Dynamical Casimir effect in nonlinear resonant cavities

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Quantization of nonstationary quantum systems

- In nonstationary quantum systems, the notions of particle and vacuum state **cannot be fixed once and forever**
- If we assume that the system is stationary in the asymptotic past and future but nonstationary at intermediate times, we can introduce **two alternative decompositions** for the quantized field:

$$\hat{\phi}(t,x) = \begin{cases} \sum_n \hat{a}_n^{\text{in}} f_n^{\text{in}}(t,x) + \text{H.c.} \\ \sum_n \hat{a}_n^{\text{out}} f_n^{\text{out}}(t,x) + \text{H.c.} \end{cases}$$

  Here, mode functions f_n^{in} and f_n^{out} **diagonalize the free Hamiltonian** in the asymptotic past and future, respectively
- In general, these mode functions do not coincide, determine different vacua, and are **related via the Bogoliubov transformations**:

$$f_n^{\text{out}} = \sum_k \left[\alpha_{kn}^* f_k^{\text{in}} - \beta_{kn} \left(f_k^{\text{in}} \right)^* \right]$$

Particle creation in nonstationary quantum systems

- The creation and annihilation operators in the asymptotic past and future are also **related via a similar transformation**:

$$\hat{a}_n^{\text{out}} = \sum_k \left[\alpha_{kn} \hat{a}_k^{\text{in}} + \beta_{kn}^* \left(\hat{a}_k^{\text{in}} \right)^\dagger \right]$$

- So, the energy level density and correlated pair density in the asymptotic past and future are different, thus **indicating a change in the quantum state**:

$$\begin{aligned} n_{pq}^{\text{out}} &= \langle 0 | \left(\hat{a}_p^{\text{out}} \right)^\dagger \hat{a}_q^{\text{out}} | 0 \rangle = \sum_n \beta_{np} \beta_{nq}^* \\ &\quad + \sum_{n,k} \left[\alpha_{np}^* \alpha_{kq} + \beta_{nq}^* \beta_{kp} \right] n_{nk}^{\text{in}} + \sum_{n,k} \beta_{np} \alpha_{kq} \, \kappa_{nk}^{\text{in}} + \sum_{n,k} \alpha_{np}^* \beta_{kq}^* \, \kappa_{nk}^{\text{in}*}, \\ \kappa_{pq}^{\text{out}} &= \langle 0 | \hat{a}_p^{\text{out}} \hat{a}_q^{\text{out}} | 0 \rangle = \sum_n \alpha_{np} \beta_{nq}^* \\ &\quad + \sum_{n,k} \left[\beta_{np}^* \alpha_{kq} + \beta_{nq}^* \alpha_{kp} \right] n_{nk}^{\text{in}} + \sum_{n,k} \alpha_{np} \alpha_{kq} \, \kappa_{nk}^{\text{in}} + \sum_{n,k} \beta_{np}^* \beta_{kq}^* \, \kappa_{nk}^{\text{in}*}. \end{aligned}$$

- The diagonal part n_{pp}^{out} has the meaning of the **number of created particles**

Loop corrections

- Usually, these phenomena are discussed in the **tree-level approximation**, where all effects are contained in the Bogoliubov coefficients
- Nevertheless, real-world systems are **usually nonlinear**, i.e., interacting
- In such systems, the initial values of n_{pq}^{in} and κ_{pq}^{in} **receive loop corrections**
- Furthermore, in some systems, these loop corrections **secularly grow** and become large even for minuscule couplings:

$$n_{pq}^{\text{in}} \sim \lambda^{a_n} t^{b_n}, \qquad \kappa_{pq}^{\text{in}} \sim \lambda^{c_n} t^{d_n}$$

- In this case, the correct values of n_{pq}^{out} and κ_{pq}^{out} are restored only after the **resummation of all loop corrections**
- Recently, such a growth was observed in the **dynamical Casimir effect**
- However, the summation was not performed in the **most interesting resonant case**, where quantum averages rapidly grow and can be measured experimentally

Tree-level case

- First of all, we consider the **free case**: the scalar dynamical Casimir effect in a linear one-dimensional cavity with perfectly reflecting walls:

$$\left(\partial_t^2 - \partial_x^2\right) \phi(t,x) = 0, \quad \phi[t, L(t)] = \phi[t, R(t)] = 0$$

- We assume that the cavity is static in the asymptotic past and future but **resonant** at intermediate times:

$$L(t) = 0, \quad R(t) = \Lambda \quad \text{for} \quad t < 0 \text{ and } t > T,$$
$$L(t) = 0, \quad R(t) = \Lambda \left[1 + \epsilon \sin\left(\frac{2\pi t}{\Lambda}\right)\right] \quad \text{for} \quad 0 < t < T$$

- The in-modes are sought in the following form:

$$f_n^{\text{in}}(t,x) = \frac{i}{\sqrt{4\pi n}} \left[e^{-i\pi n G(t+x)} - e^{-i\pi n G(t-x)}\right]$$

where the function G solves the **Moore's equation**:

$$G\left[t + R(t)\right] - G\left[t - R(t)\right] = 2 \quad \text{with i.c.} \quad G(z \le \Lambda) = z/\Lambda$$

Solution to Moore's equation

- At large evolution times, the solution to the Moore's equation quickly approaches a **"staircase" profile**[1]:

$$G(t) \approx \frac{t}{\Lambda} - \frac{1}{\pi} \arctan \frac{[1-\zeta(t)] \sin \frac{2\pi t}{\Lambda}}{[1+\zeta(t)] + [1-\zeta(t)] \cos \frac{2\pi t}{\Lambda}} + \mathcal{O}(\epsilon),$$

  where $1/\epsilon \ll t/\Lambda \ll 1/\epsilon^2$ and $\zeta(t) = e^{-2\pi\epsilon t/\Lambda}$

- For practical purposes, in this interval, $G(t)$ can be approximated by a **piecewise-linear function**:

$$G(t) \approx \begin{cases} \tau + 2\delta\xi + \delta, & \text{as } -\frac{1}{2} \le \xi < -\delta, \\ \tau + \frac{1}{2} + \frac{1-2\delta+4\delta^2}{2\delta}\xi, & \text{as } -\delta \le \xi < \delta, \\ \tau + 1 + 2\delta\xi - \delta, & \text{as } \quad \delta \le \xi < \frac{1}{2}. \end{cases}$$

- Here, we parametrize $t/\Lambda = \tau + 1/2 + \xi$, $\tau \in \mathbb{N}$, $\xi \in [-1/2, 1/2)$, and approximate the half-width of the n-th stair riser as $\delta = \frac{1}{\pi} e^{-2\pi\epsilon\tau}$

[1] J. Math. Phys. **34**, 2742 (1993); Phys. Rev. A **59**, 3049 (1999).

Bogoliubov coefficients

- Using $G(z)$, we straightforwardly calculate the **Bogoliubov coefficients**:

$$\left.\begin{matrix}\beta_{nk}\\ \alpha_{nk}\end{matrix}\right\} = \frac{1}{2}\sqrt{\frac{k}{n}}\int_{t/\Lambda-1}^{t/\Lambda+1} e^{-i\pi n G(\Lambda z)\mp i\pi k z} dz$$

- On one hand, we analytically calculate this integral for **moderate frequencies** employing the approximate form of $G(z)$:

$$\left.\begin{matrix}\beta_{nk}\\ \alpha_{nk}\end{matrix}\right\} \approx \frac{1}{\pi}\frac{1-(-1)^{nk}}{(-1)^{(k-1)/2}}\frac{\sqrt{nk}}{(n\pm 2k\delta)(k\pm 2n\delta)},$$

for $n, k \ll 1/\delta$

Bogoliubov coefficients

On the other hand, we numerically estimate the Bogoliubov coefficients and show that they **exponentially decay at high frequencies**:

$$\left.\begin{matrix}\beta_{nk} \\ \alpha_{nk}\end{matrix}\right\} \sim e^{-(n+k)\delta} \quad \text{for} \quad n, k \gg 1/\delta$$

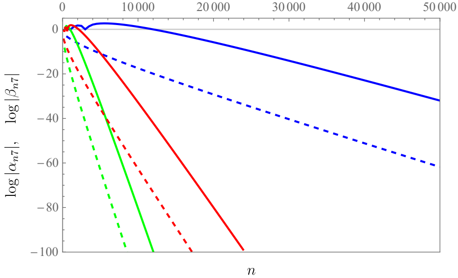

Figure: Numerically calculated Bogoliubov coefficients α_{n7} (solid lines) and β_{n7} (dashed lines) for $\delta = 1/1000\pi$ (blue), $\delta = 1/200\pi$ (red) and $\delta = 1/100\pi$ (green).

Quantum averages

- Keeping in mind the behavior of the Bogoliubov coefficients and assuming the vacuum initial state, we calculate the energy level density and correlated pair density in the asymptotic future:

$$n_{pq}^{\text{out}} \approx \kappa_{pq}^{\text{out}} \approx \frac{2}{\pi} \frac{1-(-1)^{pq}}{(-1)^{(p+q-2)/2}} \frac{1}{\sqrt{pq}} \frac{\epsilon t}{\Lambda},$$

for $p, q \ll 1/\delta$ and $n_{pq}^{\text{out}} \approx \kappa_{pq}^{\text{out}} \approx 0$ otherwise

- In particular, this approximate identity reproduces the rate of particle creation established 30 years ago:

$$\frac{d}{dt} n_p^{\text{out}} \approx \frac{2}{\pi} \frac{1-(-1)^p}{p} \frac{\epsilon}{\Lambda} \quad \text{for} \quad p \ll 1/\delta,$$

which confirms the validity of used approximations

Loop corrections and Schwinger-Keldysh technique

- Now, let us turn on a **quartic interaction**, i.e., consider the following nonlinear generalization of the free model:

$$\left(\partial_t^2 - \partial_x^2\right)\phi(t,x) = -\lambda\phi^3(t,x)$$

- In general, loop corrections to n^{in}_{pq} and κ^{in}_{pq} are conveniently calculated in the **Schwinger-Keldysh diagram technique**
- In our particular model, this technique contains two interaction vertices:

$$-i\lambda\int_{t_0}^{T}dt\int_{L(t)}^{R(t)}dx\,\phi_{cl}^3\phi_q, \quad -i\frac{\lambda}{4}\int_{t_0}^{T}dt\int_{L(t)}^{R(t)}dx\,\phi_{cl}\phi_q^3,$$

- And three propagators:

$$\begin{aligned} G_{12}^{\text{K(eldysh)}} &= -i\big\langle\phi_{cl}(t_1,x_1)\phi_{cl}(t_2,x_2)\big\rangle, \\ G_{12}^{\text{R(etarded)}} &= -i\big\langle\phi_{cl}(t_1,x_1)\phi_{q}(t_2,x_2)\big\rangle, \\ G_{12}^{\text{A(dvanced)}} &= -i\big\langle\phi_{q}(t_1,x_1)\phi_{cl}(t_2,x_2)\big\rangle, \end{aligned}$$

Propagators in the Schwinger-Keldysh technique

- The tree-level retarded propagators characterize the **particle spectrum** and do not depend on the initial state:

$$iG_{12}^{\mathrm{R,free}} = iG_{21}^{\mathrm{A,free}} = \theta(t_1 - t_2) \sum_n \left(f_{1,n}^{\mathrm{in}} f_{2,n}^{\mathrm{in}*} - \mathrm{H.c.} \right),$$

where we introduce the short notation $f_{a,n}^{\mathrm{in}} = f_n^{\mathrm{in}}(t_a, x_a)$.

- The tree-level Keldysh propagator is determined by **initial quantum averages** of interest to us:

$$iG_{12}^{\mathrm{K,free}} = \sum_{p,q} \left[\left(\frac{\delta_{pq}}{2} + n_{pq}^{\mathrm{in}} \right) f_{1,p}^{\mathrm{in}} f_{2,q}^{\mathrm{in}*} + \kappa_{pq}^{\mathrm{in}} f_{1,p}^{\mathrm{in}} f_{2,q}^{\mathrm{in}} + \mathrm{H.c.} \right].$$

- Furthermore, propagators approximately **preserve their tree-level form** as

$$\frac{t_1 + t_2}{2} \gg \frac{1}{\lambda \Lambda} \quad \text{and} \quad \frac{t_1 + t_2}{2} \gg |t_1 - t_2|.$$

- This property allows us to **extract the corrected quantum averages** in the interacting theory from the exact Keldysh propagator in the limit in question

Estimate of the exact Keldysh propagator

- To estimate the loop resummed Keldysh propagator in the interacting model with resonantly moving mirrors, we make **four crucial observations**
- *First*, we map the resonant mirror trajectories to **stationary ones**:

$$t + x = G^{-1}(\tau + \xi), \quad t - x = G^{-1}(\tau - \xi)$$

in all internal vertices of diagrams that describe loop corrections to the Keldysh propagator, e.g.:

$$\begin{aligned} V &= -i\lambda \int_{t_0}^{T} dt \int_{L(t)}^{R(t)} dx\, f_m^{\text{in}}(t,x) f_n^{\text{in}}(t,x) f_p^{\text{in}}(t,x) f_q^{\text{in}}(t,x) \\ &= -i\lambda \int_{\tau_0}^{\tau_f} d\tau \int_0^1 d\xi\, \frac{dG^{-1}(\tau-\xi)}{d\tau} \frac{dG^{-1}(\tau+\xi)}{d\tau} \times \\ &\quad \times e^{-i\pi(m+n+p+q)\tau} \frac{\sin(\pi m\xi)\sin(\pi n\xi)\sin(\pi p\xi)\sin(\pi q\xi)}{\pi^2\sqrt{mnpq}}. \end{aligned}$$

Estimate of the exact Keldysh propagator

- *Second*, we expect that the leading contribution to the loop corrections **come from large evolution times**: $t/\Lambda = \tau \gg 1/\epsilon$
- Hence, we can approximate the function $dG^{-1}(z)/dz$ with a piecewise-linear function close to a **sum of Dirac delta functions**
- *Third*, the exponential decay of the Bogoliubov coefficients imply that sums over the virtual momenta are **effectively cut off** at mode numbers $n \sim 1/\delta_s$
- Keeping in mind this cutoff, we replace the approximate delta functions with exact ones and **reduce the vertex integrals to sums**:

$$V \approx -i\lambda\Lambda^2 \sum_{s=1/\epsilon}^{\tau_f} g_m^s g_n^s g_p^s g_q^s,$$

where we introduce the notation for the **"remnant" of the initial mode** $f_n^{\text{in}}(t,x)$:

$$f_n^{\text{in}}(t,x) \to g_n^s = -i\frac{(-1)^s}{\sqrt{\pi n}}\frac{1-(-1)^n}{2}$$

Estimate of the exact Keldysh propagator

- *Fourth*, now, it is straightforward to see that **diagrams containing internal retarded/advanced propagators are approximately zero**
- Indeed, the "remnants" of internal modes are purely imaginary, and their product is purely real, so their combination $f^{\text{in}}_{1,n} f^{\text{in}*}_{2,n} - \text{H.c.} \approx 0$
- Hence, we can consider only such loop diagrams where internal vertices are connected by the Keldysh propagators alone
- There are **four such diagrams**, three of which are trivially absorbed into the renormalized mass

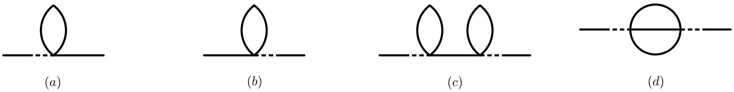

Figure: Loop corrections to the Keldysh propagator that do not contain internal retarded/advanced propagators. Solid lines denote the tree-level Keldysh propagators, half-dashed lines denote the retarded/advanced propagators.

Estimate of the exact quantum averages

- Keeping in mind these approximations, we obtain the leading loop correction to initial quantum averages:

$$\Delta n_{pq}^{\text{in}} \approx \Delta \kappa_{pq}^{\text{in}} \approx \frac{3}{5\pi} \frac{1-(-1)^{pq}}{2\sqrt{pq}} (\lambda \Lambda T)^2 \left(\frac{\epsilon T}{\Lambda}\right)^3,$$

for $p, q \ll 1/\delta_{\tau_f} \sim e^{2\pi\epsilon T/\Lambda}$

- Finally, we determine the relative correction to the quantum averages n_{pq}^{out} and κ_{pq}^{out}, which are physically meaningful in the asymptotic future:

$$\frac{\Delta n_{pq}^{\text{out}}}{n_{pq}^{\text{out}}} \approx \frac{\Delta \kappa_{pq}^{\text{out}}}{\kappa_{pq}^{\text{out}}} \approx \frac{12}{5} (\lambda \Lambda T)^2 \left(\frac{\epsilon T}{\Lambda}\right)^4$$

- So, loop contributions to the energy level density and correlated pair density **significantly exceed the tree-level expressions** in the time interval $1/\lambda\Lambda^2 \ll T/\Lambda \ll 1/(\lambda\Lambda^2)^2$ and $1/\epsilon \ll T/\Lambda \ll 1/\epsilon^2$

Discussion and open questions

- We considered the nonlinear dynamical Casimir effect in a one-dimensional cavity
- We calculated the leading loop corrections to the quantum averages (in particular, the number of created particles) generated during the resonant motion of cavity walls
- At large times, loop corrections significantly exceed the tree-level values of quantum averages
- This result encourages a careful measurement of the large-time behavior of quantum averages in the experimental implementations of the dynamical Casimir effect
- It is interesting to extend this result to other types of the resonant motion
- In addition, the nonlinear dynamical Casimir effect is very similar to a light interacting field in a rapidly expanding universe, so it would be promising to study our calculations in light of this relation