

Six-loop beta functions in general scalar theory: recent applications

Alexander Bednyakov

Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research

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(in collaboration with **A. Pikelner**, J. Henriksson, and S. Kosvous)



Outline

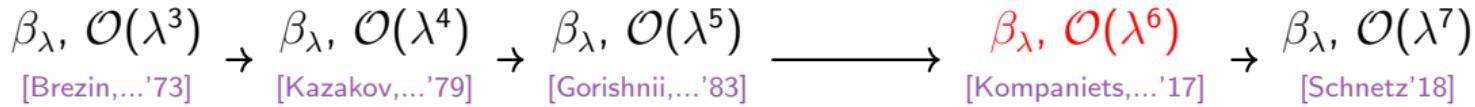
- Introduction
- From $O(N)$ -symmetric to general ϕ^4 in $d = 4$: 6 loops
 - Self-coupling beta function
 - Dummy-field method and RG functions of dimensionful parameters
 - Some results and cross-checks
- Recent applications:
 - Matching large-charge expansion and perturbative series in $O(N)$ -symmetric model at 5 and 6 loops
 - Anomalous dimensions of the low-lying operators in theories with hypercubic symmetry - 6 loop ϵ -expansion
- Conclusions and outlook

Introduction

- Renormalization Group is a powerful instrument to deal with UV and IR:
 - particle physics: high-energy asymptotics [Bogoliubov,Shirkov]
 - statistical physics: critical exponents [Vasiliev]
- RG flow in the space of couplings $\{\lambda_i\}$ and its fixed points (FPs):

$$\mu \frac{d\lambda_i}{d\mu} = \beta_i(\lambda_j), \quad \lambda_i = \lambda_i^* : \quad \beta_i(\lambda_j^*) = 0$$

- Dimensional regularization $d = 4 - 2\epsilon$ and ϵ -expansion (usually in $\overline{\text{MS}}$)
- Progress in models with order parameter being a scalar field in $\lambda\phi^4$ theory:

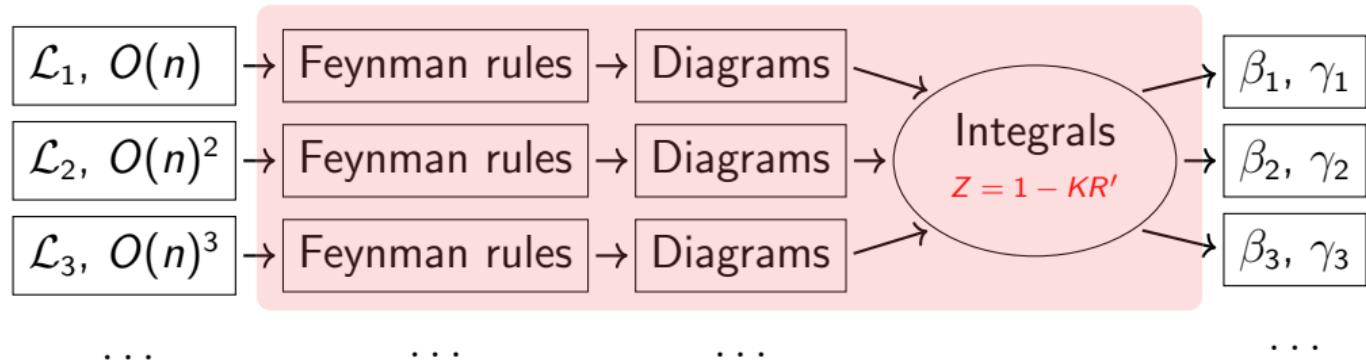


From $O(N)$ -symmetric to general ϕ^4 in $d = 4$: self-coupling

- Prominent 6-loop calculation in $O(n)$ -symmetric model [Kompaniets&Panzer'16-17]
⇒ diagram-by-diagram results of KR' -operation for 2pt and 4pt functions

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}(\phi^2)^2, \quad \phi^2 \equiv \sum_{i=1}^N \phi_i \phi_i$$

- Problem: Models with different symmetries require full (re)calculation?



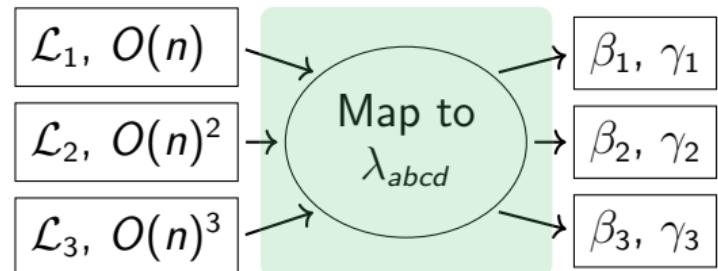
From $O(N)$ -symmetric to general ϕ^4 in $d = 4$: self-coupling

- Prominent 6-loop calculation in $O(n)$ -symmetric model [Kompaniets&Panzer'16-17]
⇒ diagram-by-diagram results of KR' -operation for 2pt and 4pt functions
- Compute renormalization constants in a general scalar theory for real ϕ_a

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_a\partial_\mu\phi_a - \frac{m_{ab}^2}{2}\phi_a\phi_b - \frac{h_{abc}}{3!}\phi_a\phi_b\phi_c - \frac{\lambda_{abcd}}{4!}\phi_a\phi_b\phi_c\phi_d - t_a\phi_a - \Lambda$$

via “simple” **replacement** of the $O(n)$ -factors in [Kompaniets&Panzer'17] by that of general theory (products of λ_{abcd} with all but four (two) indices contracted).

- **No need** to evaluate $\mathcal{O}(10^5)$ Feynman diagrams!
- **Just** substitution & index contraction via FORM [Vermaseren'92]



NB: 3-loop general ϕ^4 RG [Steudtner'20]

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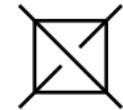
Beta function for λ_{abcd} at 6 loops

We use GraphState [Batkovic...'14] to cast the result in the following form

- Self-coupling at 6 loops

$$\beta_{abcd} = \mu \frac{\partial \lambda_{abcd}}{\partial \mu} = \sum_{l=1}^6 h^l \sum_{i=1}^{n_l} T_{i,abcd}^{(l)} C_i^{(l)}, \quad h = \frac{1}{16\pi^2}$$

where $n_l = \{1, 2, 7, 23, 110, 571\}$ counts Tensor Structures $T_{i,abcd}^{(l)}$ at l loops

Ex: $T_{7,abcd}^{(3)} \equiv \frac{1}{4!} [\lambda_{ai_1i_2i_3} \lambda_{bi_3i_4i_6} \lambda_{ci_2i_4i_5} \lambda_{di_1i_5i_6} + \text{perm.}] =$  , $\underbrace{\text{e123|e23|e3|e}}_{\text{Nickel index}}$

$$C_7^{(3)} = 12\zeta_3$$

- Field anomalous dimension (AD) at 6 loops

$$\gamma_{ab} = \sum_{l=1}^6 h^l \sum_i^{n_l} T_{i,ab}^{(l)} \tilde{C}_i^{(l)}, \quad \text{with } n_l = \{0, 1, 1, 4, 15, 69\}$$

(see supplementary files of [Bednyakov,Pikelner'21])

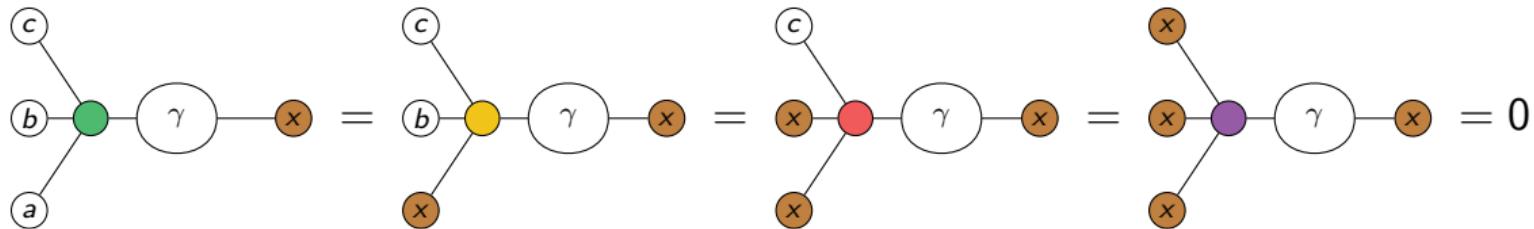
Dummy-field method and β -functions of dimensional parameters

- introduce non-propagating field(s) x_a : $\phi_a \rightarrow \phi_a + x_a$, and identify

$$\lambda_{abcx} \equiv h_{abc}, \quad \lambda_{abxx} \equiv 2m_{ab}^2, \quad \lambda_{axxx} \equiv 3!t_a, \quad \lambda_{xxxx} \equiv 4!\Lambda$$

$$\beta_\Lambda = \frac{1}{4!} \cdot \tilde{\beta}_{xxxx}, \quad \beta_a = \frac{1}{3!} \cdot \tilde{\beta}_{axxx}, \quad \beta_{ab} = \frac{1}{2} \cdot \tilde{\beta}_{abxx}, \quad \beta_{abc} = \tilde{\beta}_{abcx},$$

where in $\tilde{\beta}$ s we remove “tadpole” contributions:



6-loop RG functions in general ϕ^4 : how to use (I)

- Enumerate *real* scalars explicitly , as, e.g., in 2HDM ($a = 1, \dots, 8$):

$$\phi_a = (\text{Re}\Phi_{1,1}, \dots, \text{Im}\Phi_{2,2}), \quad \lambda_{1111} = 3\lambda_1, \dots, \quad \beta_{\lambda_1} = \frac{1}{3}\beta_{1111}, \dots$$

- For **matrix** fields ϕ decompose (alternative to multi-index notation)

$$\phi = \sum_{a=1}^{N_a} \chi_a T_a,$$

where χ_a are real fields, and there are N_a independent matrices T_a , which encode all the degrees of freedom present in ϕ .

$$m_{ab}^2 = -\tau \delta_{ab}, \quad \lambda_{abcd} = \frac{\lambda_1}{4!} [T^{ab} T^{cd} + \text{perm.}] + \frac{\lambda_2}{4!} [T^{abcd} + \text{perm.}],$$

$$T^{ab} = \text{Tr}(T_a T_b^T), \quad T^{abcd} = \text{Tr}(T_a T_b^T T_c T_d^T), \quad \text{for real } \phi,$$

$$T^{a\bar{b}} = \text{Tr}(T_a T_b^\dagger), \quad T^{a\bar{b}c\bar{d}} = \text{Tr}(T_a T_b^\dagger T_c T_d^\dagger), \quad \text{for complex } \phi.$$

- For **higher tensor** fields use **multi-index** notation, e.g., in $[O(n)]^3$ $a = \{a_1, a_2, a_3\}$.

6-loop RG functions in matrix ϕ^4 : cross-checks and “old” results

Real fields

- $\phi^T = -\phi$ 4L [Lebedev&Kompaniets'18]
5L&6L [AVB&Pikelner'21]

$$T^{ab} = \delta^{ab}, T_{ij}^a T_{kl}^a = \frac{1}{2}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})$$

- $O(n) \times O(m)$ 5L [Calabrese&Parruccini'04]
6L [Kompaniets...'20]

$$T^{ab} = \delta^{ab}, T_{\alpha i}^a T_{\beta j}^a = \delta_{\alpha\beta}\delta_{ij}$$

- Quadratic ops.
- 5L [Pelissetto&Vicari'07]
6L [AVB&Pikelner'21]

$$\begin{aligned} Q_{\alpha i \beta j}^{(1)} &= \phi_{\alpha i} \phi_{\beta j} - \phi_{\alpha j} \phi_{\beta i}, & Q_{\alpha i \beta j}^{(2)} & \\ Q_{ij}^{(3)} &= \phi_{\delta i} \phi_{\delta j} - \frac{1}{m} \delta_{ij} \phi_{\delta k} \phi_{\delta k}, & Q_{\alpha \beta}^{(4)} & \end{aligned}$$

Complex fields

- $\phi^T = -\phi$ 5L [Kalagov...'15]
6L [AVB&Pikelner'21]

$$\begin{aligned} T^{a\bar{b}} + T^{b\bar{a}} &= \delta^{ab}, T_{ij}^a T_{kl}^a = 0, \\ T_{ij}^a \bar{T}_{kl}^a &= \frac{1}{2}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) \end{aligned}$$

- $U(n) \times U(m)$ 5L [Calabrese&Parruccini'04]
6L [AVB&Pikelner'21]

$$\begin{aligned} T^{a\bar{b}} + T^{b\bar{a}} &= \delta^{ab}, T_{\alpha i}^a T_{\beta k}^a = 0, \bar{T}_{j\alpha}^a \bar{T}_{i\beta}^a = 0 \\ T_{\alpha i}^a \bar{T}_{j\beta}^a &= \delta_{\alpha\beta}\delta_{ij} \end{aligned}$$

- adj. $SU(n)$ [Litim...'20,Hnatić...'20]
+ $\text{Tr}\phi^3, \Lambda$ 5L&6L [AVB&Pikelner'21]

$$T^{ab} = \delta^{ab}, T_{ij}^a T_{kl}^a = \delta_{il}\delta_{kj} - \delta_{ij}\delta_{kl}/n$$

6-loop RG functions in general ϕ^4 : how to use (II)

To compute AD of any operator without derivatives

$$O = \sum_{a_1, \dots, a_n} P_{a_1 \dots a_n} \prod_{i=1}^n \phi_{a_i}$$

built from $n \leq 4$ scalar fields, add the following perturbation to the model Lagrangian

$$\delta L = g_O O = \underbrace{\left[g_O \sum_{a_1, \dots, a_n} P_{a_1 \dots a_n} \right]}_{g_{a_1 \dots a_n}} \prod_{i=1}^n \phi_{a_i}$$

and identify the symmetrized version of $g_{a_1 \dots a_n}$ either with m_{ab}^2 , h_{abc} or λ_{abcd} . The corresponding β -function can be used to extract β_{g_O} and AD of the operator

$$\gamma_O \equiv \frac{\partial \ln Z_O}{\partial \ln \mu} = \frac{\partial \beta_{g_O}}{\partial g_O}, \quad O_{bare} = Z_O[O]$$

NB: One can also account for multiple operators that can mix between each other.

Example: crossover exponent in $O(N)$ -symmetric model

We can consider non-singlet operators in $O(N)$ corresponding to some representations of the group, e.g., **traceless symmetric** two-component tensor operator

$$O_{ij} = \phi_i \phi_j - \frac{1}{N} \delta_{ij} \phi^2$$

$$\mathcal{L} = \frac{1}{2} \vec{\phi} (-\partial^2 + m^2) \vec{\phi} + \frac{\lambda}{4!} (\vec{\phi} \cdot \vec{\phi})^2 + \frac{1}{2} g_{\phi\phi} \mathbf{d}_{ab} \phi_a \phi_b, \quad \mathbf{d}_{aa} = 0$$

Inserting $m_{ab}^2 = m^2 \delta_{ab} + g_{\phi\phi} d_{ab}$ into our expression for β_{ab} one extracts β_{m^2} and $\beta_{g_{\phi\phi}}$. At the WF fixed point $\lambda = \lambda^* = \mathcal{O}(\epsilon)$ the **crossover exponent** is

$$\phi_c = \frac{2 - \beta_{g_{\phi\phi}}(\lambda^*)}{2 - \beta_{m^2}(\lambda^*)}$$

The latter is known at 6 loops from [Kompaniets & Wiese'19].

Matching large-charge and PT expansion in $O(N)$ -model

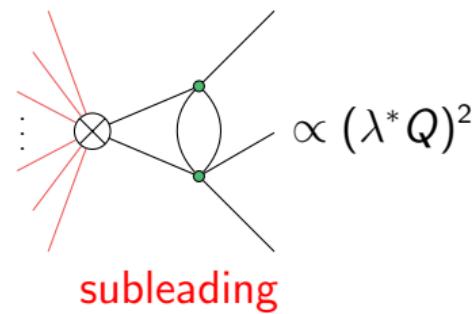
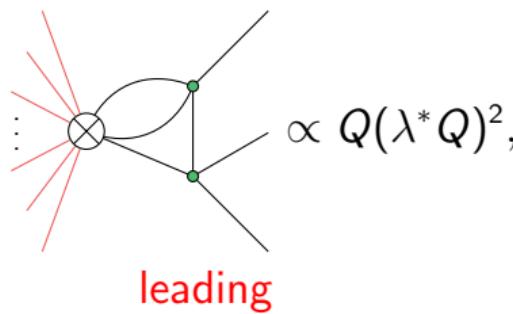
- $O(N)$ operators with **total charge** Q :

$$\varphi^Q \equiv d_{i_1 \dots i_Q} \phi_1 \dots \phi_{i_Q}, \quad d_{\dots i \dots i \dots} = 0$$

- Scaling dimensions can be computed **non-perturbatively** [Badel et al'19, Antipin et al'20]

$$\Delta_Q = \sum_{i=-1}^{\infty} (\lambda^*)^i \Delta_i(\lambda^* N) = \frac{\Delta_{-1}(\lambda^* Q)}{\lambda^*} + \Delta_0(\lambda^* Q) + \dots, \quad \lambda^* N = \text{fixed}$$

- Expanding Δ_{-1} and Δ_0 in small 't Hooft-like coupling $\lambda^* Q$, one predicts **leading** and **subleading** terms in large Q at **arbitrary high loop** :



Large- Q [Antipin et al'20] compared with explicit PT results [Jack & Jones '21] at 4 loops

Matching large- Q and PT expansions in $O(N)$ -model at 5 loops

- In PT anomalous dimensions of φ^Q are polynomials in λ , Q , and N

$$\Delta_Q = Q(1 - \epsilon) + \gamma_Q(\lambda^*) \quad \gamma_Q^{(5)} = Q^0 \left(\gamma_{0,0}^{(5)} + N\gamma_{0,1}^{(5)} + N^2\gamma_{0,2}^{(5)} + N^3\gamma_{0,3}^{(5)} + N^4\gamma_{0,4}^{(5)} \right)$$

$$\gamma_Q = Q \sum_{l=1}^{\infty} \lambda^l \gamma_Q^{(l)}, \quad + Q^1 \left(\gamma_{1,0}^{(5)} + N\gamma_{1,1}^{(5)} + N^2\gamma_{1,2}^{(5)} + N^3\gamma_{1,3}^{(5)} + N^4\gamma_{1,4}^{(5)} \right)$$

$$\gamma_Q^{(l)} = \sum_{r=0}^l Q^r \left[\sum_{s=0}^{l-r} N^s \gamma_{r,s}^{(l)} \right] \quad + Q^2 \left(\gamma_{2,0}^{(5)} + N\gamma_{2,1}^{(5)} + N^2\gamma_{2,2}^{(5)} + N^3\gamma_{2,3}^{(5)} \right)$$

$$\gamma_{0,l}^{(l)} = 0 \quad + Q^3 \left(\gamma_{3,0}^{(5)} + N\gamma_{3,1}^{(5)} + N^2\gamma_{3,2}^{(5)} \right)$$

$$+ Q^4 \left(\boxed{\gamma_{4,0}^{(5)}} + N \boxed{\gamma_{4,1}^{(5)}} \right)$$

$$+ Q^5 \boxed{\gamma_{5,0}^{(6)}}$$

- Q^5 and Q^4 coeffs are known from [Antipin et al'20].
- Lower- Q coeffs can be **fixed** by computing ADs of φ^Q for $Q \leq 4$
- Unpublished 5-loop result:) agrees with [Jin,Li'22]

Matching large- Q and PT expansions in $O(N)$ -model at 6 loops

- At 6 loops our general formula + large- Q give only 4+2 out of 7 constraints

$$\Delta_Q = Q(1 - \epsilon) + \gamma_Q(\lambda^*)$$

$$\gamma_Q^{(6)} = Q^0 \left(\gamma_{0,0}^{(6)} + N\gamma_{0,1}^{(6)} + N^2\gamma_{0,2}^{(6)} + N^3\gamma_{0,3}^{(6)} + N^4 \boxed{\gamma_{0,4}^{(6)}} + N^5 \boxed{\gamma_{0,5}^{(6)}} \right)$$

$$\gamma_Q = Q \sum_{l=1}^{\infty} \lambda^l \gamma_Q^{(l)},$$

$$+ Q^1 \left(\gamma_{1,0}^{(6)} + N\gamma_{1,1}^{(6)} + N^2\gamma_{1,2}^{(6)} + N^3\gamma_{1,3}^{(6)} + N^4 \boxed{\gamma_{1,4}^{(6)}} + N^5 \boxed{\gamma_{1,5}^{(6)}} \right)$$

$$\gamma_Q^{(l)} = \sum_{r=0}^l Q^r \left[\sum_{s=0}^{l-r} N^s \gamma_{r,s}^{(l)} \right]$$

$$+ Q^2 \left(\gamma_{2,0}^{(6)} + N\gamma_{2,1}^{(6)} + N^2\gamma_{2,2}^{(6)} + N^3\gamma_{2,3}^{(6)} + N^4 \boxed{\gamma_{2,4}^{(6)}} \right)$$

$$\gamma_{0,l}^{(l)} = 0$$

$$+ Q^3 \left(\gamma_{3,0}^{(6)} + N\gamma_{3,1}^{(6)} + N^2\gamma_{3,2}^{(6)} + N^3 \boxed{\gamma_{3,3}^{(6)}} \right) \quad \text{Large } N$$

Explicit 6L: KR' for φ^5 !

[Bednyakov' Pikelner'22]

Full 6-loop result Δ_Q
for arbitrary Q !

$$+ Q^4 \left(\gamma_{4,0}^{(6)} + N\gamma_{4,1}^{(6)} + N^2 \boxed{\gamma_{4,2}^{(6)}} \right)$$

$$+ Q^5 \left(\boxed{\gamma_{5,0}^{(6)}} + N \boxed{\gamma_{5,1}^{(6)}} \right) \quad \text{Small } J = Q/N$$

$$+ Q^6 \boxed{\gamma_{6,0}^{(6)}}. \quad \text{Large } Q$$

[Antipin et al'20]

[Derkachov, Manashov'97]

Anomalous dimensions in hypercubic theories

- $\mathcal{O}(N)$ model with cubic anisotropy (6L study [Adzhemyan et al'19]):

$$\mathcal{L} = \frac{(\partial\phi_i)^2}{2} + \frac{\lambda_{ijkl}}{4!}\phi_i\phi_j\phi_k\phi_l, \quad \lambda_{ijkl} = \frac{g_1}{3}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + g_2\delta_{ijkl}$$

- invariant under group of symmetries on an N -dim. hypercube $H_N \in O(N)$.
- four fixed points are predicted. At 1 loop ($d = 4 - \epsilon$):

$$(g_1^G, g_2^G) = (0, 0), \quad (g_1^I, g_2^I) = \left(0, \frac{\epsilon}{3}\right),$$
$$(g_1^O, g_2^O) = \left(\frac{3\epsilon}{N+8}, 0\right), \quad (g_1^H, g_2^H) = \left(\frac{\epsilon}{N}, \frac{N-4}{3N}\epsilon\right)$$

- 1-loop scaling dimensions of operators from reps. of H_N [Antipin,Bersini'19].
- **Example:** Traceless symmetric tensor $\phi_i\phi_j - \frac{1}{n}\delta_{ij}\phi^2$ splits into
 - diagonal $\delta_{ij}(\phi_i^2 - \frac{1}{n}\phi^2)$ (ϕ_{axial} exponent [Aharony'76,Aharony et al'22])
 - off-diagonal $\phi_i\phi_j$ ($i \neq j$) (ϕ_{diag} exponent [Aharony'76,Aharony et al'22])

Anomalous dimensions in hypercubic theories

Projectors: [Kousvos,Stergiou'21]

$$\phi_a \phi_b = \underbrace{P_{abij}^S}_{\nu} \phi_i \phi_j + \underbrace{P_{abij}^X}_{\phi_{\text{axial}}} \phi_i \phi_j + \underbrace{P_{abij}^Z}_{\phi_{\text{diag}}} \phi_i \phi_j$$

$$P_{abij}^S = \frac{1}{n} \left[\begin{array}{c} a \\ b \end{array} \circ \begin{array}{c} i \\ j \end{array} \right], \quad P_{abij}^X = \begin{array}{c} a \\ b \end{array} \times \begin{array}{c} i \\ j \end{array} - \frac{1}{n} \left[\begin{array}{c} a \\ b \end{array} \circ \begin{array}{c} i \\ j \end{array} \right],$$

$$P_{abij}^Z = \frac{1}{2} \left[\begin{array}{c} a \\ b \end{array} \text{---} \begin{array}{c} i \\ j \end{array} + \begin{array}{c} a \\ b \end{array} \backslash \begin{array}{c} i \\ j \end{array} - 2 \begin{array}{c} a \\ b \end{array} \times \begin{array}{c} i \\ j \end{array} \right],$$

$$P_{ab\alpha\beta}^A P_{\alpha\beta ij}^B = \delta^{AB} P_{abij}^A, \quad \text{Tr}(P^A) = \dim(A)$$

We compute critical exponents by considering

$$\delta \mathcal{L} \ni (g_Z P_{12ij}^Z + g_X P_{11ij}^X) \phi_i \phi_j$$

Anomalous dimensions in hypercubic theories (in progress)

- In [Kousvos,Stergiou'21] the projectors for **all** possible representations that can appear in the product of **two, three, or four powers** of order parameter ϕ_i .
- use the projectors and our general RGs, e.g., by

$$h_{abc} = g_{\phi_1} \underbrace{P_{122abc}^{\phi_1}}_{\phi \otimes S} + g_{\phi_2} \underbrace{P_{221abc}^{\phi_2}}_{\phi \otimes Z} + g_A \underbrace{P_{122abc}^A}_{\phi \otimes X} + g_{I_3} \underbrace{P_{123abc}^{I_3}}_{\phi \otimes Z}$$

$$\begin{aligned} \lambda_{abcd} \ni & g_{X_1} \underbrace{P_{1111abcd}^{X_1}}_{X \otimes S} + g_{X_2} \underbrace{P_{1111abcd}^{X_2}}_{X \otimes X} + g_{Z_1} \underbrace{P_{1211abcd}^{Z_1}}_{Z \otimes S} + g_{Z_2} \underbrace{P_{1211abcd}^{Z_2}}_{Z \otimes X} \\ & + g_{A_2} \underbrace{P_{1222abcd}^{A_2}}_{Z \otimes X} + g_{\bar{X}} \underbrace{P_{1221abcd}^{\bar{X}}}_{Z \otimes Z} + g_{XZ} \underbrace{P_{1233abcd}^{XZ}}_{Z \otimes Z} + g_{TotS} \underbrace{P_{1234abcd}^{TotS}}_{Z \otimes Z} \end{aligned}$$

- compute 6-loop ADs for **all** non-vanishing (eigen)operators.

[Bednyakov, Henriksson, Kousvos, Pikelner'22] (work in progress)

6-loop RG functions in general ϕ^4 : summary

- we provide convenient tool for computing 6-loop RG functions in arbitrary* scalar models **w/o necessity to evaluate Feynman diagrams** (FORM code)
- we compute
 - vacuum-energy anomalous dimension in $O(n)$ [$n = 4$ - the SM scalar sector]
 - anomalous dimensions of quadratic perturbations in the $O(n) \times O(m)$ model
 - quartic coupling β -functions in $U(n) \times U(m)$ and $[O(n)]^3$ models
 - RG functions in model with the Higgs field in the adj. rep of $SU(n)$
 - β -functions for the scalar sector of general Two-Higgs-Doublet Model

[Bednyakov,Pikelner'21]

- Anomalous dimension φ^Q -operators in $O(n)$

[Bednyakov,Pikelner'22]

- Spectrum of anomalous dimension in hypercubic theories

[Bednyakov,Henriksson,Kousvos,Pikelner,22,in progress]

6-loop RG functions in general ϕ^4 : outlook

Our results can be used

- to obtain **high-precision ϵ -expansion** for critical exponents
- as a playground for **re-summation** methods
- to facilitate **comparison/matching** with **non-perturbative** results
(large- N , large-charge, (numerical) conformal bootstrap, Monte-Carlo)

Thank you for your attention!