# Shapovalov elements for classical and quantum groups

Андрей Мудров

Centre for Fundamental Mathematics, MIPT,

October 13, 2022

,

MQFT2022, St.-Petersburg

,

Based on arXiv:2202.06220.

# Setup

Let $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ be simple complex Lie algebra

$U = U_q(\mathfrak{g})$ be its quantized universal enveloping algebra

$U \simeq U^- \otimes U^0 \otimes U^+$ (triangular decomposition)

where $U^\pm = U_q(\mathfrak{g}^\pm)$ и $U^0 = U_q(\mathfrak{h})$

$\mathrm{R}$ is its root system

$\mathrm{R}^+ \subset \mathrm{R}$ is set of positive root

$\Pi$ is basis of simple positive roots

$\rho = \frac{1}{2} \sum_{\alpha \in \mathrm{R}^+} \alpha$, Вектор Вейля

## $U$-modules

Let $V$ be a $U$ a-module

Vector $v \in V$ is of weight $\lambda \in \mathfrak{h}^*$ if

$$hv = \lambda(h)v, \quad \forall h \in \mathfrak{h}$$

Weight vector $v$ is called singular (extremal) if $\mathfrak{g}_+ v = 0$.

$V$ is of highest weight $\lambda$ if generated by a singular vector of w. $\lambda$.

Verma module $V_\lambda$ is generated by a highest vector of weight $\lambda$ freely over $U^-$, i.e. $U^- \simeq V_\lambda$.

Every module of highest weight $\lambda$ if a quotient of $V_\lambda$.

Every singular vector in $V_\lambda$ is a highest of a submodule.

## The problem: describe explicitly singular vectors in $V_\lambda$

De Concini-Kac-Kazhdan 'hyperplane'

$$\mathcal{P}_{\beta,m} = \{\lambda \in \mathfrak{h}^* | q^{2(\lambda+\rho,\beta^\vee)} = q^{m(\alpha,\alpha)}\}, \quad \beta \in \mathrm{R}^+, \quad m \in \mathbb{N},$$

$V_\lambda$ is reducible *iff* $\lambda \in \cup_{\beta,m}\mathcal{P}_{\beta,m}$.

Let $\lambda \in \mathcal{P}_{\beta,m}$, then there is a singular vector $v_{\lambda-m\beta} \in V_\lambda$,

$$v_{\lambda-m\beta} = \theta_{\beta,m} v_\lambda, \quad \theta_{\beta,m} \in U^- \simeq V_\lambda.$$

If $\beta \in \Pi$ (simple), then $\theta_{\beta,m} = f_\beta^m$ (where $f_\beta$ is root vector).

- Bernstein-Gelfand-Gelfand (1971) (reduction to products of $\theta_{\beta,m}$)
- Malikov-Feigin-Fuchs (1986) (description of $\theta_{\beta,m}$ via an interpolation procedure)
- Zhelobenko (1990) (factorization of $\theta_{\beta,m} = \theta_{\beta,1}^m$)
- Musson (2017) (Lie super algebras)
- A.M. (2015) ($U_q(\mathfrak{sl}(n))$)

## "Adjoint" $U$-module and its Hasse diagram

Let $\xi \in \mathrm{R}^+$ be maximal root.

Let $\tilde{\mathfrak{g}}$ be fin.dim $U$-module of h.w. $\xi$

Let $V$ denote $U^+$-module $\tilde{\mathfrak{g}}/\tilde{\mathfrak{g}}_+$.

$V[-\alpha] \simeq \mathbb{C}$ if $\alpha \in \mathrm{R}^+$ and $V[0] \simeq \mathfrak{h}$.

Basis in $V$:

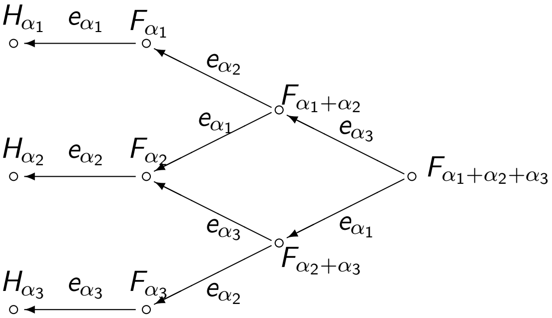
$F_\alpha \in V[-\alpha]$ with $\alpha \in \mathrm{R}^+$ and $H_\alpha = e_\alpha \cdot F_\alpha \in V[0]$ with $\alpha \in \Pi$

Hasse diagram $\mathfrak{H}(V)$ associated with $V$

- Nodes are basis elements of $V$.
- Arrow from node $a$ to node $b$ is $\alpha \in \Pi$ if $e_\alpha a \propto b$ .

# Example of Hasse diagram $\mathfrak{H}(V)$

$\mathfrak{g} = \mathfrak{sl}(4)$



# Builing bricks

Let $\hat{\mathcal{R}} = q^{-\sum_i h_i \otimes h_i} \mathcal{R} \in U^+ \otimes U^-$ be Cartan-cut R-matrix.

Set $\mathcal{C} = \frac{1}{q-q^{-1}}(\hat{\mathcal{R}} - 1 \otimes 1)$, $\qquad \lim_{q\to 1} \mathcal{C} = \sum_{\alpha \in \mathrm{R}^+} e_\alpha \otimes f_\alpha$

Put $C = (\pi \otimes \mathrm{id})(\mathcal{C}) \in \mathrm{End}(V) \otimes U^-$.

For each $\mu \in \mathfrak{h}^*$ put $h_\mu \in \mathfrak{h}$ s.t. $\lambda(h_\mu) = (\lambda, \mu)$, $\forall \lambda \in h^*$.

For each weight $\mu \in \Gamma_+ = \mathbb{Z}_+ \Pi$ put

$$\eta_\mu = h_\mu + (\mu, \rho) - \frac{1}{2}(\mu, \mu) \in \mathfrak{h} \oplus \mathbb{C}. \qquad (1)$$

Set $[z]_q = \frac{q^z - q^{-z}}{q - q^{-1}}$

# Shapovalov elements of degree 1

Write $\mu \prec \nu$ for $\mu, \nu \in \mathrm{R}^+$ if $F_\mu \succ F_\nu$.

Suppose that $\alpha \in \Pi$ and $\beta \in \mathrm{R}^+$ are such that $\alpha \prec \beta$. Define

$$\theta_{\beta,\alpha} = c_{ij} + \sum_{k\geqslant 1} \sum_{\alpha\preceq\gamma_k\prec\ldots\prec\gamma_1\prec\beta} c_{ik}\ldots c_{1j} \frac{(-1)^k q^{\eta_{\mu_k}} \ldots q^{\eta_{\mu_1}}}{[\eta_{\mu_k}]_q \ldots [\eta_{\mu_1}]_q},$$

where

$v_i = H_\alpha,\ v_j = F_\beta,$

$v_m = F_{\gamma_m}$ with $\alpha \preceq \gamma_m \prec \beta$

$\mu_m = \beta - \gamma_m,\ m = 1, \ldots, k-1.$

$\mu_k = \gamma_k.$

# Factorization of Shapovalov elements

For $\nu \in \mathfrak{h}^*$ denote by $\tau_\nu \colon U^0 \to U^0$ an automorphism

$$(\tau_\nu F)(\lambda) = \varphi(\lambda + \nu), \quad \forall F \in U^0, \quad \lambda \in \mathfrak{h}^*$$

Theorem. (A.M.,2022)

*Suppose $\beta = \ell\alpha + \ldots$, $\alpha \in \Pi$ and let $\omega_\alpha$ be fundamental weight. Then*

1. *$\theta_\beta = \theta_{\beta,\alpha}$ is a Shapovalov element of degree 1*
2. *For all $m \in \mathbb{N}$, $\theta_{\beta,m} = (\tau_{\phi_\alpha}^{m-1}\theta_\beta)\ \ldots\ (\tau_{\phi_\alpha}\theta_\beta)\,\theta_\beta$, where*

$$\phi_\alpha = \frac{(\beta,\beta)}{\ell(\alpha,\alpha)}\omega_\alpha$$