Critical exponents of model with matrix order parameter from resummation of six-loop results for anomalous dimensions

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$$S(\phi) = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4$$

RG functions are calculated up to 6 loop order[1]

- Accurate estimation of critical exponents

Exponent η	Exact Value	Monte Carlo	Conformal Bootstrap	6 loop ε-expansions
$d = 2$	0.25			0.237(27)
$d = 3$	—	0.03627(10)	0.036298(2)	0.0362(6)

- Superficial divergences of each graph is known separately

$$\widetilde{Z} = \mathcal{O}Z$$

[1] M.V. Kompaniets, E. Panzer, Physical Review D, 96 (3), 036016, 2016

$$S(\phi) = \frac{1}{2} tr(\phi(-\partial^2 + m_0^2)\phi) - \frac{g_{10}}{4!}(tr(\phi^2))^2 - \frac{g_{20}}{4!} tr(\phi^4).$$

Here $\phi_{ik} = -\phi_{ki}$, $i, k = 1, ..., n$.
Propagator:

$$\langle \phi\phi \rangle_0 = \frac{J_{ab;cd}}{(k^2 + \tau_0)}, \qquad J_{ab;cd} = \frac{1}{2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}).$$

Vertices:

$$V^{(1)}_{ab;cd;ef;mn} = \frac{1}{3}(J_{ab;cd}J_{ef;mn} + J_{ab;ef}J_{cd;mn} + J_{ab;mn}J_{cd;ef})$$

$$\begin{aligned} V^{(2)}_{ab;cd;ef;mn} = \frac{1}{6} \quad \Big(& J_{ab;ij}J_{cd;jk}J_{ef;kp}J_{mn;pi} + J_{ab;ij}J_{cd;jk}J_{mn;kp}J_{ef;pi} + \\ & + J_{ab;ij}J_{ef;jk}J_{mn;kp}J_{cd;pi} + J_{ab;ij}J_{ef;jk}J_{cd;kp}J_{mn;pi} + \\ & + J_{ab;ij}J_{mn;jk}J_{ef;kp}J_{cd;pi} + J_{ab;ij}J_{mn;jk}J_{cd;kp}J_{ef;pi} \Big) \end{aligned}$$

Functional integral converges when:

$$\begin{array}{lll} \text{even} & 2g_{10} + g_{20} > 0, & ng_{10} + g_{20} > 0 \\ \text{odd} & 2g_{10} + g_{20} > 0, & (n-1)g_{10} + g_{20} > 0 \end{array} \tag{1}$$

We using dimensional regularization $\varepsilon = 4 - d$ and $\overline{MS}$ scheme.

IR asymptotic behavior of the model is governed by the equations for invariant couplings:

$$s\partial_s \bar{g}(s, g) = \beta_g(\bar{g}), \quad \bar{g}(1, g) = g$$

IR ($s = p/\mu \to 0$) asymptotic of the Green functions is determined by fixed points g^*:

$$\beta_g(g^*) = 0$$

The type of a fixed point is determined by eigenvalues of the matrix:

$$\omega_{ik} = \partial\beta_i/\partial g_k|_{g=g^*}$$

Trivial point:

$$g_1^* = 0; \quad g_2^* = 0.$$

Point **A**:

$$g_1^* = 12\varepsilon/(n^2 - n + 16), \quad g_2^* = 0.$$

Point **B** (plus in front of square root) and **C** (minus in front of square root):

$$g_1^* = -6\varepsilon \frac{(4n^2 - 4n - 143) \pm (2n-1)\sqrt{(-8n^2 + 8n + 97)}}{(4n^4 - 8n^3 - 123n^2 + 127n + 1696)},$$

$$g_2^* = 12\varepsilon \frac{(2n-1)(n^2 - n - 20) \pm 12\sqrt{(-8n^2 + 8n + 97)}}{(4n^4 - 8n^3 - 123n^2 + 127n + 1696)}.$$

General form of scalar theory:[2]

$$S(\phi) = \frac{1}{2}\partial_\mu\phi_a\partial_\mu\phi_a + \frac{1}{2}m^2_{ab}\phi_a\phi_b + \frac{h_{abc}}{3!}\phi_a\phi_b\phi_c + \frac{g_{abcd}}{4!}\phi_a\phi_b\phi_c\phi_d$$

General expressions for of RG functions are know up to 6 loops. Using field expansion:

$$\phi_{ij} = \sum \chi_a T^a_{ij}$$

where T^a corresponds to antisymmetric generators of $SO(n)$ one can utilize general expressions to calculate 5 and 6 orders of preturbative expansions.

[2] A. Bednyakov, A. Pikelner, JHEP, 04, 233, 2021

ε-expansions for the case of fixed point C

$$
\begin{aligned}
g_1^* &= 0.818e + 0.466e^2 - 0.259e^3 + 0.485e^4 - 1.29e^5 + 4.26e^6 \\
g_2^* &= -1.09e - 0.622e^2 + 0.345e^3 - 0.646e^4 + 1.72e^5 - 5.68e^6 \\
\omega_1 &= -0.091e + 0.312e^2 - 0.230e^3 + 0.587e^4 - 1.58e^5 + 5.28e^6 \\
\omega_2 &= 1.0e - 0.570e^2 + 1.28e^3 - 3.78e^4 + 13.2e^5 - 52.2e^6 \\
\eta &= 0.021e^2 + 0.018e^3 - 0.0075e^4 + 0.020e^5 - 0.057e^6
\end{aligned}
$$

Asymptotic in order of perturbative expansion: $g_i \to z g_i$.

$$G_k^N(z, x_1, \ldots, x_k) = \oint_\gamma dz \; \frac{G_k(z, x_1, \ldots, x_k)}{(-z)^{N+1}}.$$

Rescaling field $\phi \to \sqrt{N}\phi$ and couplings $g_{1,2} \to g_{1,2}/(N\mu^\varepsilon)$

$$G_k^N(z, x_1, \ldots, x_k) \approx \int D\phi \; dz \; \phi(x_1) \ldots \phi(x_k) e^{N(S - \ln(-z))}$$

Using transformations of $SO(n)$ group one can always get filed matrix to the form:

$$\phi = \begin{pmatrix} s_1\sigma & 0 & \ldots & 0 \\ 0 & s_2\sigma & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & s_p\sigma \end{pmatrix}; \quad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$-\partial^2 s_i(\mathbf{x}) + \frac{z g_1}{3} s_i(\mathbf{x}) \sum_{j=1}^{p} s_j^2(\mathbf{x}) + \frac{z g_2}{6} s_i^3(\mathbf{x}) = 0$$

$$\int d\mathbf{x} \{ \frac{z_{st} g_1}{4!} (tr(\phi_{st}^2))^2 + \frac{z_{st} g_2}{4!} tr(\phi_{st}^4) \} = -1$$

At $d = 4$ solution of the stationary equations is a family of instantons:

$$s_i(\mathbf{x}) = \frac{\alpha_i y}{|\mathbf{x} - \mathbf{x}_0|^2 + y^2},$$

where

$$\alpha_k^2 = \frac{-48}{(2kzg_1 + zg_2)}, \quad k = 1, \ldots, p,$$

$$\alpha_m = 0, \quad m = p - k.$$

which gives asymptotic behavior of RG functions and ε-expansions:

$$\beta_i^{(N)}(g_1, g_2) = \text{Const} \cdot N! N^b (-a(g_1, g_2))^N (1 + O(\frac{1}{N})),$$

$$g_{1,2*}^{(N)} = \text{Const} \cdot N! N^{b+1} (-a(g_{1*}^{(1)}, g_{2*}^{(1)}))^N (1 + O(\frac{1}{N})),$$

$a(g_1, g_2) = max_k \; a_k(g_1, g_2) = max_k \; ((2kg_q + g_2)/4k)$;

$b = (n^2 - 2n + 22)/4$

Borel-Leroy transform:

$$f(z) = \sum_{N\geq 0} f_N z^N \quad \Rightarrow \quad F(t) = \sum_{N\geq 0} \frac{f_N}{\Gamma(N+b_0+1)}\, t^N = \sum_{N\geq 0} F_N t^N$$

Inverse:

$$f^{res}(z) = \int_0^\infty dt \, \sum_{N\geq 0} F_N e^{-t} t^{b_o} t^N$$

$$f_N \simeq \text{Const} \cdot N! N^b (-a)^N \quad \Rightarrow \quad f_N \simeq \text{Const} \cdot N^{b-b_0} (-a)^N$$

Analytical continuation by conformal mapping:

$$u(t) = \frac{\sqrt{1+at}-1}{\sqrt{1+at}+1} \quad \Leftrightarrow \quad t(u) = \frac{4u}{a(u-1)^2},$$

$$f^{res}(z) = \int_0^\infty dt \, \sum_{N\geq 0} \widetilde{F}_N e^{-t} t^{b_o} u(zt)^N$$

Table 1: Values of critical exponents at different number of loops taken into account for the fixed point C at $d = 3$

quantity	Direct		Conformal Borel			
	1 loop	2 loops	3 loops	4 loops	5 loops	6 loops
g_1^*	0.818	1.285	0.896	1.03	1.10	1.14
g_2^*	-1.09	-1.71	-1.19	-1.37	-1.47	-1.52
ω_1	-0.091	0.221	0.036	0.077	0.109	0.131
ω_2	1.00	0.430	0.724	0.762	0.781	0.787
η	0.00	0.021	0.011	0.018	0.024	0.028

Table 2: Values of critical exponents at different number of loops taken into account for the fixed point C at $d = 2$

quantity	Direct		Conformal Borel			
	1 loop	2 loops	3 loops	4 loops	5 loops	6 loops
g_1^*	1.64	3.50	1.60	2.02	2.32	2.54
g_2^*	-2.18	-4.67	-2.13	-2.69	-3.10	-3.38
ω_1	-0.181	1.07	0.112	0.245	0.377	0.499
ω_2	2.00	-0.281	1.18	1.30	1.38	1.41
η	0.00	0.083	0.024	0.048	0.074	0.101

Borel-Leroy transform:

$$f(z) = \sum_{N\geq 0} f_N z^N \quad \Rightarrow \quad F(t) = \sum_{N\geq 0} \frac{f_N}{\Gamma(N+b_0+1)}\, t^N = \sum_{N\geq 0} F_N t^N$$

Pade approximation:

$$P_{[N,M]}(t) = \frac{P_N(t)}{P_M(t)} = \frac{\sum_{i=0}^{i=N} \alpha_i t^i}{\sum_{j=0}^{j=M} \beta_j t^j}$$

Inverse:

$$f^{res}(z) = \int_0^\infty dt\ e^{-t} t^{b_o} P_{[N,M]}(zt)$$

Table 3: Coordinate g_1^* of the fixed point C at $d = 3$ obtained with different approximants $P_{[N,M]}$. The value obtained with CB scheme for comparison $g_1^* = 1.138$

N \ M	0	1	2	3	4	5	6
0	None	None	None	None	None	None	None
1	0.82	2.93	0.77	1.44	0.47	0.026	
2	1.28	1.12	1.23	1.12	1.32		
3	1.03	1.19	1.17	1.18			
4	1.51	1.16	1.18				
5	0.22	1.20					
6	4.48						

Table 4: Eigenvalue ω_2 for the point C at $d = 3$ obtained with different approximants $P_{[N,M]}$. The value obtained with CB scheme for comparison $\omega_2 = 0.787$

N \ M	0	1	2	3	4	5	6
0	None	None	None	None	None	None	None
1	1.0	0.645	1.18	0.434	-0.129	0.201	
2	0.430	0.817	0.776	0.818	0.755		
3	1.71	0.769	0.797	0.791			
4	-2.07	0.831	0.790				
5	11.1	0.708					
6	-41.1						

$$f(z) = \sum_{N \le l} f_N z^N \quad \Rightarrow \quad F(t) = \sum_{N \le l} \frac{f_N}{\Gamma(N + b_0 + 1)} t^N = \sum_{N \le l} F_N t^N$$

$$\Rightarrow \widetilde{F}(u(t)) = \sum_{N \le k} \widetilde{F}_N u(t)^N$$

$$u(t) = \frac{\sqrt{1 + at} - 1}{\sqrt{1 + at} + 1} \simeq t + O(t^2)$$

Re-expand back in terms of t

$$\widetilde{F}(u(t)) = \sum_{N \le l} F_N t^N + \sum_{N > l} F_N^r t^N$$

So that knowing first $l - 1$ coefficients f_N one can reconstruct f_l^r, and estimate its proximity to exact value f_l introducing relative discrepancy:

$$\xi_l = \frac{f_l - f_l^r}{f_l}$$

Table 5: Prediction of six loop contribution to the ε-expansions of critical exponents at the IR attractive fixed point

quantity		3 loops	4 loops	5 loops	6 loops
g_1^*	value	-174	81.8	-9.18	4.26
	ξ_6	-41.9	18.2	-3.15	
g_2^*	value	232	-109	12.2	-5.68
	ξ_6	-41.9	18.2	-3.15	
ω_1	value	-44.7	36.6	-2.50	5.28
	ξ_6	-9.47	5.93	-1.47	
ω_2	value	-104	-30.1	-54.2	-52.2
	ξ_6	0.99	0.42	0.04	
η	value	-5.77	4.54	-1.35	-0.057
	ξ_6	100	-80.7	22.7	

$$f(z) = \sum_{N \leq l} f_N z^N \quad \Rightarrow \quad F(t) = \sum_{N \leq l} \frac{f_N}{\Gamma(N + b_0 + 1)} t^N = \sum_{N \leq l} F_N t^N$$

$$\Rightarrow P_{[K,M]}(t) = \frac{\sum_{i=0}^{i=K} \alpha_i t^i}{\sum_{j=0}^{j=M} \beta_j t^j}, \quad K + M \leq N$$

Re-expand back in terms of t

$$P_{[K,M]}(t) = \sum_{N \leq l} F_N t^N + \sum_{N > l} F_N^r t^N$$

There asymptotic behavior of F_N^r at large N is uncontrolled.

Table 6: Relative discrepancy ξ_6 for Pade-Borel predictions to 6-loops contribution to g_1^*

N \ M	1	2	3	4
1	0.97	0.89	0.99	1.97
2	0.98	0.70	0.27	
3	0.50	0.11		
4	0.14			

Table 7: Relative discrepancy ξ_6 for Pade-Borel predictions to 6-loops contribution to ω_2

N \ M	1	2	3	4
1	0.99	0.92	0.71	0.49
2	0.55	0.19	0.04	
3	0.21	0.02		
4	0.05			

$$f(z) = \sum_{N \le l} f_N z^N \quad \Rightarrow \quad F(t) = \sum_{N \le l} F_N t^N = \sum_{N \le l} \frac{f_N}{\Gamma(N + b_0 + 1)} t^N$$

$$\Rightarrow \widetilde{F}(u(t)) = \sum_{N \le l} \widetilde{F}_N u(t)^N$$

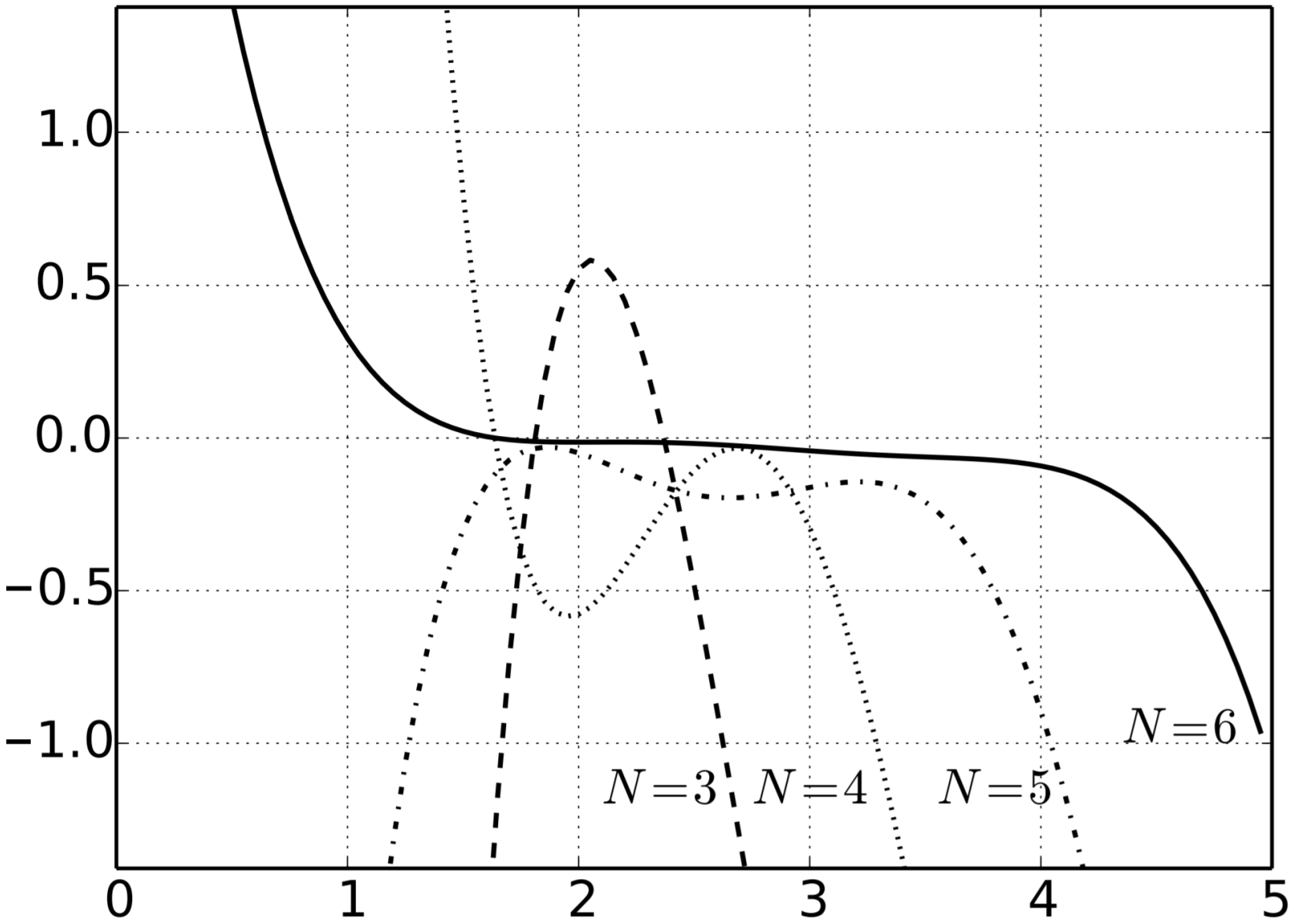
$$t \to 0 \qquad u(t) = \frac{\sqrt{1+at} - 1}{\sqrt{1+at} + 1} \simeq t + O(t^2)$$

$$t \to \infty \qquad u(t) = \frac{\sqrt{1+at} - 1}{\sqrt{1+at} + 1} \simeq 1 + O(\frac{1}{t})$$

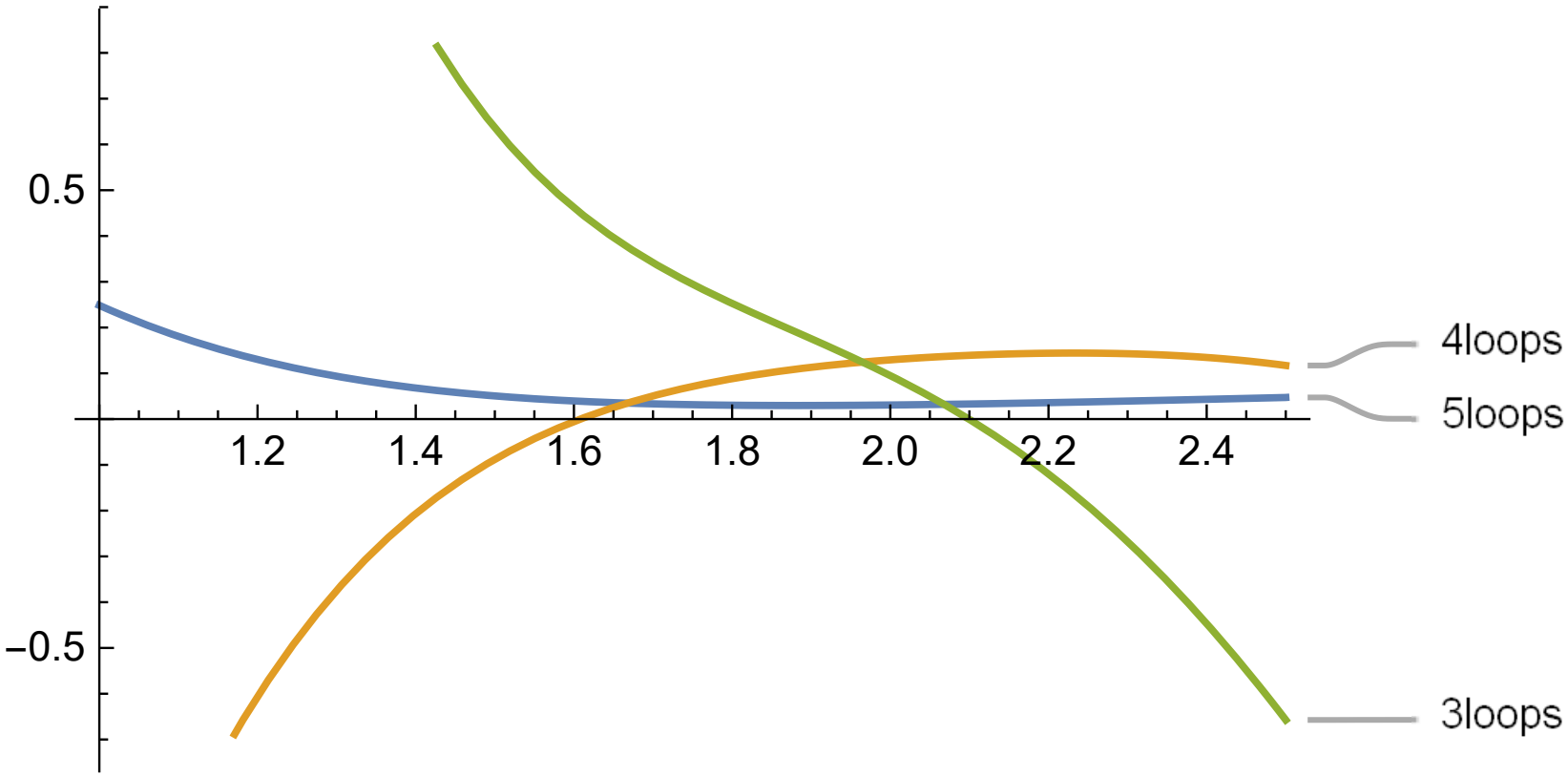
$$\widetilde{F}(u(t)) = \left(\frac{t}{u(t)}\right)^{\nu} \left(\sum_{N \le l} \widetilde{F}_N u(t)^N\right)$$

So that now $F^r_{N>l}$ depend on ν, which is parameter governing large z behaviour

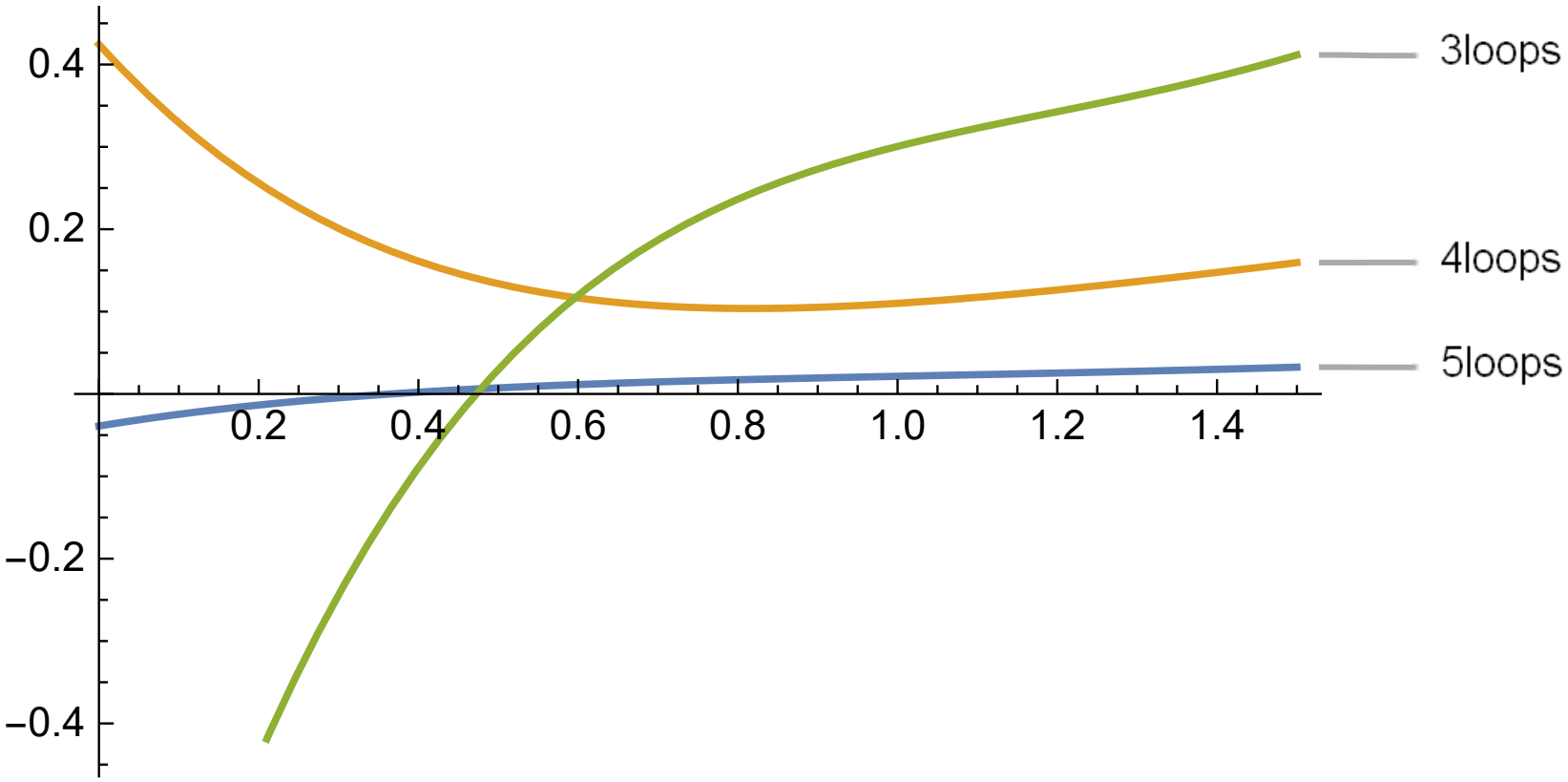
$$f^{res}(z) = \int_0^\infty dt \, \left(\frac{tz}{u(t)}\right)^{\nu} \left(\sum_{N \ge 0} \widetilde{F}_N e^{-t} t^{b_o} u(zt)^N\right)$$



From M.V. Kompaniets, Journal of Physics: Conference Series 762 (2016)



Dependence of ξ_6 for the coordinate g_1^* of the IR attractive fixed point on parameter ν



Dependence of ξ_6 for the eigenvalue ω_2 for the IR attractive fixed point on parameter ν

Rescale couplings:

$$\beta(g_1, g_2) = \sum_{i,j} \beta_{i,j}\, g_1^i g_2^j \quad \Rightarrow \quad \beta(z) = \beta(zg_1, zg_2) = \sum_i \beta_i(g_1, g_2) z^i$$

$$\beta(z)|_{z=1} = \beta(g_1, g_2)$$

$$B(t) = \sum \frac{\beta_N(g1, g2)}{\Gamma(N + b_0 + 1)}\, t^N = \sum B_N t^N$$

At each point of plane of couplings the most relevant instanton shell be used:

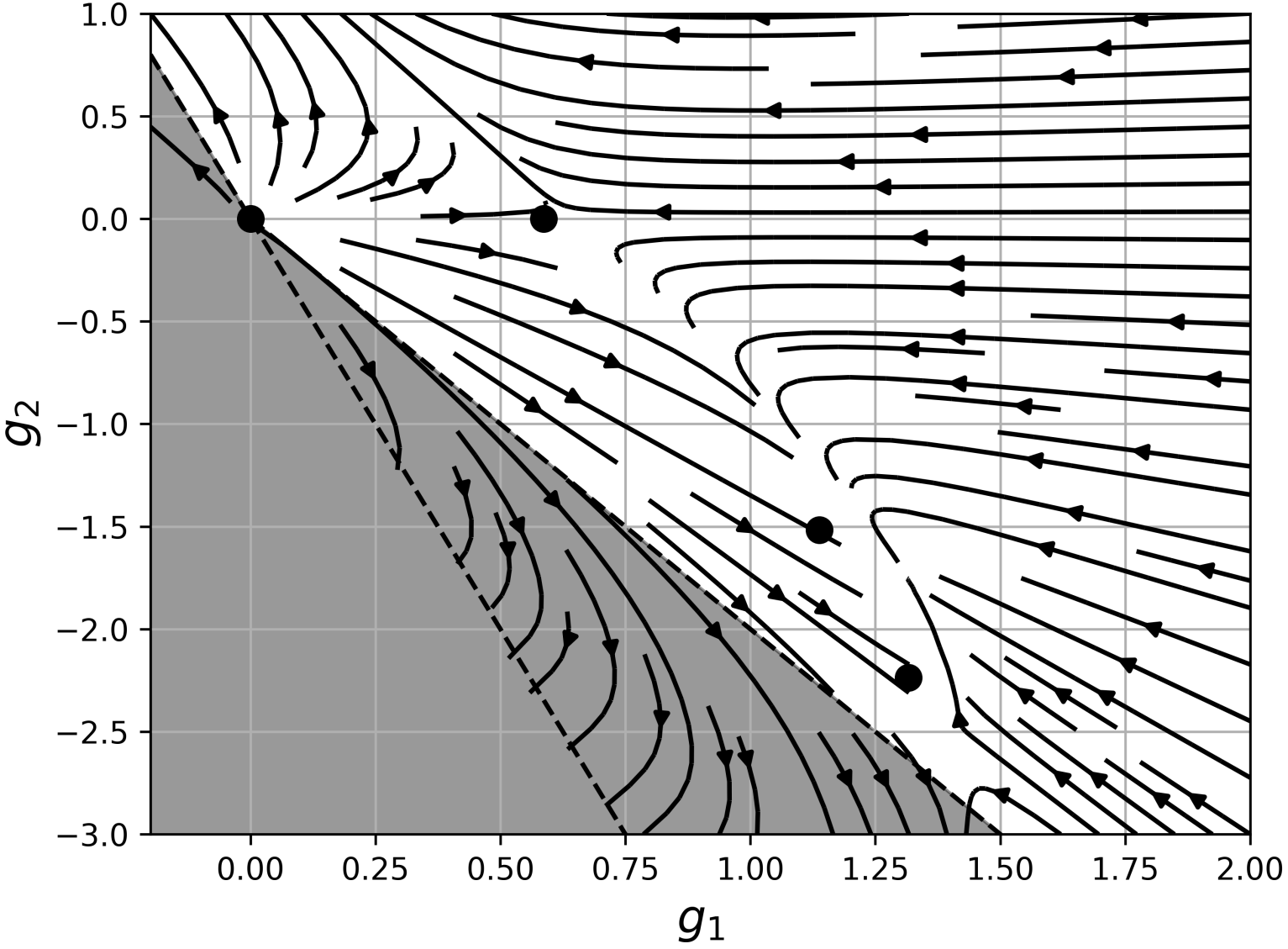
$$u(t) = \frac{\sqrt{1 + a(g_1, g_2)t} - 1}{\sqrt{1 + a(g_1, g_2)t} + 1} \quad \Leftrightarrow \quad t(u) = \frac{4u}{a(g_1, g_2)(u-1)^2},$$

$$\beta^{res}(g_1, g_2) = \beta^{res}(z = 1) = \sum \widetilde{B}_N \int_0^\infty dt\; e^{-t} t^{b_o} u(t)^N$$

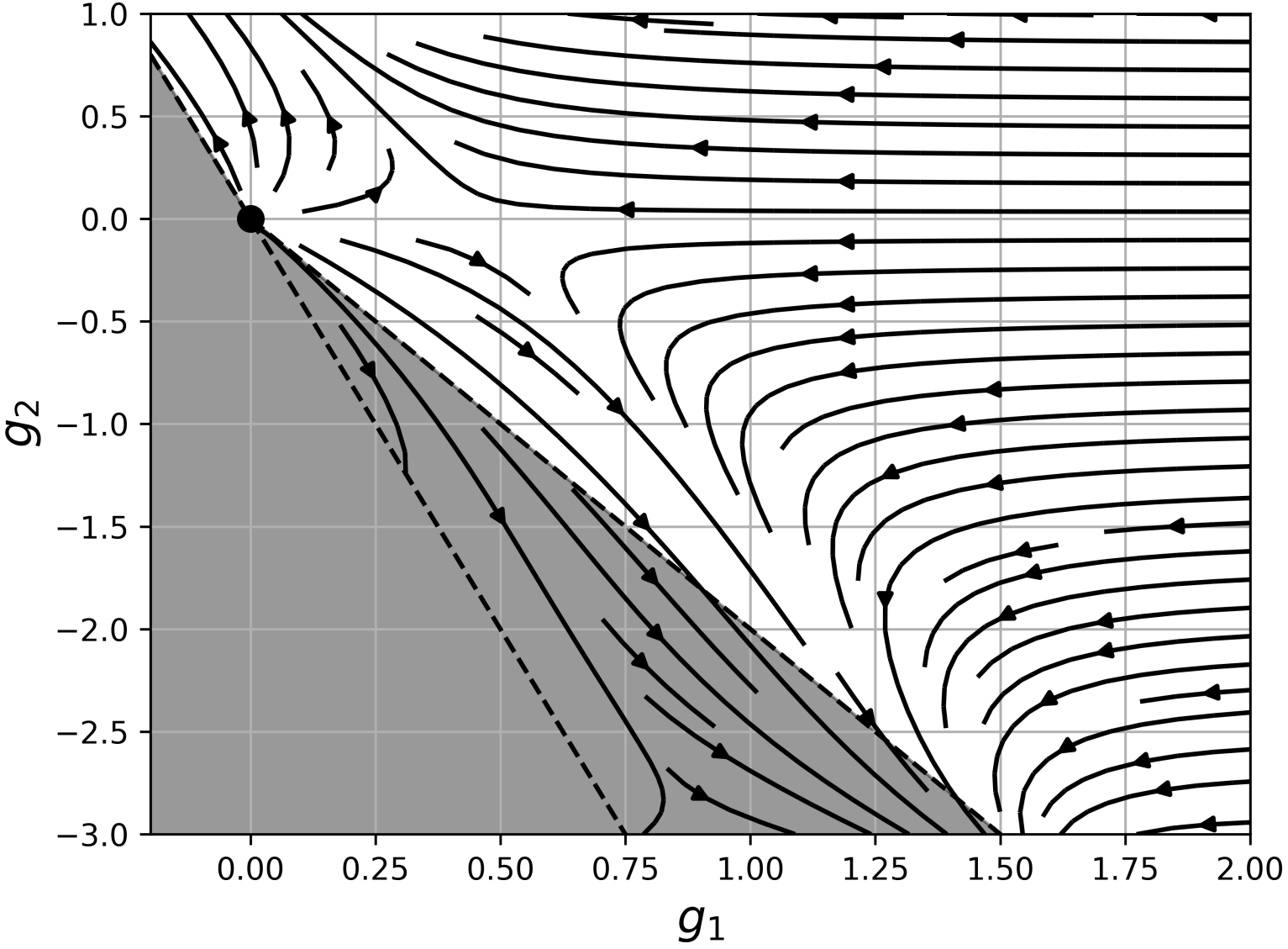
Solve numerically:

$$s\partial_s \bar{g}(s, g) = \beta_g^{res}(\bar{g}), \quad \bar{g}(1, g) = g$$

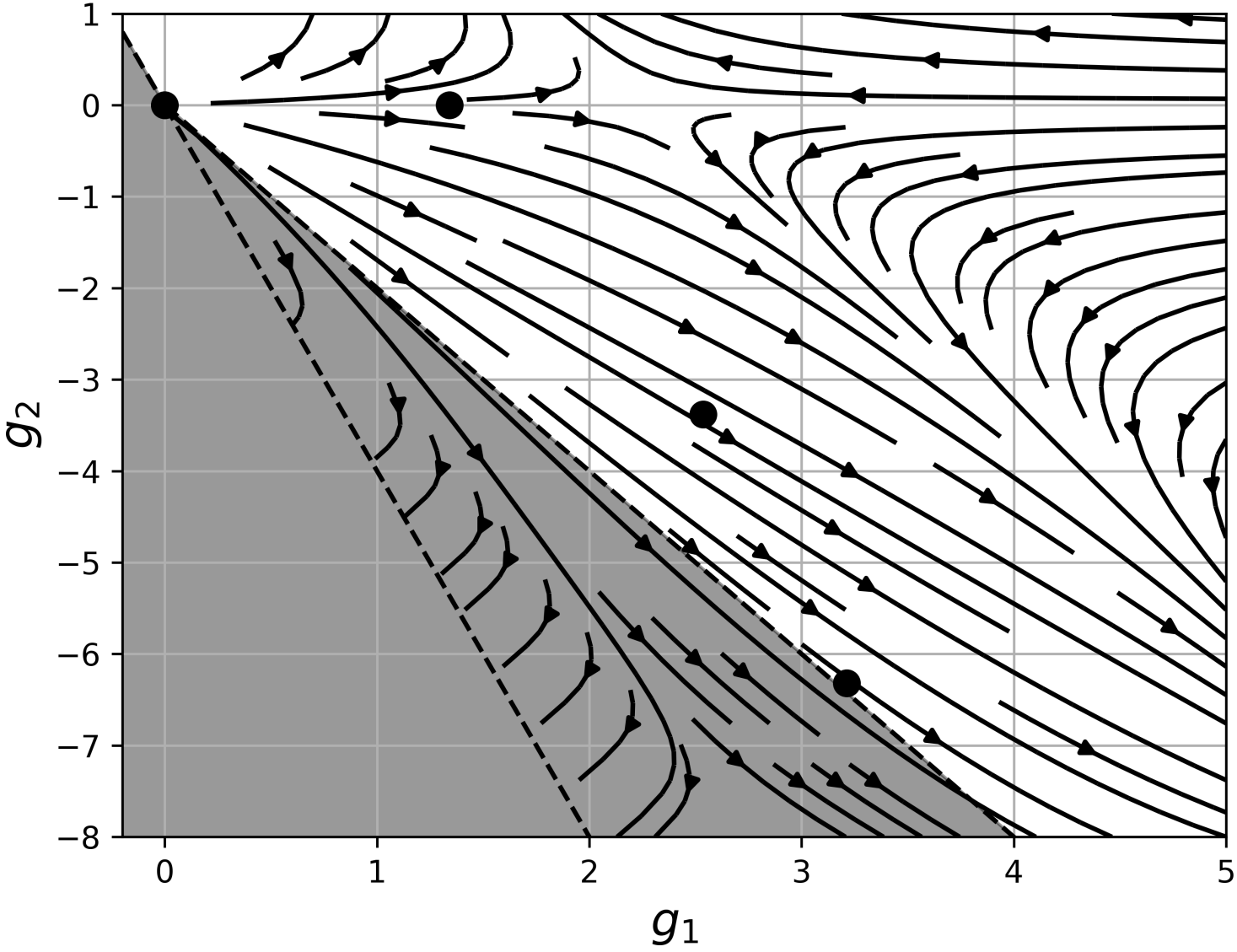
RG flows resummed with Conformal Borel scheme at $n = 4$ and $d = 3$



RG flows resummed with Conformal Borel scheme at $n = 5$ and $d = 3$



RG flows resummed with Conformal Borel scheme at $n = 4$ and $d = 2$



Thank you for attention!