Multiloop RG functions and their transcendental structure: an update

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Two qualifications

First,

"transcendental" part of the talk is based on works P. Baikov, K.Ch.: arXiv:1804.10088, arXiv:1808.00237, arXiv:1908.03012, a paper in preparation

Second,

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PART I:
Mini-review of the current status of multiloop RG business

Last 40+ years $\rightarrow$ revolution in our ability to deal with multiloop RG functions as well as with generic (that is multiscale) Feynman Integrals (FI's)

The field is currently huge and will concentrate only on RG functions, namely QCD β-function and the one for the ϕ^4-model. The (minimally) renormalized RG-functions are fully saturated by the simplest family of FI's, namely massless 2-point integrals (p-integrals, for brevity). These are essentially scaleless and can be analytically computed to pretty high loop orders (4-loop in QCD and other non-abelian theories) and 7-loop and even more for scalar models (see below).

The starting and (currenly) final points of the revolution for the QCD β-function are (imho!)

3-loop term /Tarasov, Vladimirov, Zharkov, (1980)/

⋮

5-loop term /Baikov, K.Ch., Kühn, (2017); SU(3) gauge group/ (a bit later the results were checked and extended for generic gauge group by two independent teams)

And for the ϕ^4-beta-function:
4-loop term: /Kazakov, Tarasov, Vladimirov (1979)/

⋮

6-loop term: /Kompaniets, Panzer (2917)/; also the talk by Sasha Bednjakov

7-loop term: /Schnetz (2018)/

Main (but not all) theoretical tools behind the revolition (first universal ones):

- Dim. Reg. / Bollini, Giambiagi (1972); t' Hooft & Veltman (1972)/ (exactly 50 years ago!)
- IBP method /Vasiliev, Pis'mak, Khonkonen (1981)/
  (first example in a somewhat disguised form)
  +
  F. Tkachov (1981); K. Ch. and F. Tkachov (1981) (systematic algorithm for 3-loop p-integrals)
- DRR (dimensional recurrence relations) /Derkachov, Honkonen, Pis'mak, (1999); Tarasov (1996); Lee (2010)/
  combined with IBP leads to powerful recursive eqs. for evaluation of FI's/
- differential equations for FI's (with IBP relations happens to be remarkably powerfull (and presently very polular) way to compute master FI's) /Kotikov (1991)/
- Laporta approach to solve IBP: along with differential equations and DRR is currently used by dosens teams all over the world for evaluation of extremely complicated multiscale FI's)

Main (but not all) theoretical tools behind the revolition (universal ones, continuation):

- **Baikov's representtaion of FI's and (based on it) method of solution of IBP relations (that is the problem of reduction to masters) with the help of $1/D$ -expansion of the corresponing coefficient functions /Baikov (1996 ... 2022)'**

- **asymptic expansions (aka method of regions)**
  euclidean case:/G. Pivovarov, F. Tkachov (1984)/
  generic case: /Beneke V. Smirnov, (1997)/ (effectively separates scales in multiscale FI's; the heart of the so-called method of effective theories like HQET, decoupling of heavy particles, etc.)

- **Finally: Computer Algebra: from legendary SCHOONSCHIP (Veltman) to Mathematica and, especially, FORM (Vermaseren)**

Tools specific for computing RG functions:

- **"IR-reduction" $\longrightarrow$ most useful trick to automatically reduce # of loops by one in computing Z-factors (read any β-function and anomalous dimension in any theory) /Vladimirov (1978), K.Ch., F. Tkachov (1982); K. Ch., V. Smirnov (1984)/**
- **the method of uniqueness /Vasiliev, Pis'mak, Honkonen (1981); Usyukina (1983), Kazakov (1983)/**
- **the Glue-and-cut method /K.Ch, F. Tkachov (1981); Isaev, Gorishny (1994)/: combined with IBP leads to powerfull way of analytical computing of master L-loop p-integrals (Baikov, K.Ch. (2010))**

  The most recent achievement of the method is analytical calculations of *all* master p-integrals on 5-loop level: Georgoudis, Gonçalves, Panzer, Pereira, A. Smirnov, V. Smirnov, (2021) These results open a way to analytical calculation of arbitrary 6-loop RG-functions!
- **conformal bootstrap method /Vasiliev, Pis'mak, Honkonen (1981) . . . Ciuchini, Derkachov, Gracey, Manashov (1999)/;**
  very usefull for finding leading (and sometimes subleading) terms in large n_f-limit
- **(rather new) method of graphical functions /Schnetz (2014); Borinsky, Schnetz (2016–2022)/** The graphical function method unifies and extends many known techniques to evaluate and relate p-integrals into one framework. Examples are IBP, the cut-and-glue identity and the magic conformal identities /Drummond, Henn, Smirnov , Sokatchev (2007)/

  It has been used manily for the scalar p-integrals (the ϕ^4-model at 6 7 and (partially) 8 loops (!))

PART II:

Transcendental structure of RG functions

or the puzzle of π-dependent terms

QCD β-function in FIVE loops: Zeta's

In general any 5-loop beta in any theory will have the following "transcendental structure" (an obvious outcome of our knowledge of the corresponding masters)

1 and 2 loops: rational

3 loops: rationals + ζ_3

4 loops: rationals + $\zeta_3 + \zeta_4 + \zeta_5$

5 loops: rationals + $\zeta_3 + \zeta_4 + \zeta_5 + \zeta_6 + \zeta_3^2 + \zeta_7$

$$\begin{aligned}
\beta_1 &= \frac{1}{4}\Big\{11 - \frac{2}{3}n_f,\Big\}, \quad \beta_2 = \frac{1}{4^2}\Big\{102 - \frac{38}{3}n_f\Big\}, \quad \beta_3 = \frac{1}{4^3}\Big\{\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2\Big\}, \\
\beta_4 &= \frac{1}{4^4}\Big\{\left(\frac{149753}{6} + 3564\zeta_3\right) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right) n_f \\
&\qquad + \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right) n_f^2 + \frac{1093}{729}n_f^3\Big\}
\end{aligned}$$

QCD β-function in FIVE loops: Zeta's

In reality, the QCD β-function displays a delayed appearance of zeta's (well-known at 3 and 4 loops) which happens also in 5 loops.

1 and 2 loops: rational

3 loops: rationals + $\cancel{\zeta_3}$

4 loops: rationals + ζ_3 + $\cancel{\zeta_4 + \zeta_5}$

5 loops: rationals + $\zeta_3 + \zeta_4 + \zeta_5$ $\cancel{+\zeta_6 + \zeta_3^2 + \zeta_7}$

$$\beta_1 = \frac{1}{4}\left\{11 - \frac{2}{3}n_f,\right\}, \quad \beta_2 = \frac{1}{4^2}\left\{102 - \frac{38}{3}n_f\right\}, \quad \beta_3 = \frac{1}{4^3}\left\{\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2\right\},$$

$$\beta_4 = \frac{1}{4^4}\left\{\left(\frac{149753}{6} + 3564\zeta_3\right) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right)n_f + \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right)n_f^2 + \frac{1093}{729}n_f^3\right\}$$

$$
\begin{aligned}
4^5\,\beta_5 =\; & \frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\zeta_5 \\
+\quad & n_f\left[-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5\right] \\
+\quad & n_f^2\left[\frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5\right] \\
+\quad & n_f^3 \boxed{\left[-\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5\right]} + n_f^4 \boxed{\left[\frac{1205}{2916} - \frac{152}{81}\zeta_3\right]}
\end{aligned}
$$

all possible irrationalities do appear in separate diagrams contributing to the β at 5 loops (about two and half million ($2.5 \cdot 10^6$))

A related puzzle

The seminal calculation /Gorishnii, Kataev, Larin/ of the $\mathcal{O}(\alpha_s^3)$ Adler function demonstrated for the first time a mysterious complete cancellation of **all** contributions proportional to ζ_4 (abounding in separate diagrams) while odd zetas ζ_3 and ζ_5 survive! The result is π-**free** ($\zeta_4 = \frac{\pi^4}{90}$ and $\zeta_6 = \frac{\pi^6}{945}$)

$$d_2 = -\frac{3}{32}C_F^2 + C_F T_f\left[\zeta_3 - \frac{11}{8}\right] + C_F C_A\left[\frac{123}{32} - \frac{11\zeta_3}{4}\right],$$

$$d_3 = -\frac{69}{128}C_F^3 + C_F^2 T_f\left[-\frac{29}{64} + \frac{19}{4}\zeta_3 - 5\zeta_5\right] + C_F T_f^2\left[\frac{151}{54} - \frac{19}{9}\zeta_3\right] + C_F^2 C_A\left[-\frac{127}{64} - \frac{143}{16}\zeta_3 + \frac{55}{4}\zeta_5\right]$$
$$+ C_F T_f C_A\left[-\frac{485}{27} + \frac{112}{9}\zeta_3 + \frac{5}{6}\zeta_5\right] + C_F C_A^2\left[\frac{90445}{3456} - \frac{2737}{144}\zeta_3 - \frac{55}{24}\zeta_5\right],$$

the authors wrote: f "We would like to stress the cancellations of ζ_4 in the final results for $R(s)$. It is very interesting to find the origin of the cancellation of ζ_4 in the physical quantity." f

The situation got even more interesting about 20 years later: the $\mathcal{O}({\alpha_s}^4)$ contributions to the Adler function and to the coefficient function (CF) of C_{Bjp} the Bjorken sum rule /Baikov, Kühn, K. Ch. (2009-2010)/ were found to be

completely π-free★

★ **we do not consider any powers of π which are routinely generated during the procedure of analytical continuation to the Minkowskian (negative) values of the momentum transfer Q^2)**

	d_4	$(1/C^{Bjp})_4$
C_F^4	$\frac{4157}{2048} + \frac{3}{8}\,\zeta_3$	$\frac{4157}{2048} + \frac{3}{8}\,\zeta_3$
$n_f \frac{d_F^{abcd} d_F^{abcd}}{d_R}$	$-\frac{13}{16} - \zeta_3 + \frac{5}{2}\,\zeta_5$	$-\frac{13}{16} - \zeta_3 + \frac{5}{2}\,\zeta_5$
$\frac{d_F^{abcd} d_A^{abcd}}{d_R}$	$\frac{3}{16} - \frac{1}{4}\,\zeta_3 - \frac{5}{4}\,\zeta_5$	$\frac{3}{16} - \frac{1}{4}\,\zeta_3 - \frac{5}{4}\,\zeta_5$
$C_F T_f^3$	$-\frac{6131}{972} + \frac{203}{54}\,\zeta_3 + \frac{5}{3}\,\zeta_5$	$-\frac{605}{972}$
$C_F^2 T_f^2$	$\frac{5713}{1728} - \frac{581}{24}\,\zeta_3 + \frac{125}{6}\,\zeta_5 + 3\,\zeta_3^2$	$\frac{869}{576} - \frac{29}{24}\,\zeta_3$
$C_F T_f^2 C_A$	$\frac{340843}{5184} - \frac{10453}{288}\,\zeta_3 - \frac{170}{9}\,\zeta_5 - \frac{1}{2}\,\zeta_3^2$	$\frac{165283}{20736} + \frac{43}{144}\,\zeta_3 - \frac{5}{12}\,\zeta_5 + \frac{1}{6}\,\zeta_3^2 EQN$
$C_F^3 T_f$	$\frac{1001}{384} + \frac{99}{32}\,\zeta_3 - \frac{125}{4}\,\zeta_5 + \frac{105}{4}\,\zeta_7$	$-\frac{473}{2304} - \frac{391}{96}\,\zeta_3 + \frac{145}{24}\,\zeta_5$
$C_F^2 T_f C_A$	$\frac{32357}{13824} + \frac{10661}{96}\,\zeta_3 - \frac{5155}{48}\,\zeta_5 - \frac{33}{4}\,\zeta_3^2 - \frac{105}{8}\,\zeta_7$	$-\frac{17309}{13824} + \frac{1127}{144}\,\zeta_3 - \frac{95}{144}\,\zeta_5 - \frac{35}{4}\,\zeta_7$
$C_F T_f C_A^2$	$-\frac{(\cdots)}{(\cdots)} + \frac{8609}{72}\,\zeta_3 + \frac{18805}{288}\,\zeta_5 - \frac{11}{2}\,\zeta_3^2 + \frac{35}{16}\,\zeta_7$	$-\frac{(\cdots)}{(\cdots)} - \frac{59}{64}\,\zeta_3 + \frac{1855}{288}\,\zeta_5 - \frac{11}{12}\,\zeta_3^2 + \frac{35}{16}\,\zeta_7$
$C_F^3 C_A$	$-\frac{253}{32} - \frac{139}{128}\,\zeta_3 + \frac{2255}{32}\,\zeta_5 - \frac{1155}{16}\,\zeta_7$	$-\frac{8701}{4608} + \frac{1103}{96}\,\zeta_3 - \frac{1045}{48}\,\zeta_5$
$C_F^2 C_A^2$	$-\frac{592141}{18432} - \frac{43925}{384}\,\zeta_3 + \frac{6505}{48}\,\zeta_5 + \frac{1155}{32}\,\zeta_7$	$-\frac{435425}{55296} - \frac{1591}{144}\,\zeta_3 + \frac{55}{9}\,\zeta_5 + \frac{385}{16}\,\zeta_7$
$C_F C_A^3$	$\frac{(\cdots)}{(\cdots)} - \frac{(\cdots)}{(\cdots)}\,\zeta_3 - \frac{77995}{1152}\,\zeta_5 + \frac{605}{32}\,\zeta_3^2 - \frac{385}{64}\,\zeta_7$	$\frac{(\cdots)}{(\cdots)} - \frac{(\cdots)}{(\cdots)}\,\zeta_3 - \frac{12545}{1152}\,\zeta_5 + \frac{121}{96}\,\zeta_3^2 - \frac{385}{64}$

Transcedentals: odd zetas: $\zeta_3, \zeta_5, \zeta_7$ BUT NOT even one ζ_4 or ζ_6 (both appear eventually in *every* separate input diagram /from about 20 thousand!/ There exist many more such examples, but sometimes π-dependent terms do appear in physical quantities but starting from order α_s^5)

Recently there has happened a breakthrough* in our understanding of the transcendental structure of all RG-functions, including β_{QCD}, as well as the Adler function and similar objects like C_{Bjp}. As a result, we do understand now and can even predict the exact form of π-dependent terms in RG-functions *in terms of π-independent ones*. Here are some examples:

4-loops:

$$\boldsymbol{\beta}_4^{\zeta_4} = \boldsymbol{\beta}_1\, \boldsymbol{\beta}_3^{\zeta_3} \;\; (= \mathbf{0} \;\text{ for QCD as }\; \boldsymbol{\beta}_3^{\zeta_3} \equiv \mathbf{0})$$

5-loops (the relation below is, in fact, sitting /in a disguised form!/ in an important paper
Jamin and Miravitllas, *Absence of even-integer ζ-function values in Euclidean physical quantities in QCD*, 1711.00787
which has triggered our work on the π-dependence of RG-functions)

$$\boldsymbol{\beta}_5^{\zeta_4} = \frac{\mathbf{9}}{\mathbf{8}} \boldsymbol{\beta}_1\, \boldsymbol{\beta}_4^{\zeta_3}$$

where ($\boldsymbol{F}^{\zeta_i} = \lim_{\zeta_i \to 0} \frac{\partial}{\partial \zeta_i} \boldsymbol{F}$):

the above formulas are exact and valid in any 1-charge model!

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* K.Ch. and P. Baikov, eprints 1908.03012, 1808.00237 and 1804.10088

The factorization in the second formula is not trivial at all:

$$\frac{\partial}{\partial\zeta_4}\,\beta_5 \quad = \quad \overset{\beta_1}{\frac{9}{8}\Big(\frac{2}{3}\,n_f - 11\Big)}\,\overset{(\partial/\partial\zeta_3)\beta_4}{\Big(-\frac{6472}{81}\,n_f^2 + \frac{6508}{27}\,n_f - 3564\Big)}$$

while for a case of a generic gauge group it takes the form:

$$\begin{aligned}\frac{\partial}{\partial\zeta_4}\,\beta_5 \quad = \quad & \overset{\beta_1}{\frac{9}{8}\Big(\frac{4}{3}\,n_f\,T_F - \frac{11}{3}\,C_A\Big)} \star \overset{(\partial/\partial\zeta_3)\beta_4}{\Big(\frac{44}{9}\,C_A^4 - \frac{136}{3}\,C_A^3\,n_f\,T_F} \\ & + \frac{656}{9}\,C_A^2\,C_F\,n_f\,T_F - \frac{224}{9}\,C_A^2\,n_f^2\,T_F^2 - \frac{352}{9}\,C_A\,C_F^2\,n_f\,T_F \\ & - \frac{448}{9}\,C_A\,C_F\,n_f^2\,T_F^2 + \frac{704}{9}\,C_F^2\,n_f^2\,T_F^2 - \frac{704}{3}\,\frac{d_A^{abcd}\,d_A^{abcd}}{N_A} \\ & + \frac{1664}{3}\,\frac{d_F^{abcd}\,d_A^{abcd}}{N_A}\,n_f - \frac{512}{3}\,\frac{d_F^{abcd}\,d_F^{abcd}}{N_A}\,n_f^2\Big)\end{aligned}$$

π-structure of p-integrals

We will call a (bare) L-loop p-integral $F(Q^2, \epsilon)$ *π-safe* if the π-dependence of its pole in ϵ *and* constant part can be completely absorbed into the properly defined "hatted" odd zetas.

The first observation of a non-trivial class of π-safe p-integrals — all 3-loop ones — was made in /Broadhurst (1999)/ An extension of the observation on the class of all 4-loop p-integrals was performed in /Baikov, K.Ch. (2010)/ Here it was shown that, given an arbitrary 4-loop p-integral, its pole in ϵ *and* constant part depend on even zetas *only* via the following combinations:

$$\hat{\zeta}_3 := \zeta_3 + \frac{3\epsilon}{2}\zeta_4 - \frac{5\epsilon^3}{2}\zeta_6, \ \ \hat{\zeta}_5 := \zeta_5 + \frac{5\epsilon}{2}\zeta_6 \quad \text{and} \quad \hat{\zeta}_7 := \zeta_7.$$

Exact meaning: for any 4-loop p-integral F_4:

$$F_4(\zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7) = F_4(\hat{\zeta}_3, 0, \hat{\zeta}_5, 0, \hat{\zeta}_7) + \mathcal{O}(\epsilon) \qquad \star$$

A generalization of the $\star$ for L=5 has been recently constructed in /Georgoudis, Goncalves, Panzer, Pereira, [1802.00803]/ (and confirmed independently by us)

The hatted representation clearly shows that π-dependent terms are in a sense governed by π-independent ones. The are all suppressed by some powers of ϵ and thus are predictable from the knowledge of lower orders. It is $exactly$ the same mechanics which was thoroughly explained in yesterday's talk by Dima Kazakov!

Thus I will skip some purely technical details and will show you some predictions and how they corresppomd to the results of real calculation.

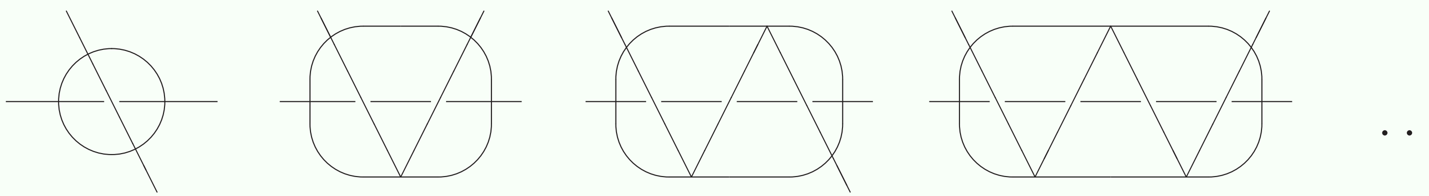
But before I should say few words about the very idea of π-(in)-dependence as in higher orders it is not completely trivial

What is π independence?

In higher loops it is not obvious any more, e.g. :

$$\frac{3}{5}\zeta_{5,3} + \zeta_3\,\zeta_5 - \frac{29}{20}\zeta_8 \equiv \zeta_{6,2} - \zeta_{3,5} \quad \textbf{with} \quad \zeta_{n_1,n_2} := \sum_{i>j>0} \frac{1}{i^{n_1} j^{n_2}}$$

A famous series of primitive (no subdivergences and hence scheme–independent contributions to the β–function of ϕ^4 theory) are diagrams from so-called 'zig–zag' series (D. Broadhurst, hep-ph/9504352)



$Z(3) = 6\zeta_3$ $\quad Z(4) = 20\zeta_5$ $\quad Z(5) = \frac{441}{8}\zeta_7$ $\quad Z(6) = 168\zeta_9$

clearly shows that ζ_{2L-3} are associated with primitive L-loop diagrams (or, equivalently) with finite p-integrals. Thus, it is natural (and is eventually justified by our results) to consider finite p-integrals as $\pi-$independent *by definition* /D. Broadhurst/

While famous? Because they were important milestones of multiloop calculations in the ϕ^4-model during last 40+ years

L=3 /Rosner 1976/

L=4 /Kazakov, Tarasov, Vladimirov (1979) /

L=5 /K. Ch. Gorishny, Larin, F. Tkachov (1983), Kazakov (1984)/

Model independent predictions for β and γ for any 1-charge theory

$$\beta_4^{\zeta_4} = \beta_1 \beta_3^{\zeta_3}$$

$$\beta_5^{\zeta_4} = \frac{1}{2}\beta_3^{\zeta_3}\beta_2 + \frac{9}{8}\beta_1\beta_4^{\zeta_3}$$

$$\beta_5^{\zeta_6} = \frac{15}{8}\beta_1\beta_4^{\zeta_5}$$

$$\beta_5^{\zeta_3\zeta_4} = 0$$

$$\beta_6^{\zeta_4} = \frac{3}{4}\beta_2\beta_4^{\zeta_3} + \frac{6}{5}\beta_1\beta_5^{\zeta_3}$$

$$\beta_6^{\zeta_6} = \frac{5}{4}\beta_2\beta_4^{\zeta_5} + 2\beta_1\beta_5^{\zeta_5} - \beta_1^3\beta_3^{\zeta_3}$$

$$\beta_6^{\zeta_3\zeta_4} = \frac{12}{5}\beta_1\beta_5^{\zeta_3^2}$$

$$\gamma_4^{\zeta_4} = -\frac{1}{2}\beta_3^{\zeta_3}\gamma_1 + \frac{3}{2}\beta_1\gamma_3^{\zeta_3}$$

$$\gamma_5^{\zeta_4} = -\frac{3}{8}\beta_4^{\zeta_3}\gamma_1 + \frac{3}{2}\beta_2\gamma_3^{\zeta_3} - \beta_3^{\zeta_3}\gamma_2 + \frac{3}{2}\beta_1\gamma_4^{\zeta_3}$$

$$\gamma_5^{\zeta_6} = -\frac{5}{8}\beta_4^{\zeta_5}\gamma_1 + \frac{5}{2}\beta_1\gamma_4^{\zeta_5}$$

$$\gamma_5^{\zeta_3\zeta_4} = 0$$

$$\gamma_6^{\zeta_4} = \frac{3}{2}\beta_3^{(1)}\gamma_3^{\zeta_3} - \frac{3}{10}\beta_5^{\zeta_3}\gamma_1 - \frac{3}{4}\beta_4^{\zeta_3}\gamma_2 + \frac{3}{2}\beta_2\gamma_4^{\zeta_3} - \frac{3}{2}\beta_3^{\zeta_3}\gamma_3^{(1)} + \frac{3}{2}\beta_1\gamma_5^{\zeta_3}$$

$$\gamma_6^{\zeta_6} = -\frac{1}{2}\beta_5^{\zeta_5}\gamma_1 - \frac{5}{4}\beta_4^{\zeta_5}\gamma_2 + \frac{5}{2}\beta_2\gamma_4^{\zeta_5} + \frac{5}{2}\beta_1\gamma_5^{\zeta_5} + \frac{3}{2}\beta_1^2\beta_3^{\zeta_3}\gamma_1 - \frac{5}{2}\beta_1^3\gamma_3^{\zeta_3}$$

$$\gamma_6^{\zeta_3\zeta_4} = -\frac{3}{5}\beta_5^{\zeta_3^2}\gamma_1 + 3\beta_1\gamma_5^{\zeta_3^2}$$

$$\beta_6^{\zeta_8} = \frac{14}{5}\beta_1\beta_5^{\zeta_7}$$

$$\beta_6^{\zeta_3\zeta_6} = 0$$

$$\beta_6^{\zeta_4\zeta_5} = 0$$

$$\gamma_6^{\zeta_8} = -\frac{7}{10}\beta_5^{\zeta_7}\gamma_1 + \frac{7}{2}\beta_1\gamma_5^{\zeta_7}$$

$$\gamma_6^{\zeta_3\zeta_6} = 0$$

$$\gamma_6^{\zeta_4\zeta_5} = 0$$

The above constraints have been sucessfully checked on the following examples:

L=4 and 5: numerous QCD RG functions (including gauge-dependent ones taken in the Landau gauge) recently computed in

/K.Ch, Falcioni, Herzog and J Vermaseren [1709.08541] .

L=4,5 and 6: β-function and ADs of $O(n)\ \phi^4$ model recently computed in
Batkovich, K. Ch. and Kompaniets, [1601.01960] (γ_2 only)
Schnetz, [1606.08598] ($\beta, \gamma_2, \gamma_m$)
Kompaniets and Panzer, [1705.06483] ($\beta, \gamma_2, \gamma_m$)

Predictions for 6-loop QCD RG functions:

$$\beta_6 \stackrel{\pi}{=} \boxed{\frac{608}{405}\, n_f^5\, \zeta_4} + n_f^4\,(\frac{164792}{1215}\,\zeta_4 - \frac{1840}{27}\,\zeta_6) + n_f^3\,(-\frac{4173428}{405}\,\zeta_4 + \frac{1800280}{243}\,\zeta_6)$$
$$+n_f^2\,(\frac{68750632}{405}\,\zeta_4 - \frac{13834700}{81}\,\zeta_6) + n_f\,(-\frac{146487538}{135}\,\zeta_4 + \frac{40269130}{27}\,\zeta_6)$$
$$+99\,(44213\,\zeta_4 - 64020\,\zeta_6)$$

$$\gamma_6^m \stackrel{\pi}{=} \boxed{\frac{320}{243}\, n_f^5\, \zeta_4 + n_f^4\,(-\frac{90368}{405}\,\zeta_4 + \frac{22400}{81}\,\zeta_6)}$$
$$+n_f^3\,(-\frac{92800}{27}\,\zeta_3\,\zeta_4 - \frac{2872156}{405}\,\zeta_4 + \frac{503360}{243}\,\zeta_6)$$
$$+n_f^2\,(\frac{661760}{9}\,\zeta_3\,\zeta_4 + \frac{155801234}{405}\,\zeta_4 - \frac{378577520}{729}\,\zeta_6 + \frac{12740000}{81}\,\zeta_8)$$
$$+n_f\,(-\frac{1413280}{3}\,\zeta_3\,\zeta_4 - \frac{4187656168}{1215}\,\zeta_4 + \frac{5912758120}{729}\,\zeta_6 - \frac{96071360}{27}\,\zeta_8)$$
$$+3194400\,\zeta_3\,\zeta_4 + \frac{272688530}{81}\,\zeta_4 - \frac{6778602160}{243}\,\zeta_6 + 15889720\,\zeta_8$$

boxed terms are in FULL AGREEMENT with the well-known results by /Gracey (1996)/ and /Ciuchini, Derkachov, Gracey and Manashov (1999-2000)/

all other terms are new

New developments: 6 (and 7 loops)

In order to construct the hatted represenation of master FI's at $L > 5$ we neeed the knowledge of the corresponding master integrals which is basically absent. Bur we do know many terms of ϵ-expansions of 4-loop masters from a remarkable work

R. N. Lee, A. V. Smirnov and V. A. Smirnov, *Master Integrals for Four-Loop Massless Propagators up to Transcendentality Weight Twelve, Nucl. Phys.* **B856 (2012) 95–110,**

Volodja Smirnov extended for us the results to the weight thirteen

Armed with these result we have constructed the hatted representation for master p-integrals at L=6 and L=7 and have found the corresponding predictions for L=7 and L=8 RG functions

Hatted form for the 6-loop case /transcendental level ≤ 11/

$$\underbrace{\hat{\zeta}_3 := \boxed{\zeta_3} + \frac{3\epsilon}{2}\zeta_4}_{L=3} \qquad \underbrace{-\frac{5\epsilon^3}{2}\zeta_6}_{\delta(L=4)} \qquad \underbrace{+\frac{21\epsilon^5}{2}\zeta_8}_{\delta(L=5)} \qquad \underbrace{-\frac{153\epsilon^7}{2}\zeta_{10}}_{\delta(L=6)}, \tag{1}$$

$$\underbrace{\hat{\zeta}_5 := \boxed{\zeta_5} + \frac{5\epsilon}{2}\zeta_6}_{(L=4)} \qquad \underbrace{-\frac{35\epsilon^3}{4}\zeta_8}_{\delta(L=5)} \qquad \underbrace{+63\epsilon^5\zeta_{10}}_{\delta(L=6)}, \tag{2}$$

$$\underbrace{\hat{\zeta}_7 := \boxed{\zeta_7}}_{L=4} \qquad \underbrace{+\frac{7\epsilon}{2}\zeta_8}_{\delta(L=5)} \qquad \underbrace{-21\epsilon^3\zeta_{10}}_{\delta(L=6)}, \tag{3}$$

$$\underbrace{\hat{\varphi} := \boxed{\varphi} - 3\epsilon\,\zeta_4\,\zeta_5 + \frac{5\epsilon}{2}\zeta_3\,\zeta_6}_{L=5} \qquad \underbrace{-\frac{24\,\epsilon^2}{47}\zeta_{10} + \epsilon^3\,(-\frac{35}{4}\zeta_3\zeta_8 + 5\zeta_5\zeta_6)}_{\delta(L=6)}, \tag{4}$$

$$\underbrace{\hat{\zeta}_9 := \boxed{\zeta_9}}_{L=5} \qquad \underbrace{+\frac{9}{2}\epsilon\,\zeta_{10}}_{\delta(L=6)}, \tag{5}$$

$$\underbrace{\hat{\zeta}_{7,3} := \boxed{\zeta_{7,3} - \frac{793}{94}\zeta_{10}} + 3\epsilon(-7\zeta_4\zeta_7 - 5\zeta_5\zeta_6)}_{L=6}, \tag{6}$$

$$\underbrace{\hat{\zeta}_{11} := \boxed{\zeta_{11}}}_{L=6}, \tag{7}$$

$$\underbrace{\hat{\zeta}_{5,3,3} := \boxed{\zeta_{5,3,3} + 45\zeta_2\zeta_9 + 3\zeta_4\zeta_7 - \frac{5}{2}\zeta_5\zeta_6}}_{L=6}. \tag{8}$$

The boxed terms are in agreement with the results of F. Brown, D. Broadhurst, D. Kreimer, E, Panzer, O. Schnetz ...

Now we can upgrade our formulas for π-dependent terms in AD's and β-functions at the next 7-loop level!

$$\beta_7^{\zeta_4} = \frac{3}{8}\,\beta_4^{\zeta_3}\beta_3^{(1)} + \frac{9}{10}\,\beta_2\beta_5^{\zeta_3} - \frac{1}{2}\,\beta_3^{\zeta_3}\beta_4^{(1)} + \frac{5}{4}\,\beta_1\beta_6^{\zeta_3},$$

$$\beta_7^{\zeta_6} = \frac{5}{8}\,\beta_4^{\zeta_5}\beta_3^{(1)} + \frac{3}{2}\,\beta_2\beta_5^{\zeta_5} + \frac{25}{12}\,\beta_1\beta_6^{\zeta_5} - 2\beta_1^2\beta_3^{\zeta_3}\beta_2 - \frac{5}{4}\,\beta_1^3\beta_4^{\zeta_3},$$

$$\beta_7^{\zeta_3\zeta_4} = \frac{9}{5}\,\beta_2\beta_5^{\zeta_3^2} - \frac{1}{8}\,\beta_3^{\zeta_3}\beta_4^{\zeta_3} + \frac{5}{2}\,\beta_1\beta_6^{\zeta_3^2},$$

$$\beta_7^{\zeta_8} = \frac{21}{10}\,\beta_2\beta_5^{\zeta_7} + \frac{35}{12}\,\beta_1\beta_6^{\zeta_7} - \frac{7}{24}\,\beta_1(\beta_3^{\zeta_3})^2 + \frac{7}{4}\,\beta_1^2\beta_5^{\zeta_3^2} - \frac{35}{8}\,\beta_1^3\beta_4^{\zeta_5},$$

$$\beta_7^{\zeta_3\zeta_6} = \frac{5}{8}\beta_3^{\zeta_3}\beta_4^{\zeta_5} + \frac{25}{12}\beta_1\beta_6^{\zeta_3\zeta_5} + \frac{25}{12}\beta_1\beta_6^{\phi},$$

$$\beta_7^{\zeta_4\zeta_5} = -\frac{1}{2}\beta_3^{\zeta_3}\beta_4^{\zeta_5} + \frac{5}{4}\beta_1\beta_6^{\zeta_3\zeta_5} - \frac{5}{2}\beta_1\beta_6^{\phi},$$

$$\beta_7^{\zeta_{10}} = \frac{15}{4}\beta_1\beta_6^{\zeta_9},$$

$$\beta_7^{\zeta_4\zeta_3^2} = \frac{15}{4}\beta_1\beta_6^{\zeta_3^3},$$

$$\beta_7^{\zeta_4\zeta_7} = \beta_7^{\zeta_5\zeta_6} = \beta_7^{\zeta_3\zeta_8} = 0.$$

Tests of pur predictions for AD's at L=7 loop: I

We have checked that the π-dependent contributions to the terms of order $n_f^6 \alpha_s^7$ in the the QCD β-function as well as to the terms of order $n_f^6 \alpha_s^7$ and of order $n_f^5 \alpha_s^7$ contributing to the quark mass AD, all computed in

J. Gracey, *The QCD Beta function at $\mathcal{O}(1/N_f)$*, *Phys.Lett.* B373 (1996) 178–184, [hep-ph/9602214].

M. Ciuchini, S. E. Derkachov, J. Gracey and A. Manashov, *Quark mass anomalous dimension at O(1/N(f)**2) in QCD*, *Phys.Lett.* B458 (1999) 117–126, [hep-ph/9903410].

M. Ciuchini, S. E. Derkachov, J. Gracey and A. Manashov, *Computation of quark mass anomalous dimension at O(1 / N**2(f)) in quantum chromodynamics*, *Nucl.Phys.* B579 (2000) 56–100, [hep-ph/9912221].

are in agreement with our predictions

Tests of pur predictions for AD's at L=7 loop, cont-ed

Significantly more complicated test is provided by the recent calculation of the full 7-loop RG functions in the φ^4-model

O. Schnetz, ***Numbers and Functions in Quantum Field Theory*, *Phys. Rev.* D97 (2018) 085018, [**`1606.08598`**]**

We have reproduced successfully all π-dependent constants appearing in the β-function and anomalous dimensions γ_m and γ_2 of the $O(n)$ φ^4 at 7 loops

Oliver Schnetz, PRD 97 (2018): 7! loop result for ϕ^4 RG functions:

$$\beta = \Big(\frac{195654269}{23040} + \frac{15676169}{720}\zeta(3) - \frac{316009}{3840}\pi^4 \frac{18326039}{480}\zeta(5) - \frac{129631}{5040}\pi^6$$
$$+ \frac{516957}{20}\zeta(3)^2 - \frac{4453}{60}\pi^4\zeta(3) + \frac{1536173}{20}\zeta(7) - \frac{20425591}{1260000}\pi^8$$
$$+ 116973\zeta(3)\zeta(5) + \frac{947214}{25}\zeta(5,3) - \frac{1010}{63}\pi^6\zeta(3) + \frac{613}{5}\pi^4\zeta(5) + 4176\zeta(3)^3$$
$$+ \frac{547118}{3}\zeta(9) - \frac{45106}{43659}\pi^{10} - 48\pi^4\zeta(3)^2 + \frac{84231}{2}\zeta(3)\zeta(7) - \frac{273030}{7}\zeta(5)^2$$
$$+ \frac{8460}{7}\zeta(7,3) - \frac{174}{25}\pi^8\zeta(3) + \frac{6227}{35}\pi^6\zeta(5) - \frac{56043}{25}\pi^4\zeta(7)$$
$$- 504387\pi^2\zeta(9) + 46845\zeta(3)^2\zeta(5) + 27216\zeta(3)\zeta(5,3) - \frac{336258}{5}\zeta(5,3,3)$$
$$+ \frac{52756839}{10}\zeta(11) + 24P_{7,11}\Big) g^8 + \ldots$$

All π-dependent terms follow from $\beta/.\pi \to 0$: first (partial) check of both the 7-loop β for the ϕ^4-model and on the hatted representation of $\mathcal{P}_6$. The same is true for γ_m, γ_2 and the 6-loop self-energy

The most advanced for today full result for the field anomalous dimension of the ϕ^4-model /O. Schnetz, 20221/

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(-169/2*zeta[3]^3 + 642917/1920*zeta[3]^2 + 8/15*Pi^4*zeta[3]^2 +
807/2*zeta[3]*zeta[5] + 105/2*zeta[3]*zeta[7] - 10429/21*zeta[5]^2 +
115767719/414720*zeta[3] + 1837/8640*Pi^4*zeta[3] +
635/2268*Pi^6*zeta[3] - 1274869/11520*zeta[5] - 523/360*Pi^4*zeta[5] -
6018361/5760*zeta[7] - 608849/432*zeta[9] + 3801/100*zeta[3, 5] -
94/7*zeta[3, 7] + 22553/1964655*Pi^10 + 672397/6480000*Pi^8 +
549949/829440*Pi^4 + 39437/90720*Pi^6 + 1506066907/6635520)*g^8
+ \dots
```

we have checked that all π-dependent terms are in the full aghreement to our prtedictions!

Conclusions

- all π-dependent terms in a generic (L+1)-loop $\overline{\text{MS}}-$ (or, equivalently, G-) anomalous dimension γ are completely fixed by π-independent contributions to γ (and corresponding β) with loop number L or less *provided* the (all) L-loop p-master integrals are π-safe

- The π-safeness holds for L=4 and L=5 and, probably, for L=6. It is known that for L=7 the property (partially) stops to be valid* and, thus, our predictions should be modified (at astronomically large for QCD level of L=8 RG functions). In fact, we have done the modifications (with some reasonable assumptions); all possible checks at at L=7 /for p-integrals/ currently not so many!) are fulfilled

- All available results at 5 (QCD), and 6 and 7 and 8 loops (large n_f QCD and the ϕ^4-model) do meet all our constraints

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* communicated to us by Oliver Schnetz

(the problem is an appearence of the ζ_{12} as indepenent term of some 7-loop finite p-integral, see works by (F.Brown, O.Schnetz, E.Panzer . . . on Feynman periods)

QCD β-function in FIVE loops: result

$$\mu^2 \frac{\partial}{\partial \mu^2} a_s = \beta(a_s)\, a_s, \quad a_s \equiv \frac{\alpha_s}{\pi}, \quad \beta(a_s) = \sum_{i \geq 1} \beta_i\, a_s^i$$

$$\begin{aligned}
4^5 \beta_5 = \; & \frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\zeta_5 \\
+ \quad & n_f \left[-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5 \right] \\
+ \quad & n_f^2 \left[\frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5 \right] \\
+ \quad & n_f^3 \boxed{\left[-\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5 \right]} + n_f^4 \boxed{\left[\frac{1205}{2916} - \frac{152}{81}\zeta_3 \right]}
\end{aligned}$$

n_f^4 **term is in FULL AGREEMENT with the 20 years old result by John Gracey (in the framework of the conformal bootstrap method of A. Vasiliev, Yu. Pis'mak and J. Honkonen (1981))**

n_f^3 **term is in FULL AGREEMENT with a result by Th. Luthe, A. Maier, P. Marquard and Y. Schröder (made within the "massive way")**

New Representation of FI's /due to Baikov/:

Feynman parameters:

$$\frac{1}{m^2 - p^2} \approx \int d\ \alpha\ \mathrm{e}^{i\alpha(m^2 - p^2)}$$

New parameters:

$$\frac{1}{m^2 - p^2} \approx \int \frac{d\ x}{x}\ \delta(x - (m^2 - p^2))$$

Now for a given topology one can make loop integrations once and forever with the result:

$$\boxed{F(\underline{n}) \quad \sim \quad \int \ldots \int \frac{\mathbf{d}x_1 \ldots \mathbf{d}x_N}{x_1^{n_1} \ldots x_N^{n_N}} \left[P(\underline{x})\right]^{(D-h-1)/2} ,}$$

where $P(\underline{x})$ is a polynomial on $x_1, \ldots, x_N$ (and masses and external momenta)

New representation obviously meets the same set IBP'id as the original integral but it has much more flexibility ... and haw been efficiently used tp solve IBP