields $\Phi = \{h, h', \mathbf{v}, \mathbf{v}'\}$ and the action functional

$$S(\Phi) = C_0 h' h'/2 + h'\{-\nabla_t h + \nu_{\|0}\partial_{\|}^2 h + \nu_{\perp 0}\partial_{\perp}^2 h - \partial_{\|} h^2/2\} + \\ + D_0 \mathbf{v}'^2/2 + \mathbf{v}'\{-\nabla_t \mathbf{v} + \nu_0 \partial^2 \mathbf{v}\}. \quad (6)$$

$$D_0(k) = D_{10} + D_{20} k^{4-d-y}.$$

Relationship of amplitudes and coupling constants:

$$g_0 = \frac{C_0}{\mu^\varepsilon \nu_{\perp 0}^{3/2} \nu_{\|0}^{3/2}} \sim \Lambda^\varepsilon, \quad w_0 = \frac{D_{10}}{\mu^\varepsilon \nu_0^3} \sim \Lambda^\varepsilon, \quad u_0 = \frac{D_{20}}{\mu^y \nu_0^3} \sim \Lambda^y.$$

Calculation of the UV divergence indices

The UV divergence index δ_Γ for a Green's function Γ that involves N_h fields h, $N_{h'}$ fields h', etc., coincides with the total canonical dimension d_Γ of that function in the frequency-momentum representation, taken at the logarithmic dimension [10]:

$$\begin{aligned}\delta_\Gamma &= d_\Gamma|_{d=4} = (d + 2 - d_h N_h - d_{h'} N_{h'} - d_v N_{\mathbf{v}} - d_{\mathbf{v}'} N_{\mathbf{v}'})|_{d=4} = \\ &= 6 - N_h - 3N_{h'} - N_{\mathbf{v}} - 3N_{\mathbf{v}'}.\end{aligned}$$

The real divergence index δ'_Γ is given by the expression:

$$\delta'_\Gamma = 6 - N_h - 4N_{h'} - N_{\mathbf{v}} - 4N_{\mathbf{v}'}.$$

New coupling constants: $x_{1,0} = \nu_{\|0}/\nu_0$ and $x_{2,0} = \nu_{\perp 0}/\nu_0$.

The renormalized action functional

The divergence of the expression is transferred to the renormalization constant Z_i. The rest of the terms are finite.

$$S_R(\Phi) = w\widetilde{\nu}^3\mu^\epsilon(\boldsymbol{v}' \cdot \boldsymbol{v}')/2 + u\widetilde{\nu}^3\mu^y(\boldsymbol{v}' \cdot \widetilde{D}_2\boldsymbol{v}')/2 + g\nu_\perp^{3/2}\nu_\|^{3/2}\mu^\epsilon h'h'/2+ \\ + h'\{-\nabla_t h + Z_1\nu_\|\partial_\|^2 h + Z_2\nu_\perp\partial_\perp^2 h - \partial_\| h^2/2\} + (\boldsymbol{v}' \cdot \{-\nabla_t \boldsymbol{v} + Z_3\widetilde{\nu}\partial^2\boldsymbol{v}\}), \tag{7}$$

where $\widetilde{D}_2 = k^{4-d-y}$.

The coupling constants and other parameters are related to their counterparts as follows:

$$g_0 = Z_g g\mu^\varepsilon, \quad w_0 = Z_w w\mu^\varepsilon, \quad x_{1,0} = Z_{x_1}x_1, \quad x_{2,0} = Z_{x_2}x_2,$$
$$\nu_0 = \nu Z_\nu, \quad \nu_{\|0} = \nu_\| Z_{\nu_\|}, \quad \nu_{\perp 0} = \nu_\perp Z_{\nu_\perp}.$$

$$Z_{\boldsymbol{v}} = Z_{\boldsymbol{v}'} = Z_h = Z_{h'} = 1, \quad Z_g = Z_{\nu_\|}^{-3/2}Z_{\nu_\perp}^{-3/2}, \quad Z_w = Z_u = Z_{\widetilde{\nu}}^{-3}, \\ Z_{\nu_\|} = Z_1, \quad Z_{\nu_\perp} = Z_2, \quad Z_{\widetilde{\nu}} = Z_3. \tag{8}$$

Among them, the expression for Z_g and the relations $Z_h = Z_{h'} = 1$ are one-loop. Decomposition parameters: $\varepsilon = 4 - d$ and y.

Diagram technique

In the frequency-momentum $(\omega - \boldsymbol{k})$ representation, the bare propagators for the model (6) are:

$$\langle hh' \rangle_0 = \langle h'h \rangle_0^* = \frac{1}{-i\omega + \epsilon(\mathbf{k})}, \quad \langle h'h' \rangle_0 = 0, \quad \langle hh \rangle_0 = \frac{C_0}{\omega^2 + \epsilon^2(\mathbf{k})},$$
$$\langle \boldsymbol{v}_i \boldsymbol{v}'_j \rangle_0 = \langle \boldsymbol{v}'_i \boldsymbol{v}_j \rangle_0^* = \frac{P_{ij}(\mathbf{k})}{-i\omega + \widetilde{\nu}_0 k^2}, \quad \langle \boldsymbol{v}'_i \boldsymbol{v}'_j \rangle_0 = 0, \quad \langle \boldsymbol{v}_i \boldsymbol{v}_j \rangle_0 = \frac{D_0 P_{ij}(\mathbf{k})}{\omega^2 + \widetilde{\nu}_0^2 k^4}, \tag{9}$$

where $\epsilon(\mathbf{k}) = \nu_{\|0} k_\|^2 + \nu_{\perp 0} \mathbf{k}_\perp^2$.

$\langle hh' \rangle_0 =$ , $\quad \langle hh \rangle_0 =$,

$\langle \boldsymbol{v}_i \boldsymbol{v}'_j \rangle_0 =$ , $\quad \langle \boldsymbol{v}_i \boldsymbol{v}_j \rangle_0 =$ .

The three vertices $-h'\partial_\| h^2/2$, $-h'(\boldsymbol{v}\boldsymbol{\partial})h$ and $-\boldsymbol{v}'(\boldsymbol{v}\boldsymbol{\partial})\boldsymbol{v}$ correspond to the vertex factors $ik_\|^{h'}$, $ik_j^{h'}$ and $i(k_n^{\boldsymbol{v}'}\delta_{sj} + k_s^{\boldsymbol{v}'}\delta_{nj})$, respectively.

Available symmetries

Firstly, the counterterms $(\boldsymbol{v}\partial)h'$ and $(\boldsymbol{v'}\partial)h$ vanish due to **transversality** of the fields $\boldsymbol{v}$ and $\boldsymbol{v'}$.

Secondly, there is **passivity of the field** h which means that the dynamic of the field $\boldsymbol{v}$ is not affected by the field h. The full Green's functions with $N_h = 0$, $N_{h'} \geq 1$ $(\forall\, \mathrm{N}_{\boldsymbol{v}}, N_{\boldsymbol{v'}})$ and 1-irreducible Green's functions with $N_h \geq 1$, $N_{h'} = 0$ $(\forall\, \mathrm{N}_{\boldsymbol{v}}, N_{\boldsymbol{v'}})$ translate to the vanishing. Thus, the counterterms $(\boldsymbol{v}'\boldsymbol{v})\partial_{||}h$ and $h(\boldsymbol{v'}\partial)h$ should be dropped.

Lastly, the action functional (6) is invariant with respect to **the Galilean transformation**

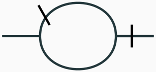
$$
\begin{aligned}
\boldsymbol{v}(t,\boldsymbol{x}) &\to \boldsymbol{v}(t,\boldsymbol{x}+\boldsymbol{u}\,t) - \boldsymbol{u}, &\qquad \boldsymbol{v}'(t,\boldsymbol{x}) &\to \boldsymbol{v}'(t,\boldsymbol{x}+\boldsymbol{u}\,t),\\
h(t,\boldsymbol{x}) &\to h(t,\boldsymbol{x}+\boldsymbol{u}\,t), &\qquad h'(t,\boldsymbol{x}) &\to h'(t,\boldsymbol{x}+\boldsymbol{u}\,t)
\end{aligned}
$$

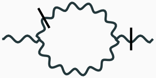
with a constant vector $\boldsymbol{u}$.

The Green functions

Table 1: Counterterms for the model (6)

№	Γ	δ_Γ	δ'_Γ	counterterm
1	$\langle h' \rangle$	3	3	h'
2	$\langle \boldsymbol{v}' \rangle$	3	3	$\boldsymbol{v}'$
3	$\langle \boldsymbol{v}'\boldsymbol{v} \rangle$	2	1	$\boldsymbol{v}'\partial^2\boldsymbol{v}$
4	$\langle \boldsymbol{v}'\boldsymbol{v}\boldsymbol{v} \rangle$	1	0	$\boldsymbol{v}'(\boldsymbol{v}\cdot\partial)\boldsymbol{v}$
5	$\langle h'h \rangle$	2	1	$h\partial^2 h'$
6	$\langle h'\boldsymbol{v} \rangle$	2	1	$h'\partial\boldsymbol{v}$
7	$\langle h'hh \rangle$	1	0	$h^2\partial^2 h'$
8	$\langle h'\boldsymbol{v}h \rangle$	1	0	$h'(\boldsymbol{v}\cdot\partial)h$

$\langle hh' \rangle_{1-\mathrm{ir}} = i\omega - \nu_{\|} p_{\|}^2 Z_1 - \nu_{\perp} p_{\perp}^2 Z_2 +$  $+$  ,

$\langle \boldsymbol{v}\boldsymbol{v}' \rangle_{1-\mathrm{ir}} = i\omega - \widetilde{\nu} p^2 Z_3 +$  .

The Hwa-Kardar model and the Navier-Stokes equation only with local contribution

Local case

The Hwa-Kardar model with the motion environment modeled by the Navier-Stokes equation with a local contribution (i.e. $D_0 = D_{10}$, and there is no non-local contribution):

$$\begin{aligned} S(\Phi) &= C_0 h' h'/2 + h'\{-\nabla_t h + \nu_{\|0}\partial_{\|}^2 h + \nu_{\perp 0}\partial_{\perp}^2 h - \partial_{\|} h^2/2\} + \\ &\qquad + D_0 \boldsymbol{v}'^2/2 + \boldsymbol{v}'\{-\nabla_t \boldsymbol{v} + \nu_0 \partial^2 \boldsymbol{v}\}. \end{aligned} \tag{10}$$

The renormalized constants:

$$Z_1 = 1 - \frac{1}{\varepsilon}\left[g\frac{3}{16} + w f_1(x_1, x_2)\right], \; Z_2 = 1 - \frac{w}{\varepsilon} f_2(x_1, x_2), \; Z_3 = 1 - \frac{1}{\varepsilon}\frac{w}{8} \; .$$

$$f_1(x_1, x_2) \equiv \frac{1}{2\,x_1\left(\sqrt{1+x_1} + \sqrt{1+x_2}\right)^2}\left(1 + 2\sqrt{\frac{1+x_1}{1+x_2}}\right),$$

$$f_2(x_1, x_2) \equiv \frac{1}{6\,x_2\left(\sqrt{1+x_1} + \sqrt{1+x_2}\right)^2}\left(5 + 4\sqrt{\frac{1+x_1}{1+x_2}}\right).$$

Renormalization functions

Renormalization functions (anomalous dimensions γ and β-functions):

$$\gamma_Q = \widetilde{\mathcal{D}}_\mu \ln Z_Q, \quad \beta_r = \widetilde{\mathcal{D}}_\mu r.$$

Here Q is a given quantity with renormalization constant Z_Q and r is a coupling constant. Differential operator $\widetilde{\mathcal{D}}_\mu$

$$\widetilde{\mathcal{D}}_\mu = \mu\partial_\mu|_{\{g_0, w_0, x_{10}, x_{20}, \nu_0\}} \tag{11}$$

emerges from the relation $\widetilde{\mathcal{D}}_\mu F = 0$ for a physical quantity F that encapsulates the fact that F cannot depend on renormalization mass μ (which is not an observable).

$$\begin{aligned} \gamma_1 = g\frac{3}{16} + wf_1(x_1, x_2), \quad \gamma_2 = wf_2(x_1, x_2), \quad \gamma_3 = \frac{w}{8}, \\ \beta_g = -g\left[\varepsilon - \frac{3}{2}\gamma_1 - \frac{3}{2}\gamma_2\right], \quad \beta_w = -w\left[\varepsilon - 3\gamma_3\right], \\ \beta_{x_1} = -x_1\left[\gamma_1 - \gamma_3\right], \quad \beta_{x_2} = -x_2\left[\gamma_2 - \gamma_3\right]. \end{aligned} \tag{12}$$

Fixed points

Taking into account that charges x_1, x_2 are positive, we find the following fixed points:

FP 1 – a Gaussian (trivial) fixed point:

$$g^* = 0; \quad w^* = 0; \quad x_1^* \neq 0; \quad x_2^* \neq 0; \quad \lambda_i = \{0, 0, -\varepsilon, -\varepsilon\}.$$

FP 2 – a line of fixed points, parametrized by one of the coordinates (e.g., $\{g^*(x_2^*), w^*(x_2^*), x_1^*(x_2^*), x_2^*\}$) and determined by the following equations:

$$w_* = 8\epsilon/3, \; f_2(x_1^*, x_2^*) = 1/8, \; g_* = \frac{128}{9}\epsilon\left(\frac{1}{8} - f_1(x_1^*, x_2^*)\right),$$
$$\lambda_i = \{0, \varepsilon, \lambda_3, \lambda_4\}.$$

FP 2a – pure turbulence point:

$$w_* = 8\epsilon/3, \quad g_* = 0, \quad x_1^* = x_2^* = \frac{\sqrt{13}-1}{2}, \quad \lambda_i = \left\{0, \varepsilon, \frac{47+\sqrt{13}}{162}\varepsilon, \frac{13-\sqrt{13}}{18}\varepsilon\right\}.$$

Fixed points

The eigenvalues λ_3 and λ_4 change along the line FP 2 and can also be parametrized. Both eigenvalues are non-negative for the permitted values of x_2^* and positive values of ε.

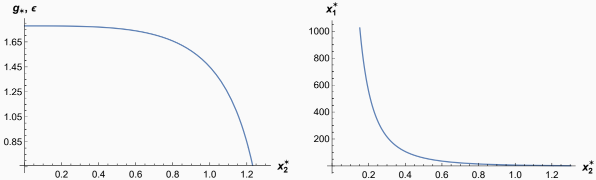


Figure 1: Dependence of charges g_* и x_1^* from copling constant x_2^*.

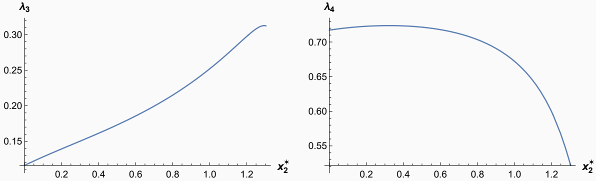


Figure 2: Eigenvalues λ_3 and λ_4 parameterized by x_2^*.

Scaling regimes

The equation for critical IR scaling:

$$\left(-D_x-\Delta_\omega D_t+d_g^p D_g+d_w^p D_w+d_{x_1}^p D_{x_1}+d_{x_2}^p D_{x_2}-n_{\mathbf{v}}\Delta_{\mathbf{v}}-n_{\mathbf{v}'}\Delta_{\mathbf{v}'}-n_h\Delta_h-n_{h'}\Delta_{h'}\right)W^R=0,$$

$$\Delta_\omega=2-\gamma_{\widetilde{\nu}}^*,\quad \Delta_{\mathbf{v}}=d_{\mathbf{v}}^p+\Delta_\omega d_{\mathbf{v}}^\omega+\gamma_{\mathbf{v}}^*,$$

where $\gamma_{\widetilde{\nu}}^*=\gamma_{\widetilde{\nu}}(w_*,g_*,x_1^*,x_2^*)$ and $\gamma_{\mathbf{v}}^*=\gamma_{\mathbf{v}}(w_*,g_*,x_1^*,x_2^*)$ and similarly for $\Delta_h,\Delta_{\mathbf{v}'},\Delta_{h'}$.

Table 2: Fixed stable points of the model (10)

Fixed point	Eigenvalues	Critical dimension		
		Δ_ω	$\Delta_{\mathbf{v}}$, Δ_h	$\Delta_{\mathbf{v}'}$, $\Delta_{h'}$
$g_*=0, w_*=0, \forall\, x_1^*, x_2^*$	$0, 0, -\epsilon, -\epsilon$	2	1	$d-1$
$w_*=8\epsilon/3...$	$0, \epsilon, \lambda_3(x_2^*), \lambda_4(x_2^*)$	$2-\epsilon/3$	$1-\epsilon/3$	$d-1+\epsilon/3$

Unstable points

Unstable points were found in other systems:
$y_{1,2} = x_{1,2}^{-1}$; $u_{1,2} = w x_{1,2}^{-1}$; $u = w x_1^{-1} x_2^{-1}$.

For example: $g_* = 32\varepsilon/9, w_* = 0, y_1^* = 0, \forall y_2^*, \quad \lambda_i = \{0, -\varepsilon, 2\varepsilon/3, \varepsilon\}$.
This point belongs to the class of universality of the pure Hwa-Kardar equation without turbulent motion of the medium.

And some words about the common model

The full model

This work is in progress.

$$\begin{aligned}\gamma_1 = g\frac{3}{16} + (w+u)\,f_1(x_1,x_2), \quad \gamma_2 = (w+u)\,f_2(x_1,x_2), \quad \gamma_3 = \frac{w}{8}, \\ \beta_g = -g\left[\varepsilon - \frac{3}{2}\gamma_1 - \frac{3}{2}\gamma_2\right], \quad \beta_w = -w\left[\varepsilon - 3\gamma_3\right], \\ \beta_{x_1} = -x_1\left[\gamma_1 - \gamma_3\right], \quad \beta_{x_2} = -x_2\left[\gamma_2 - \gamma_3\right], \\ \beta_u = -u\left[y - 3\gamma_3\right].\end{aligned} \tag{13}$$

Conclusion

Conclusion

- A field theory equivalent to the original stochastic problem was constructed;
- After analyzing the canonical dimensions and counterterms, it was found that it is renormalizable;
- For the particular model (the Hwa-Kardar and Navier-Stokes equations with the local contribution in the correlator), the (parametrized) coordinates of the fixed points and the corresponding critical dimensions (in the one-loop approximation or exactly) are found.
- For the particular model one of the regims is not implemented.

The End.

Thank you for your attention!

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