Effects of turbulent environment on self-organized critical behavior and nonconventional scaling behavior

N.V. Antonov, N.M. Gulitskiy, P.I. Kakin

Saint Petersburg State University

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Physics of Elementary Particles and Statistical Physics

In this talk we deal with two areas of physic: statistical physics and high energy physics:

- Hwa-Kardar equation describing the growth of the surface;
- stochastic description of the system;
- functional integration and calculation of Feynman graphs;
- renormalization group (RG).

The problem under consideration is growth of the surface under turbulent motion of the environment and mutual influence of the isotropy of the system and anisotropy of the turbulent flow.

Plan of the talk

The main steps (general scheme) are following:

- Stochastic formulation of Hwa-Kardar model;
- quantum field action and Feynman diagrams;
- divergences of the diagrams;
- renormalization, RG, RG flow and fixed points;
- critical dimensions at different fixed points.

Hwa-Kardar model



T. Hwa and M. Kardar, Phys. Rev. Let., Vol.62, №16 (1989).

Stochastic equation (*f* is a random force)

$$\partial_t h(t,x) =
u_{\parallel} \partial_{\parallel}^2 h(t,x) +
u_{\perp} \partial_{\perp}^2 h(t,x) - \frac{1}{2} \partial_{\parallel} h^2(t,x) + f(t,x).$$

Anisotropy of the system

$$\mathbf{x} = \mathbf{x}_{\perp} + \mathbf{n} x_{\parallel}, \quad \mathbf{n} \perp \mathbf{x}_{\perp} \qquad (\hat{T} \equiv \mathbf{n}).$$

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Hwa-Kardar model: turbulent mixing

Introducing of velocity field: anisotropic one

$$\partial_t \to
abla_t \equiv \partial_t + v \partial_{\parallel};$$

 $\mathbf{v}(t, x) = \mathbf{n} v(t, x_{\perp}), \quad \partial_{\parallel} v = 0$

In the real life **v** obeys Navier-Stokes equation, but calculation should be very complicated. Simplified model: **v** is a Gaussian variable with given correlator function

$$ig\langle \mathbf{v}(t,x)\mathbf{v}(t',x')ig
angle = \int rac{dkd\omega}{(2\pi)^d} e^{ik(x-x')-i\omega(t-t')} B_{\mathbf{v}}(\omega,k),$$

 $B_{\mathbf{v}}(\omega,k) = 2\pi\delta(k_{\parallel})B_0 rac{k_{\perp}^{5-d-(\xi+\eta)}}{\omega^2 + [\alpha\nu_{\perp}k_{\perp}^{2-\eta}]^2}.$

arXiv:2005.04756

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Hwa-Kardar model: turbulent mixing

Introducing of velocity field: isotropic one

$$\langle v_i(t,\mathbf{x})v_j(t',\mathbf{x}')\rangle = \delta(t-t')D_{ij}(\mathbf{x}-\mathbf{x}'),$$

$$D_{ij}(\mathbf{r}) = B_0 \int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{k^{d+\xi}} P_{ij}(\mathbf{k}) \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}).$$

arXiv:2009.00302

Hwa-Kardar model: choise of the random force

Thermal noise

$$\left\langle f(\mathbf{x})f(\mathbf{x}')
ight
angle = D_0\,\delta(t-t')\delta^{(d)}(\mathbf{x}-\mathbf{x}'),\quad D_0>0;$$

rapid correlations in both space and time.

Static noise (optional, do not consider here)

$$\langle f(x)f(x')\rangle = D_0 \,\delta^{(d)}(\mathbf{x}-\mathbf{x}'), \quad D_0 > 0;$$

external average influence of the environment to the landscape is a constant.

Action functional Graphs Fixed points

Action functional: General rules

Theorem: any stochastic equation of the type

$$\partial_t \phi(x) = U(x,\phi) + f(x), \quad \langle f(x)f(x') \rangle = D(x,x'),$$

where $\phi(x) = \phi(t, \mathbf{x})$ is a random field, $U(x, \phi)$ is a *t*-local functional depending on the fields and their derivatives, f(x) is a random force, **is equivalent to quantum field model** of the double set of fields $\widetilde{\phi} = \{\phi, \phi'\}$ and action functional

$$S[\varphi] = \underbrace{\frac{1}{2}\varphi'D\varphi'}_{\text{noise term}} + \varphi' \underbrace{[-\partial_t \varphi + U]}_{\text{dynamics}},$$

integration over t and \mathbf{x} implied.

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Action functional: General rules

What does it mean:

- ► statistical average is equivalent to functional integration with weight exp S[φ];
- classical random field \rightarrow quantum field;
- we may use all techniques from quantum field theory: Feynman graphs, renormalization group, operator product expansion, *etc*.

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Actions functional

Quantum field action:

$$S(\Phi) = \frac{1}{2}h'D_0h' + h'(-\nabla_t h + \nu_{\parallel_0}\partial_{\parallel}^2 h + \nu_{\perp_0}\partial_{\perp}^2 h - \frac{1}{2}\partial_{\parallel}h^2) + S_v.$$

All integrations are implied:

$$h'D_0h'\equiv D_0\int dt\int d^dx\,h'(t,x)h'(t,x).$$

Action functional Graphs Fixed points

Feynman rules and divergent functions

According to general rules both models contain three propagators

$$\langle hh' \rangle_0 =$$
_____, $\langle hh \rangle_0 =$ _____, $\langle vv \rangle_0 =$ ______

and two vertices

$$-\frac{1}{2}h'\partial_{\parallel}h^{2} = h^{2}\partial_{\parallel'}h' = +; \quad -h'(v\partial_{\parallel})h = h(v\partial_{\parallel})h' = +i_{2},$$

The propagators are

$$\langle hh' \rangle_0 = \frac{1}{-i\omega + \epsilon(k)}, \quad \langle hh \rangle_0 = \frac{D_0}{\omega^2 + \epsilon^2(k)}; \quad \epsilon(k) \equiv \nu_{\parallel_0} k_{\parallel}^2 + \nu_{\perp_0} k_{\perp}^2.$$

Logarithmic dimension is d = 4 and the only divergent Green function is $\langle h'h \rangle$.

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Effects of turbulent environment on self-organized critical behavior

Action functional Graphs Fixed points

Dimensions and scales

The key difference between anisotropic velocity ensemble

$$\mathbf{v}(t,x) = \mathbf{n} v(t,x_{\perp})$$

and isotropic one (under consideration)

$$\langle v_i(t,\mathbf{x})v_j(t',\mathbf{x}')
angle = \delta(t-t')D_{ij}(\mathbf{x}-\mathbf{x}')$$

is possibility/impossibility to introduxe two independent momentum scales.

In first case any quantity F is described by three canonical dimensions

$$[F] \sim [T]^{-d_F^{\omega}} [L_{\parallel}]^{-d_F^{\parallel}} [L_{\perp}]^{-d_F^{\perp}},$$

in second case only by two:

$$[F] \sim [T]^{-d_F^\omega} [L]^{-d_F^k}.$$

Action functional Graphs Fixed points

Couplings and diagrams

This fact drastically changes RG analysis: we have one more dimensionless quantity, i.e., one more "coupling constant"

$$u_0=\nu_{\parallel 0}/\nu_{\perp 0}.$$

All together three couplings are

$$D_0 = g_0(
u_{\parallel_0}
u_{\perp_0})^{3/2}, \quad B_0 = w_0
u_{\parallel_0}
u_{\perp_0}^2, \quad u_0 =
u_{\parallel 0}/
u_{\perp_0}.$$

Answers for the graphs are



Action functional Graphs Fixed points

Renormalization constants Z and β -functions

The renormalization constants \boldsymbol{Z} are

$$\begin{split} & Z_{\nu_{\parallel}} = 1 - \frac{3}{8u} \frac{x}{\xi} - \frac{3}{16} \frac{g}{\varepsilon}, \\ & Z_{\nu_{\perp}} = 1 - \frac{3}{8} \frac{x}{\xi}, \quad \text{where} \quad x = uw. \end{split}$$

The β -functions are

$$\beta_g = g\left(-\varepsilon + \frac{9}{32}g + \frac{9}{16}\frac{x}{u} + \frac{9}{16}x\right),$$

$$\beta_x = x\left(-\xi + \frac{3}{8}x\right),$$

$$\beta_u = u\left(-\frac{3}{16}g - \frac{3}{8}\frac{x}{u} + \frac{3}{8}x\right).$$

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Fixed points and scaling regimes

Four fixed points depending on the parameters ξ and η



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Critical dimensions: Fixed point FP2

The equation of critical scaling is a combination of equations of canonical scaling with RG equation taken at fixed point.

$$\left(\mathcal{D}_{k_{\parallel}}+\mathcal{D}_{k_{\perp}}+\Delta_{\omega}\mathcal{D}_{\omega}-d_{G}^{k}-\Delta_{\omega}d_{G}^{\omega}-\gamma_{G}^{*}
ight)G^{R}=0,$$

where $\Delta_{\omega}=2-\gamma_{\nu_{\perp}}^{*}.$ The critical dimension Δ_{F} of a quantity F reads

$$\Delta_F = d_F^k + \Delta_\omega d_F^\omega + \gamma_F^*.$$

Usually this step is very straightforward, and for the fixed point FP2 they are

$$\Delta_h = 1 - \xi, \quad \Delta_v = 1 - \xi, \quad \Delta_{h'} = 3 - \varepsilon + \xi, \quad \Delta_\omega = 2 - \xi.$$

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Critical dimensions: Fixed points FP3 and FP4

The points FP3 and FP4 has the coordinate $\alpha^* = 1/u^* = 0$. Direct substitution of the coordinates g^* , x^* , α^* into $\gamma_{\nu_{\perp}}$ gives us trivial answer: $\gamma_{\nu_{\perp}}^* = 0$ and, consequently, $\Delta_F = d_F^k + 2d_F^{\omega}$ are simply canonical dimensions.

To obtain nontrivial corections we should expand $\beta_{\alpha}!$

$$\left(\mathcal{D}_{k_{\parallel}}+\mathcal{D}_{k_{\perp}}+\Delta_{\omega}\mathcal{D}_{\omega}-\lambda^{*}\mathcal{D}_{\alpha}-d_{G}^{k}-\Delta_{\omega}d_{G}^{\omega}-\gamma_{G}^{*}\right)G^{R}=0,$$

where $\lambda = \partial \beta_{\alpha} / \partial \alpha$ at $\alpha = 0$ and λ^* denotes $\lambda(g^*, x^*)$.

This trick allows us to determine nontrivial one-loop corrections to canonical dimensions at fixed points FP3 and FP4 and, moreover, reproduce well-known one-loop answers for pure Hwa-kardar model [Phys. Rev. Lett. 62, 1813 (1989); Phys. Rev. A 45, 7002 (1992)], which corresponds in our terminology to fixed point FP3.

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Critical dimensions: Fixed point FP3

For point FP3 general solution of the set of canonical equations and RG equation is an arbitrary function of three independent variables

$$z_1 = rac{\omega}{
u_\perp k_\perp^2}, \quad z_2 = rac{k_\parallel}{k_\perp}, \quad ext{and} \quad z_3 = lpha \left(rac{k_\perp}{\mu}
ight)^{2arepsilon/3}.$$

Additional symmetry means that $[F] \sim [T]^{-d_F^{\omega}} [L_{\parallel}]^{-d_F^{\parallel}} [L_{\perp}]^{-d_F^{\perp}}$.

The variable z_1 satisfies this requirement, variables z_2 and z_3 do not, so we should construct $z_0 = z_2 z_3^{-1/2}$ which has the needed canonical dimensions and is the second solution

$$z_0 = rac{k_{\parallel}}{k_{\perp}^{\Delta_{\parallel}} \alpha^{\wp}} \mu^{arepsilon/3} \quad ext{with} \quad \wp = rac{3}{2arepsilon} \left(\Delta_{\parallel} - 1
ight), \quad ext{where} \quad \Delta_{\parallel} = 1 + arepsilon/3.$$

Action functional Graphs Fixed points

Dimensional transmutation

The ratio u of the two diffusivity coefficients ν_{\parallel} and ν_{\perp} acquires a nontrivial canonical dimension.

$$z_0 = rac{k_\parallel}{k_\perp^{\Delta_\parallel} lpha^\wp} \, \mu^{arepsilon/3} :$$

this regime corresponds to the scaling behavior where the coordinates x_{\parallel} and \mathbf{x}_{\perp} are simultaneously rescaled with nontrivial exponent $\Delta_{\parallel} \neq 1$ while the IR irrelevant parameters are kept fixed.

The model obtains new canonical symmetry that corresponds to independent canonical dimensions d^{\parallel} and d^{\perp} .

Action functional Graphs Fixed points

Critical dimensions: Fixed point FP4

For point FP4 general solution is an arbitrary function of three independent variables

$$\hat{z}_1 = \frac{\omega}{\nu_\perp k_\perp^2} \left(\frac{k_\perp}{\mu}\right)^{\xi}, \quad \hat{z}_2 = \frac{k_\parallel}{k_\perp}, \quad \hat{z}_3 = \alpha \left(\frac{k_\perp}{\mu}\right)^{2\varepsilon/3 - 2\xi}$$

Any special case like can be obtained by combining \hat{z}_1 , \hat{z}_2 and \hat{z}_3 , but to perform it we need additional information about dependence of scaling function on these variables.

Conclusion

We applied methods of **quantum field theory** (functional integration, calculation of Feynman graphs and renormalization group) to the Hwa-Kardar model which describes the profile of a surface.

- The key point is the possibility to reformulate initial stochastic problem into some quantum field theory.
- We coupled isotropic velocity ensemble with anisotropic initial stochastic equation. As a consequence, one more coupling constant and corresponding β-function should be considered.
- As a result we should expand β-function to obtain nontrivial corrections to critical dimensions.
- ► For more detailes see Universe 6, 145 (2020).

Thank you for your attention!