Schwarzians with extended supersymmetry

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Main result

We revisit the construction of supersymmetric Schwarzians using nonlinear realizations. We show that supersymmetric Schwarzians can be systematically obtained as certain projections of Maurer-Cartan forms of superconformal groups after imposing simple conditions on them. Likewise, we also present the supersymmetric Schwarzian actions, defined as the integrals of products of Cartan forms. In contrast with the previous attempts to obtain the super-Schwarzians within nonlinear realizations technique, our set of constraints do not impose any restriction on the super-Schwarzians.
We also provide new bosonic versions of the Schwarzians.

Plan

- Introduction
- Three steps towards Schwarzian. $\mathcal{N} = 0$ case
- New version of the bosonic Schwarzians
- $\mathcal{N} = 2$ supersymmetric Schwarzian
- $\mathcal{N}$-extended supersymmetric Schwarzian
- Conclusion

The Schwarzian derivative $\{t,\tau\}$ defined as

$$\{t,\tau\} = \frac{\dddot{t}}{\dot{t}} - \frac{3}{2}\left(\frac{\ddot{t}}{\dot{t}}\right)^2, \quad \dot{t} = \partial_\tau t,$$

itself appears in seemingly unrelated fields of physics and mathematics. The action of the bosonic Schwarzian mechanics reads

$$S_{schw}[t] = -\frac{1}{2}\int d\tau \left(\{t,\tau\} + 2m^2\dot{t}^2\right).$$

Remarkably, the equation of motion of this higher-derivative action is equivalent just to

$$\frac{d}{d\tau}\left[\{t,\tau\} + 2m^2\dot{t}^2\right] = 0.$$

The characteristic feature of the Shwarzian derivative (1) is its invariance under $Sl(2,\mathbb{R})$ Möbius transformations acting on $t[\tau]$ via

$$t \to \frac{at+b}{ct+d}.$$

The presence of $m^2\dot{t}^2$ term in the action $S_{schw}[t]$ modify the realization of $Sl(2,\mathbb{R})$ symmetry. The simplest way to understand the modification is to notice that the action $S_{schw}[t]$ can be represented as

$$S_{schw}[t] = -\frac{1}{2}\int d\tau\,\{F,\tau\}\,, \qquad F[\tau] = \tan\left(mt[\tau]\right),$$

and, therefore, the action (1) possesses the $Sl(2,\mathbb{R})$ invariance via

$$F \to \frac{aF+b}{cF+d}$$

with $F[\tau]$ defined in (1).
Being invariant under $d=1$ conformal transformation, the Schwarzian derivative naturally appears in the transformations of the conformal stress tensor $T(z)$ (A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, 1984)

$$T(z) = \left(\frac{d\tilde{z}}{dz}\right)^2 \tilde{T}(\tilde{z}) + \{\tilde{z}, z\}\,. \tag{1}$$

The $\mathcal{N}{=}1,2,3,4$ supersymmetric generalization of the Schwarzian derivative are present in the transformation properties of the current superfield $J^{(\mathcal{N})}(Z)$ generating $\mathcal{N}$ - extended superconformal transformations (K. Schoutens,1988). Thus, we have complete zoo of the supersymmetric Schwarzians.

The treatment of the supersymmetric Schwarzians as the anomalous terms in the transformations of the currents superfield $J^{(\mathcal{N})}(Z)$ leads to the conclusion that the structure of the (super)Schwarzians is completely defined by the conformal symmetry and, therefore, it should exist a different, probably purely algebraic, way to define the (super)Schwarzians. The main property of the (super)Schwarzians which define their structure, is their invariance with respect to (super)conformal transformations. The suitable way to construct (super)conformal invariants is the method of nonlinear realizations equipped by the inverse Higgs phenomenon. Such approach, demonstrated how the Schwarzians can be obtained via the non-linear realizations approach, was initiated in Anton Galajinsky paper (A. Galajinsky, 2019) and then it was applied to different super-conformal algebra in the series of his papers.Later on, this approach has been extended to the cases of non-relativistic Schwarzians and Carroll algebra(J. Gomis, D. Hidalgo, P. Salgado-Rebolledo, 2021).
The preference of the non-linear realizations approach for construction of the Schwarzians with respect to approach related with superconformal transformations, is much more wide area of its applications. Indeed, the non-linear realization method works perfectly for any (super)algebra and the set of invariant Cartan forms can be easily obtained.

Thus, the main questions in such approach are

- What is the role and source of the "boundary" time τ and its supersymmetric partners?
- Which constraints have to be imposed on the Cartan forms? Which forms nullified and how to construct the action from the surviving forms?
- Which additional technique can be used to simplify the calculations?

Of course, these questions was already partially analyzed and answered. However, some important properties and statements were missing. Moreover, the constraints proposed in the previous papers looks like the results of illuminating guess. The main puzzle is the fact that the constraints were imposed on the fermionic projections of the forms, but not on the forms themselves. Thus, the questions why it is so and what happens with the full Cartan forms after imposing of such constraints have been not fully analyzed. Finally, in the cases of more complicate superconformal group the calculations quickly become a rather cumbersome and the standard technique does not help.

Step one

The bosonic conformal group in $d = 1$ is infinite-dimensional. Its finite dimensional $sl(2,\mathbb{R})$ subalgebra spanned by the Hermitian generators of translation P, dilatation D and conformal boost K, can be fixed by the following relations

$$\mathrm{i}\,[D, P] = P, \quad \mathrm{i}\,[D, K] = -K, \quad \mathrm{i}\,[K, P] = 2D.$$

If we parameterized the $SL(2,\mathbb{R})$ - group element g as

$$g = e^{\mathrm{i}t\left(P+m^2K\right)}\,e^{\mathrm{i}zK}\,e^{\mathrm{i}uD},$$

then the Cartan forms

$$g^{-1}dg = \mathrm{i}\omega_P P + \mathrm{i}\omega_D D + \mathrm{i}\omega_K K$$

read

$$\omega_P = e^{-u}dt, \quad \omega_D = du - 2zdt, \quad \omega_K = e^{u}\left(dz + z^2dt + m^2dt\right).$$

The infinitesimal $sl(2,\mathbb{R})$ transformations

$$g \;\to\; g' = e^{\mathrm{i}aP}\;e^{\mathrm{i}bD}\;e^{\mathrm{i}cK}\;g$$

leaving the Cartan forms invariant read

$$\delta t = a\,\frac{1+\cos(2mt)}{2} + b\,\frac{\sin(2mt)}{2m} + c\,\frac{1-\cos(2mt)}{2m^2}, \quad \delta u = \frac{d}{dt}\delta t, \quad \delta z = \frac{1}{2}\frac{d}{dt}\delta u - \frac{d}{dt}\delta t\, z.$$

Step two

All Cartan forms are invariant with respect to $sl(2,\mathbb{R})$ transformations. Notice, within the nonlinear realization approach we implicitly mean that the "coordinates" u and z are functions depending on time t. However, neither "time" t, nether its differentials dt are invariant under $sl(2,\mathbb{R})$ transformations. Thus, to get the invariants one has to introduce the "invariant time" τ and parameterize the form ω_P as

$$\omega_P = e^{-u} dt = d\tau \quad \Rightarrow \quad \dot{t} = e^{u},\ \dot{u} = \frac{\ddot{t}}{\dot{t}},\ \ddot{u} = \frac{\dddot{t}}{\dot{t}} - \left(\frac{\ddot{t}}{\dot{t}}\right)^2.$$

Let us stress again that the τ is a new "invariant time" which completely inert under $sl(2,\mathbb{R})$ transformations. Correspondingly, the rest $sl(2,\mathbb{R})$ forms now read

$$\omega_D = \left(\dot{u} - 2e^{u} z\right) d\tau, \quad \omega_K = e^{u}\left(\dot{z} + e^{u}\left(z^2 + m^2\right)\right) d\tau.$$

Now, nullifying the form ω_D we will express the field $z(\tau)$ in terms of dilaton $u(\tau)$ and then in terms of new time τ

$$\omega_D = 0 \quad \Rightarrow \quad z = \frac{1}{2} e^{-u}\dot{u} = \frac{\ddot{t}}{2\dot{t}^2}.$$

This is particular case of the Inverse Higgs phenomenon (E.A. Ivanov, V.I. Ogievetsky, 1975).

Step three

After Second step we are leaving with only one field - "old time" $t(\tau)$ and only one invariant - form ω_K which now reads (The form $\omega_P = d\tau$ is also invariant. However, adding this form to the action evidently does not produce new equations of motion)

$$\omega_K = \frac{1}{2}\left[\ddot{u} - \frac{1}{2}\dot{u}^2 + 2m^2 e^{2u}\right] d\tau = \frac{1}{2}\left[\frac{\dddot{t}}{\dot{t}} - \frac{3}{2}\left(\frac{\ddot{t}}{\dot{t}}\right)^2 + 2m^2\dot{t}^2\right] d\tau$$

Thus, the Schwarzian action (1) can be re-obtained within our approach as

$$\mathcal{S}[t] = -\int \omega_K.$$

It proves useful to re-write the form ω_K and, therefore, the Schwarzian action in terms of dilaton $u(t)$ and "old time" variable t

$$\mathcal{S}[u] = -\int \omega_K = \int dt\left(\left(\frac{dy}{dt}\right)^2 - m^2 y^2\right), \qquad y(t) = e^{\frac{1}{2}u(t)}.$$

Thus, formally speaking, the action of Schwarzian mechanics is just the action of one dimensional harmonic oscillator rewritten in terms of time variable t depending on new inert time variable τ.

As the first example of the application of the proposed approach, let us consider the nonlinear realization of the Maxwell algebra in $d = 1$.
The Maxwell algebra contains the Hermitian generators of translation P, analogue of the dilatation - central charge generator Z, analogue of the conformal boost K, and the generator of $U(1)$ rotations obeying the following relations

$$\mathrm{i}\,[J, P] = P, \quad \mathrm{i}\,[J, K] = -K, \quad \mathrm{i}\,[K, P] = 2Z.$$

If we parameterized the Maxwell - group element g as

$$g = e^{\mathrm{i}t\left(P+qJ+m^2K\right)}e^{\mathrm{i}zK}e^{\mathrm{i}uZ}e^{\mathrm{i}\phi J},$$

then the Cartan forms read

$$\omega_P = e^{-\phi}dt, \quad \omega_Z = du - 2zdt, \quad \omega_K = e^{\phi}\left(dz - qzdt + m^2dt\right), \quad \omega_J = d\phi.$$

The constraints

$$\omega_P = d\tau, \quad \omega_Z = 0$$

result in the following relations

$$\dot{t} = e^{\phi}, \quad z = \frac{\dot{u}}{2\dot{t}}.$$

Finally,

$$\omega_K = \dot{t}\left[\frac{1}{2}\left(\frac{\ddot{u}}{\dot{t}} - \frac{\dot{u}\ddot{t}}{\dot{t}^2}\right) + m^2\dot{t} - \frac{1}{2}q\dot{u}\right].$$

This is exactly flat space analogue of the Schwarzian constructed in H. Afshar and H.A. Gonzalez, D. Grumiller, D. Vassilevich, 2020.

As the next non-trivial example we consider the nonlinear realization of the algebra $su(1,2)$ - bosonic analogue of the $\mathcal{N}=2$ superconformal algebra in $d=1$.
The $su(1,2)$ algebra in the preferred basis includes the following generators:

- the generators P, D, K, forming $sl(2,\mathbb{R})$ subalgebra
- the generators $Q, \overline{Q}$, and $S, \overline{S}$ - the bosonic analogs of the supersymmetric and conformal supersymmetry generators
- $U(1)$ generator U

The generators P, D, K and U are Hermitian, while the Q and S-generators obey the following conjugation rules $(Q)^\dagger = \overline{Q},\ (S)^\dagger = \overline{S}$. The non-zero commutators read

$$
\begin{aligned}
&\mathrm{i}\,[P,K] = -2D,\ \mathrm{i}\,[P,D] = -P,\ \mathrm{i}\,[K,D] = K,\\
&\mathrm{i}\,[P,S] = -Q,\ \mathrm{i}\left[P,\overline{S}\right] = -\overline{Q},\quad \mathrm{i}\,[K,Q] = S,\ \mathrm{i}\left[K,\overline{Q}\right] = \overline{S},\\
&\mathrm{i}\,[D,Q] = \frac{1}{2}Q,\ \mathrm{i}\left[D,\overline{Q}\right] = \frac{1}{2}\overline{Q},\quad \mathrm{i}\,[D,S] = -\frac{1}{2}S,\ \mathrm{i}\left[D,\overline{S}\right] = -\frac{1}{2}\overline{S},\\
&[U,Q] = Q,\ \left[U,\overline{Q}\right] = -\overline{Q},\quad [U,S] = S,\ \left[U,\overline{S}\right] = -\overline{S},\\
&\left[Q,\overline{Q}\right] = -\gamma P,\ \mathrm{i}\left[Q,\overline{S}\right] = -\frac{3}{2}\gamma U - \mathrm{i}\gamma D,\quad \left[S,\overline{S}\right] = -\gamma K,\ \mathrm{i}\left[S,\overline{Q}\right] = \frac{3}{2}\gamma U - \mathrm{i}\gamma D.
\end{aligned}
$$

We parametrize the group element in a standard way as

$$g = e^{\mathrm{i}tP} e^{\mathrm{i}\left(\phi Q + \bar{\phi}\overline{Q}\right)} e^{\mathrm{i}\left(vS + \bar{v}\overline{S}\right)} e^{\mathrm{i}zK} e^{\mathrm{i}uD} e^{\mathrm{i}\varphi U}.$$

The Cartan forms read

$$\begin{aligned}
\omega_P &= e^{-u}\left(dt + \frac{\mathrm{i}}{2}\gamma\left(\phi d\bar{\phi} - \bar{\phi} d\phi\right)\right) \equiv e^{-u}\triangle t, \\
\omega_D &= du - \mathrm{i}\gamma\left(\bar{v} d\phi - v d\bar{\phi}\right) - 2z\triangle t, \\
\omega_K &= e^{u}\left[dz + \left(z^2 + \frac{\gamma^2}{4}v^2\bar{v}^2\right)\triangle t - \mathrm{i}\gamma z\left(v d\bar{\phi} - \bar{v} d\phi\right) + \frac{\mathrm{i}}{2}\gamma\left(v d\bar{v} - \bar{v} dv\right)\right. \\
&\qquad \left. -\frac{\gamma^2}{2}v\bar{v}\left(v d\bar{\phi} + \bar{v} d\phi\right)\right] \\
\omega_Q &= e^{-\frac{u}{2} - \mathrm{i}\varphi}\left[d\phi - v\triangle t\right], \quad \bar{\omega}_Q = e^{-\frac{u}{2} - \mathrm{i}\varphi}\left[d\bar{\phi} - \bar{v}\triangle t\right], \\
\omega_S &= e^{\frac{u}{2} - \mathrm{i}\varphi}\left[dv - \left(z + \frac{\mathrm{i}}{2}\gamma v\bar{v}\right)\left(d\phi - v\triangle t\right) - \mathrm{i}\gamma v^2 d\bar{\phi}\right], \\
\bar{\omega}_S &= e^{\frac{u}{2} + \mathrm{i}\varphi}\left[d\bar{v} - \left(z - \frac{\mathrm{i}}{2}\gamma v\bar{v}\right)\left(d\bar{\phi} - \bar{v}\triangle t\right) + \mathrm{i}\gamma \bar{v}^2 d\phi\right], \\
\omega_U &= d\varphi - \frac{3}{2}\gamma\left(v d\bar{\phi} + \bar{v} d\phi - v\bar{v}\triangle t\right).
\end{aligned}$$

The constraints

$$\omega_D = \omega_Q = \bar{\omega}_Q = 0$$

lead to the following expressions

$$v = e^{-u}\dot{\phi},\ \bar{v} = e^{-u}\dot{\bar{\phi}}, \qquad z = \frac{1}{2}e^{-u}\dot{u}.$$

The equations of motion follow from the constraints

$$\omega_K = \omega_S = \bar{\omega}_S = 0$$

$$\begin{aligned}
\ddot{\phi} &= \dot{u}\dot{\phi} + \mathrm{i}e^{-u}\gamma\dot{\phi}^2\dot{\bar{\phi}}, \quad \ddot{\bar{\phi}} = \dot{u}\dot{\bar{\phi}} - \mathrm{i}e^{-u}\gamma\dot{\phi}\dot{\bar{\phi}}^2, \\
\ddot{u} &= \frac{1}{2}\left(\dot{u}^2 - e^{-2u}\gamma^2\dot{\phi}^2\dot{\bar{\phi}}^2\right).
\end{aligned}$$

The simplest conserved current reads

$$\frac{d}{d\tau}\left(e^{-2u}\dot{\phi}\dot{\bar{\phi}}\right) = 0.$$

The $\mathcal{N}$=2 super-Schwarzian has been introduced in J.D. Cohn,*N = 2 super Riemann surfaces*, (1987) and then it was re-obtaines in K. Schoutens, *$O(N)$-Extended superconformal field theory in superspace*, (1988) . The treatment of the $\mathcal{N}$=2 super-Schwarzian within the nonlinear realization of the $su(1,1|1)$ supergroup was initiated in A. Galajinsky, *Super-Schwarzians via nonlinear realizations*, (2020). The consideration performed in this paper correctly reproduced $\mathcal{N}$=2 super-Schwarzian but unfortunately the constraints used there imposed the further constraint on the super-Schwarzian to be a constant. Now, I will demonstrate that our variant of the constraints correctly reproduce $\mathcal{N}$=2 super-Schwarzian, expressed all $su(1,1|1)$ Cartan forms in terms of this super-Schwarzian and its derivatives. Finally, we will show that imposing the constraints on the full Cartan forms makes possible to utilize the Maurer-Cartan equations which drastically simplify all calculations.

In the case of $\mathcal{N}$=2 supersymmetry we are dealing with the $\mathcal{N}$=2 superconformal algebra $su(1,1|1)$ defined by the following relations

$$
\begin{aligned}
&\mathrm{i}\,[D,P]=P, \quad \mathrm{i}\,[D,K]=-K, \quad \mathrm{i}\,[K,P]=2D,\\
&\left\{Q,\overline{Q}\right\}=2P, \quad \left\{S,\overline{S}\right\}=2K, \quad \left\{Q,\overline{S}\right\}=-2D+2J, \left\{\overline{Q},S\right\}=-2D-2J,\\
&\mathrm{i}\,[J,Q]=\frac{1}{2}Q,\ \mathrm{i}\left[J,\overline{Q}\right]=-\frac{1}{2}\overline{Q}, \quad \mathrm{i}\,[J,S]=\frac{1}{2}S,\ \mathrm{i}\left[J,\overline{S}\right]=-\frac{1}{2}\overline{S},\\
&\mathrm{i}\,[D,Q]=\frac{1}{2}Q,\ \mathrm{i}\left[D,\overline{Q}\right]=\frac{1}{2}\overline{Q}, \quad \mathrm{i}\,[D,S]=-\frac{1}{2}S,\ \mathrm{i}\left[D,\overline{S}\right]=-\frac{1}{2}\overline{S},\\
&\mathrm{i}\,[K,Q]=-S,\ \mathrm{i}\left[K,\overline{Q}\right]=-\overline{S}, \quad \mathrm{i}\,[P,S]=Q,\ \mathrm{i}\left[P,\overline{S}\right]=\overline{Q}.
\end{aligned}
$$

Defining the "inert" element $g_0 = e^{\mathrm{i}\tau P}\, e^{\theta Q + \bar\theta \overline{Q}}$ and calculating the "inert" Cartan forms

$$\Omega_0 = g_0^{-1} dg_0 = \mathrm{i}\left(d\tau - \mathrm{i}(\theta d\bar\theta + \bar\theta d\theta)\right) P + d\theta Q + d\bar\theta \overline{Q} \equiv \mathrm{i}\triangle\tau P + d\theta Q + d\bar\theta\overline{Q},$$

one may easily construct the covariant derivatives

$$\mathcal{D}_\tau = \partial_\tau,\; \mathcal{D} = \frac{\partial}{\partial\theta} - i\bar\theta\frac{\partial}{\partial\tau},\; \overline{\mathcal{D}} = \frac{\partial}{\partial\bar\theta} - i\theta\frac{\partial}{\partial\tau}, \qquad \left\{\mathcal{D}, \overline{\mathcal{D}}\right\} = -2i\partial_\tau$$

Thus, from now we will treated all fields as the superfields depending on the coordinates of "inert" superspace $\left\{\tau, \theta, \bar\theta\right\}$.
Similarly to the previously considered cases, we choose the following parametrization of the general element of the $\mathcal{N}$=2 superconformal group $SU(1,1|1)$

$$g = e^{it(P+m^2K)}\, e^{\xi Q + \bar\xi\overline{Q}}\, e^{\psi S + \bar\psi\overline{S}} e^{izK} e^{iuD} e^{\phi J}$$

where the parameters $t, \xi, \bar\xi, \psi, \bar\psi, z, u$ and ϕ are, as we stated above, the superfunctions depending on $\{\tau, \theta, \bar\theta\}$.
The Cartan forms

$$g^{-1}dg = i\omega_P P + \omega_Q Q + \bar\omega_Q \overline{Q} + i\omega_D D + \omega_J J + \omega_S S + \bar\omega_S \overline{S} + i\omega_K K$$

explicitly read

$$
\begin{aligned}
&\omega_P \equiv e^{-u}\triangle t = e^{-u}\left(dt - i(\xi d\bar\xi + \bar\xi d\xi)\right),\\
&\omega_Q = e^{-\frac{u}{2}+i\frac{\phi}{2}}\left(d\xi + \psi\,\triangle t\right),\ \bar\omega_Q = e^{-\frac{u}{2}-i\frac{\phi}{2}}\left(d\bar\xi + \bar\psi\,\triangle t\right),\\
&\omega_D = du - 2z\,\triangle t - 2i(d\xi\bar\psi + d\bar\xi\psi),\ \omega_J = d\phi - 2\psi\bar\psi\,\triangle t + 2(d\bar\xi\psi - d\xi\bar\psi) - 2m^2\xi\bar\xi dt\\
&\omega_S = e^{\frac{u}{2}+i\frac{\phi}{2}}\left(d\psi - i\psi\bar\psi d\xi + z\,(d\xi + \psi\,\triangle t)\ - m^2\,(1-\mathrm{i}\bar\xi\psi)\,\xi dt\right),\\
&\bar\omega_S = e^{\frac{u}{2}-i\frac{\phi}{2}}\left(d\bar\psi + i\psi\bar\psi d\bar\xi + z\,(d\bar\xi + \bar\psi\,\triangle t)\ - m^2\,(1-\mathrm{i}\xi\bar\psi)\,\bar\xi dt\right),\\
&\omega_K = e^{u}\left(dz + z^2\triangle t - i(\psi\,d\bar\psi + \bar\psi\,d\psi) + 2iz\,(d\xi\,\bar\psi + d\bar\xi\psi) + m^2\,\left(1+\mathrm{i}\left(\psi\bar\xi + \bar\psi\xi\right)\right)^2\,dt\right).
\end{aligned}
$$

Now,identifying the forms $\omega_P, \omega_Q, \bar\omega_Q$ with $\triangle\tau$, $d\theta$ and $d\bar\theta$ we will get the following equations

$$
e^{-u}\triangle t = e^{-u}\left(dt + \mathrm{i}\left(d\bar\xi\xi + d\xi\bar\xi\right)\right) = \triangle\tau \qquad \Rightarrow \qquad \left\{\begin{array}{l} \dot t + \mathrm{i}\left(\dot{\bar\xi}\xi + \dot\xi\bar\xi\right) = e^u,\\ Dt + \mathrm{i}D\xi\,\bar\xi = 0,\\ \overline{D}t + \mathrm{i}\overline{D}\bar\xi\,\xi = 0,\end{array}\right.
$$

$$
e^{-\frac{1}{2}(u-\mathrm{i}\,\phi)}\left(d\xi + \psi\triangle t\right) = d\theta \qquad \Rightarrow \qquad \left\{\begin{array}{l} \dot\xi + e^u\psi = 0,\\ D\xi = e^{\frac{1}{2}(u-\mathrm{i}\,\phi)},\\ \overline{D}\xi = 0,\end{array}\right.
$$

$$
e^{-\frac{1}{2}(u+\mathrm{i}\,\phi)}\left(d\bar\xi + \bar\psi\triangle t\right) = d\bar\theta \qquad \Rightarrow \qquad \left\{\begin{array}{l} \dot{\bar\xi} + e^u\bar\psi = 0,\\ \overline{D}\bar\xi = e^{\frac{1}{2}(u+\mathrm{i}\,\phi)},\\ D\bar\xi = 0.\end{array}\right.
$$

Finally, one has to nullify the form ω_D :

$$\omega_D = du - 2e^u z \triangle\tau - 2i(e^{\frac{1}{2}(u-\mathrm{i}\,\phi)} d\theta\bar\psi + e^{\frac{1}{2}(u+\mathrm{i}\,\phi)} d\bar\theta\psi) = 0 \quad \Rightarrow \quad \begin{cases} \dot u - 2e^u z = 0, \\ Du = 2\mathrm{i}\, e^{\frac{1}{2}(u-\mathrm{i}\,\phi)}\bar\psi, \\ \overline{D}u = 2\mathrm{i}\, e^{\frac{1}{2}(u+\mathrm{i}\,\phi)}\psi. \end{cases}$$

From these relations one may obtained several important consequences. In particular, we have

$$Du = \mathrm{i}D\phi, \overline{D}u = -\mathrm{i}\overline{D}\phi, \quad \Rightarrow \quad \left[D,\overline{D}\right] u = -2\dot\phi, \; \left[D,\overline{D}\right]\phi = 2\dot u,$$

$$D\bar\psi = 0, \quad \overline{D}\psi = 0, \quad \psi = -\frac{\dot\xi}{D\xi\overline{D}\bar\xi}, \; \bar\psi = -\frac{\dot{\bar\xi}}{D\xi\overline{D}\bar\xi},$$

$$D\xi\,\overline{D}\bar\xi = e^u, \; \dot u = \frac{D\dot\xi}{D\xi} + \frac{\overline{D}\dot{\bar\xi}}{\overline{D}\bar\xi}, \quad \frac{\overline{D}\bar\xi}{D\xi} = e^{\mathrm{i}\phi}, \quad \dot\phi = \mathrm{i}\left(\frac{D\dot\xi}{D\xi} - \frac{\overline{D}\dot{\bar\xi}}{\overline{D}\bar\xi}\right).$$

Now, one may check that the form ω_J reads

$$\omega_J = \mathrm{i}\left[\frac{D\dot\xi}{D\xi} - \frac{\overline{D}\dot{\bar\xi}}{\overline{D}\bar\xi} - 2\mathrm{i}\frac{\dot\xi\dot{\bar\xi}}{D\xi\overline{D}\bar\xi} + 2\mathrm{i}m^2\xi\bar\xi D\xi\overline{D}\bar\xi\right]\triangle\tau \equiv \mathrm{i}\,\triangle\tau\,\mathcal{S}_{\mathcal{N}=2}.$$

Thus we see, that $\mathcal{N}=2$ Schwarzian $\mathcal{S}_{\mathcal{N}=2}$ appears automatically.

One may check that the other Cartan forms, $\omega_S, \bar{\omega}_S$ and ω_K can be also expressed in terms of the $\mathcal{N}$=2 Schwarzian only

$$\begin{aligned}
\omega_P &= \triangle\tau, \omega_Q = d\theta,\ \bar{\omega}_Q = d\bar{\theta}, \quad \omega_J = \mathrm{i}\mathcal{S}_{\mathcal{N}=2}\triangle\tau, \\
\omega_S &= -\frac{1}{2}\mathcal{S}_{\mathcal{N}=2}\, d\theta - \frac{\mathrm{i}}{2}\overline{D}\mathcal{S}_{\mathcal{N}=2}\triangle\tau, \quad \bar{\omega}_S = \frac{1}{2}\mathcal{S}_{\mathcal{N}=2}\, d\bar{\theta} + \frac{\mathrm{i}}{2}D\mathcal{S}_{\mathcal{N}=2}\triangle\tau, \\
\omega_K &= \frac{1}{2}D\mathcal{S}_{\mathcal{N}=2}d\theta - \frac{1}{2}\overline{D}\mathcal{S}_{\mathcal{N}=2}d\bar{\theta} + \frac{1}{4}\left(\mathrm{i}\left[D,\overline{D}\right]\mathcal{S}_{\mathcal{N}=2} - \mathcal{S}^2_{\mathcal{N}=2}\right)\triangle\tau.
\end{aligned}$$

The transformation laws of the basic superfields t, ξ, $\bar{\xi}$, are induced by left multiplication $g' = g_0 g$. In the case of superconformal transformations $g_0 = e^{\epsilon Q + \bar{\epsilon}\overline{Q}} e^{\varepsilon S + \bar{\varepsilon}\overline{S}}$ the transformation laws of t and ξ, $\bar{\xi}$ read

$$\delta t = \mathrm{i}\left(\bar{\epsilon}\xi + \epsilon\bar{\xi}\right)\cos(mt) - \mathrm{i}\frac{\sin(mt)}{m}\left(\bar{\varepsilon}\xi + \varepsilon\bar{\xi}\right),$$

$$\delta\xi = \cos(mt)\epsilon + \mathrm{i}\epsilon m\sin(mt)\xi\bar{\xi} - \frac{\sin(mt)}{m}\varepsilon + \mathrm{i}\varepsilon\cos(mt)\xi\bar{\xi},$$

$$\delta\bar{\xi} = \cos(mt)\bar{\epsilon} - \mathrm{i}\bar{\epsilon}m\sin(mt)\xi\bar{\xi} - \frac{\sin(mt)}{m}\bar{\varepsilon} - \mathrm{i}\bar{\varepsilon}\cos(mt)\xi\bar{\xi}.$$

The modified $\mathcal{N}$=2 Schwarzian $\mathcal{S}_{\mathcal{N}=2}$

$$\mathcal{S}_{\mathcal{N}=2} = \frac{D\dot{\xi}}{D\xi} - \frac{\overline{D}\dot{\bar{\xi}}}{\overline{D}\bar{\xi}} - 2\mathrm{i}\frac{\dot{\xi}\dot{\bar{\xi}}}{D\xi\overline{D}\bar{\xi}} + 2\mathrm{i}m^2\xi\bar{\xi}D\xi\overline{D}\bar{\xi}$$

is invariant with respect to these transformations.

Thus one can expect that the proper Schwarzian action reads

$$S_{N2schw} = -\frac{\mathrm{i}}{2}\int d\tau\, d\theta\, d\bar\theta \mathcal{S} = -\frac{1}{2}\int \omega_J \wedge \omega_Q \wedge \bar\omega_Q = \mathrm{i}\int \omega_P \wedge \omega_S \wedge \bar\omega_Q .$$

The component action is

$$\begin{aligned} S_{N2schw} \quad = \quad & -\frac{1}{2}\int d\tau \left[\frac{\partial_\tau^2\left(\dot t + \mathrm{i}\dot\xi\bar\xi + \mathrm{i}\dot{\bar\xi}\xi\right)}{\dot t + \mathrm{i}\dot\xi\bar\xi + \mathrm{i}\dot{\bar\xi}\xi} - \frac{3}{2}\frac{\left(\partial_\tau\left(\dot t + \mathrm{i}\dot\xi\bar\xi + \mathrm{i}\dot{\bar\xi}\xi\right)\right)^2}{\left(\dot t + \mathrm{i}\dot\xi\bar\xi + \mathrm{i}\dot{\bar\xi}\xi\right)^2} + 2\mathrm{i}\frac{\ddot\xi\dot{\bar\xi}\ddot{\bar\xi}\dot\xi}{\left(\dot t + \mathrm{i}\dot\xi\bar\xi + \mathrm{i}\dot{\bar\xi}\xi\right)^2} \right. \\ & \left. -\frac{1}{2}\dot\phi^2 - 2\frac{\dot\phi\,\dot\xi\,\dot{\bar\xi}}{\dot t} + 2m^2\frac{\dot t^3}{\dot t + \mathrm{i}\dot\xi\bar\xi + \mathrm{i}\dot{\bar\xi}\xi} + 2m^2\dot\phi\dot t\xi\bar\xi + 4m^2\xi\bar\xi\dot\xi\dot{\bar\xi} \right] \end{aligned}$$

- Within our approach we can construct $\mathcal{N}$=1, 2, 3, 4 supersymmetric extension of the Schwarzian basing on the supergroups $OSp(1|2), SU(1,1|1), OSp(3|2), SU(1,1|2)$ and $D(1,2;\alpha)$.
- The further extension to the groups $SU(1,1|>2)$ does not work
- The approach works fine for the supergrop $OSp(N|2)$
- It is interesting to analyze the supersymmetric versions of the Maxwell algebra