Strongly Nonlinear Diffusion in Turbulent Environment: A Problem with Infinitely Many Couplings

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The general advection-diffusion equation:

$$\nabla_t \theta = \partial_i J_i + f, \qquad \nabla_t = \partial_t + (v_i \partial_i), \qquad \{i = 1, \dots, d\}\,. \tag{1}$$

The random noise $f = f(x)$ is taken to be Gaussian with zero mean, δ-correlated in time, with the pair correlation function:

$$\langle f(x) f(x') \rangle_f = \delta(t - t') \int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d}\, D_f(k)\, \exp\{\mathrm{i}\mathbf{k}(\mathbf{x} - \mathbf{x}')\}, \tag{2}$$

$$D_f(k) = D_0\, k^{2-d-y}, \quad D_0 > 0.$$

The velocity field $\mathbf{v} = \{v_i(x)\}$, $i = 1, \dots, d$ is described by the Kazantsev–Kraichnan “rapid-change ensemble”. It means that the velocity field is taken Gaussian, with zero mean and the given pair correlation function

$$\langle v_i(x) v_j(x') \rangle_v = \delta(t - t') B_0 \int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d}\, \frac{1}{k^{d+\xi}}\, P_{ij}(\mathbf{k})\, \exp\{\mathrm{i}\mathbf{k}(\mathbf{x} - \mathbf{x}')\}, \tag{3}$$

$$B_0 > 0, \qquad B_0 = w\nu_0, \qquad 0 < \xi < 2, \qquad P_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}.$$

There are physical reasons to believe that the structure and correlation functions can exhibit scaling behavior in the IR region:

$$C_{2n}(t,r) = \langle [\theta(t,\mathbf{x}) - \theta(0,\mathbf{0})]^{2n} \rangle \simeq r^{-2n\Delta_\theta} F_{2n}(tr^{\Delta_t}), \quad r = |\mathbf{x}|. \tag{4}$$

$$G(t,r) = \langle \theta(t,\mathbf{x})\theta'(0,\mathbf{0}) \rangle \simeq r^{-\Delta_\theta - \Delta_{\theta'}} F(tr^{\Delta_t}). \tag{5}$$

It is the IR asymptotics that arose the interest, so only the first term can be left in the expression for the current $J_i = \partial_i V(\theta) + O(\partial^3)$.
Furthermore, function $V(\theta)$ have the explicit form:

$$V(\theta) = \sum_{n=1}^{\infty} \frac{1}{n!} \lambda_{n0}\, \theta^n. \tag{6}$$

Dimensional analysis shows that to study nonlinear diffusion, all terms in this sum must be taken into account.

In this case, the stochastic advection-diffusion equation takes the form:

$$\nabla_t \theta(x) = \partial^2 V(\theta(x)) + f(x). \tag{7}$$

According to the general De Dominicis–Janssen theorem the original stochastic problem is equivalent to the field theoretic model of an extended set of fields $\Phi = \{\theta', \theta, \mathbf{v}\}$ with the action functional $\mathcal{S}(\Phi) = \mathcal{S}_\theta(\Phi) + \mathcal{S}_v(\mathbf{v})$, where

$$\mathcal{S}_\theta(\Phi) = \frac{1}{2}\theta' D_f \theta' + \theta' \left[-\nabla_t \theta + \partial^2 V(\theta)\right], \tag{8}$$

$$\mathcal{S}_v(\mathbf{v}) = -\frac{1}{2}\int dt \int d\mathbf{x} \int d\mathbf{x}' v_i(t,\mathbf{x}) D^{-1}_{ij}(\mathbf{x}-\mathbf{x}') v_j(t,\mathbf{x}'). \tag{9}$$

Here D^{-1}_{ij} is kernel of the integration operation that is reversed to $D_{ij} = \langle v_i v_j \rangle_v$.

The following steps were taken there:

(1) It was shown that the constructed model is multiplicatively renormalizable. Also we derived the one-loop counterterm in closed form and calculate the full set of renormalization constants Z_n in the one-loop approximation.

(2) We wrote out the RG functions (anomalous dimensions γ and β functions).

(3) The RG equations was written there. We showed that their solutions are two attractors in the form of a pair of two-dimensional surfaces of fixed points in the infinite-dimensional parameter space. For each of these surfaces, there was investigated the critical dimensions of fields and frequencies that arise in formulas (4) and (5).

Canonical dimension of any value F:

$$[F] \sim [T]^{-d_F^\omega} [L]^{-d_F^k} . \qquad (10)$$

The complete canonical dimension is defined as $d_F = d_F^k + 2d_F^\omega$.
It is possible to introduce new parameters (couplings) according to the following relations:

$$\lambda_{n0} = g_{n0}\nu_0^{(n+1)/2}, \quad \lambda_n = g_n \nu^{(n+1)/2} \mu^{y(n-1)/2}, \quad n > 1. \qquad (11)$$

Then

F	θ	θ'	$\mathbf{v}$	m, μ	B_0	λ_{n0}
d_F^k	$1 - y/2$	$d - 1 + y/2$	-1	1	$\xi - 2$	$y\,(n-1)\,/2 - (n+1)$
d_F^ω	$-1/2$	$1/2$	1	0	1	$(n+1)\,/2$
d_F	$-y/2$	$d + y/2$	1	1	ξ	$y\,(n-1)\,/2$
F	λ_n	ν_0, ν	w_0	g_{n0}	w	g_n
d_F^k	$-\,(n+1)$	-2	ξ	$y\,(n-1)\,/2$	0	0
d_F^ω	$(n+1)\,/2$	1	0	0	0	0
d_F	0	0	ξ	$y\,(n-1)\,/2$	0	0

Table №1: Canonical dimensions of the model (8), (9).

Superficial UV divergences can appear in the 1-irreducible Green's functions with the formal index of divergence

$$\delta = (d+2) - d_{\theta'} N_{\theta'} - d_{\theta} N_{\theta} - d_{\mathbf{v}} N_{\mathbf{v}} \tag{12}$$

When analyzing the divergences of this model, one should take into account the following considerations:

1. All the 1-irreducible functions without the fields θ' in fact vanish.

2. In any 1-irreducible diagram each external field θ', attached to one of the vertices $\theta'(v_i\partial_i)\theta$ or $\theta'\partial^2\theta^n$, "releases" the corresponding external momentum, and the real index of divergence: $\delta - 2N_{\theta'} \leq \delta' \leq \delta - N_{\theta'}$.

3. The counterterm $\theta'\partial_t\theta$, allowed by the formal index δ, is in fact forbidden by the item (2): it does not contain a spatial gradient. On the other hand, because of the Galilean symmetry of the model there are also forbidden the counterterm $\theta'(v_i\partial_i)\theta$.

So, the superficial UV divergences in fact are only presented in the functions $\langle\theta'\theta\ldots\theta\rangle$. For all of them $\delta = 2$, $\delta' = 0$, and the counterterms can be reduced to the form $\theta'\partial^2\theta^n$. All such terms are already presented in the action functional (8), therefore, the model is multiplicatively renormalizable.

Renormalized action functional:

$$\mathcal{S}_R(\Phi) = \frac{1}{2}\theta' D_f \theta' + \theta' \left\{-\nabla_t\theta + \partial^2 V_R(\theta)\right\} + \mathcal{S}_{vR}(\mathbf{v}), \tag{13}$$

$$V_R(\theta) = \sum_{n=1}^{\infty} \frac{1}{n!} Z_n \lambda_n \theta^n \quad B_0 = B = w\nu\mu^{\xi}.$$

The functional (13) is obtained from (8) using the relations:

$$\lambda_{n0} = \lambda_n Z_n \ (n \geq 1), \quad \nu_0 = \nu Z_\nu, \quad w_0 = w\mu^{\xi} Z_w, \tag{14}$$

$$g_{n0} = g_n \mu^{(n-1)y/2} Z_{g_n} \ (n \geq 2).$$

The constants Z_w, Z_{g_n} and Z_ν can be expressed in terms of Z_n:

$$Z_\nu = Z_1, \quad Z_{g_n} = Z_n Z_1^{-(n+1)/2}, \quad Z_w = Z_1^{-1}. \tag{15}$$

Generating functional of 1-irreducible Green's functions in the renormalized theory:

$$\Gamma_R(\Phi) = \sum_{p=0}^{\infty} \Gamma^{(p)}(\Phi), \quad \Gamma^{(0)}(\Phi) = \mathcal{S}_R(\Phi), \quad \Gamma^{(1)}(\Phi) = -(1/2)\,\mathrm{Tr}\,\ln(W/W_0). \tag{16}$$

Here W is a linear operator with the kernel

$$W(x,x') = -\delta^2\, S_R(\Phi)\,/\delta\Phi(x)\delta\Phi(x').$$

For the Z_n only the UV-divergent part $\Gamma^{(1)}(\Phi)$ is needed there. In particular, it causes that (16) is sufficient to know only in the 1st order in the elements $W^{(\theta\theta)}$, $W^{(\theta v)}$ and $W^{(v\theta)}$ that are linear in θ'. All these considerations make it possible to explicitly write out the divergent part $\Gamma^{(1)}(\Phi)$ as poles in y and ξ (in the MS scheme):

$$\Gamma^{(1)}(\Phi) = \frac{a_d}{4y}\left(\frac{\mu}{m}\right)^y \int dx\, \theta'(x)\, \partial^2\, F\,(\theta(x)) + a_d\, \frac{(d-1)}{2d}\, \frac{w\nu}{\xi}\, \left(\frac{\mu}{m}\right)^{\xi} \int dx\, \theta'(x)\, \partial^2 \theta(x). \tag{17}$$

Here $a_d = S_d/(2\pi)^d$, where $S_d = 2\pi^{d/2}/\Gamma(d/2)$. Function $F(\theta)$ is defined as

$$F(\theta) = \mu^{-y}\,\frac{V''(\theta)}{V'(\theta)}. \tag{18}$$

Potential function $V(\theta)$ is given by the expression for $V_R(\theta)$ from (13) with the substitution $Z_n = 1$.

At the moment, it is not possible to continue the analysis in terms of closed functional representations

It remains only to expand $F(\theta)$ back in the powers of θ

$$F(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!}\,\mu^{y(n-1)/2}\,\nu^{(n+1)/2}\,r_n\,\theta^n. \tag{19}$$

So, from the requirement that the poles in y and ξ should be canceled in the loop expansion (16) we get

$$Z_1 = 1 - \frac{a_d}{4}\frac{r_1}{y} - \frac{a_d(d-1)}{2d}\frac{w}{\xi}, \quad Z_n = 1 - \frac{a_d}{4}\frac{r_n}{y}\frac{1}{g_n} \quad (n>1). \tag{20}$$

The RG equation for a multiplicatively renormalizable theory:

$$\left\{ \mathcal{D}_\mu - \beta_w \partial_w - \sum_{n=2}^{\infty} \beta_n \partial_{g_n} - \gamma_\nu \right\} G(e; \dots) = 0. \tag{21}$$

The anomalous dimensions γ and β functions are defined as

$$\gamma_F = \widetilde{\mathcal{D}}_\mu \ln Z_F \quad \forall \mathrm{F}, \quad \beta_w = \widetilde{\mathcal{D}}_\mu \, w, \quad \beta_n = \widetilde{\mathcal{D}}_\mu \, g_n. \tag{22}$$

According to the definitions (22)and the expressions (14), (15) all RG functions can be expressed in terms of the anomalous dimensions γ_n:

$$\gamma_\nu = -\gamma_w = \gamma_1, \qquad \gamma_{g_n} = \gamma_n - (n+1)\gamma_1/2, \tag{23}$$

$$\beta_w = w[-\xi + \gamma_1], \qquad \beta_n = g_n[-(n-1)y/2 - \gamma_n + (n+1)\gamma_1/2]. \tag{24}$$

Substitution of the one-loop expressions (20) into definitions (22) gives:

$$\gamma_1 = \widetilde{\mathcal{D}}_\mu \ln Z_1 = \frac{a_d}{4} r_1 + \frac{a_d(d-1)}{2d} w, \qquad \gamma_n = \widetilde{\mathcal{D}}_\mu \ln Z_n = \frac{a_d}{4} \frac{r_n}{g_n}. \tag{25}$$

Attractors of the RG equation:

$$\beta_w(w^*, g_n^*) = 0, \quad \beta_n(w^*, g_n^*) = 0 \quad (n > 1). \tag{26}$$

For the β_w function, from (24) and (25) we get:

$$\beta_w = w\left[-\xi + \frac{a_d}{4}(g_3 - g_2^2) + \frac{a_d(d-1)}{2d}w\right]. \tag{27}$$

Thus, the first equation in (26) has two solutions: $w^* = 0$ and

$$w^* = 2d\,[\xi - a_d(g_3 - g_2^2)/4]/(a_d(d-1)).$$

For the first case $w^* = 0$ equation (24) gives:

$$\beta_n = -g_n(n-1)y/2 + (a_d/8)\,[-2r_n + (n+1)g_n r_1]\,. \tag{28}$$

Whereas for $w^* \neq 0$ the equation implies:

$$\beta_n = g_n\,[-(n-1)y/2 + (n+1)\xi/2] - (a_d/4)\,r_n. \tag{29}$$

In both cases, the successive substitution of solutions of the equations $\beta_w = 0$, $\beta_k = 0$ with $k \leq n$ into the remaining equations $\beta_k = 0$ with $k > n$ allows us to express all the quantities g_n^* with $n > 3$ through two parameters g_2^* and g_3^*.

So we conclude that the attractor for the system (26) consists of a pair of two-dimensional surfaces.

For the dynamical models critical dimension of the quantity F:

$$\Delta_F = d_F^k + \Delta_\omega d_F^\omega + \gamma_F^*, \quad \Delta_\omega = 2 - \gamma_\nu^*. \tag{30}$$

From the data of Table №1 and the ratios $\gamma_\theta = \gamma_{\theta'} = \gamma_m = 0$ we get

$$\Delta_\theta = (1 - y/2) - (1/2)\Delta_\omega, \tag{31}$$

$$\Delta_{\theta'} = (d - 1 + y/2) + (1/2)\Delta_\omega.$$

The critical dimension Δ_ω depends on the exact solution of the RG equation.

For the surface with $w^* = 0$ we have

$$\Delta_\omega = 2 - a_d\,(g_3 - g_2^2)/4, \tag{32}$$

and, respectively, conclude that the critical dimensions (31) are nonuniversal. It means that they depend on the specific choice of a fixed point on the attractor surface.
For the second sheet with $w^* \neq 0$:

$$\Delta_\omega = 2 - \xi. \tag{33}$$

So

$$\Delta_\theta = \frac{1}{2}\,(\xi - y) \qquad \Delta_{\theta'} = d + \frac{1}{2}\,(y - \xi)\,. \tag{34}$$

For $w^* \neq 0$ the dimensions are universal. In both cases, these dimensions are subject to exact relations: $\Delta_{\theta'} + \Delta_\theta = d$ и $2\Delta_{\mathbf{v}} = -\xi + \Delta_\omega$.

Let us consider the spreading of a cloud of admixed particles in a turbulent medium. The effective radius of the cloud of such particles at time

$$R^2(t) = \int d\mathbf{x}\, x^2 \langle \theta(t, \mathbf{x}) \theta'(0, \mathbf{0}) \rangle. \tag{35}$$

Substituting the scaling representation (5) and taking into account the ratio $\Delta_\theta + \Delta_{\theta'} = d$, one can easily arrive at the following propagation law:

$$R^2(t) \propto t^{2/\Delta_\omega} \tag{36}$$

or, equivalently:

$$dR^2(t)/dt \propto R^{2-\Delta_\omega}(t) = R^\xi(t), \tag{37}$$

where we used expression (33) derived for the case $w^* \neq 0$. The special choice $\xi = 4/3$ in the Kazantsev–Kraichnan ensemble corresponds to the assumptions of the Kolmogorov–Obukhov theory of turbulence. As result, we have the statement of Richardson's "four-thirds" law: $dR^2/dt \propto R^{4/3}$.

(1) The multiplicative renormalizability of the model requires taking into account an infinite number of the interaction terms in the original diffusion equation. The result is an infinite set of coupling constants.

(2) The one-loop counterterm was explicitly constructed in closed form, which made it possible to find the complete set of γ and β functions in the one-loop approximation.

(3) The RG equations have two attractors in the form of a pair of two-dimensional surfaces of fixed points ($w^* = 0$ and $w^* \neq 0$).

(4) The first surface corresponds to the critical dimensions Δ_θ, $\Delta_{\theta'}$ and Δ_ω, which are nonuniversal. The same critical dimensions for the second surface turned out to be universal and were found explicitly (see (34)).

As a further study, we can consider the following questions that remained open in the framework of the work under discussion:

(1) The results obtained in this work can be compared with the results in the “problem of turbulent advection of a passive scalar impurity”. There instead of the Kraichnan ensemble, turbulence was described by the stochastic Navier-Stokes equation. It is proposed to consider an intermediate ensemble of Gaussian velocities with a finite correlation time.

(2) An infinite number of functions β_n, and the necessity (for the lack of a better option) of introduction the coefficients r_n. It would be desirable to carry out the RG analysis in terms not of an infinite set of functions, but of the functional $\beta(V)$.

Thank a lot for your attention!

Prepared a presentation and performed:

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