

# Field-theoretic models of turbulence: From Kolmogorov power laws to intermittency

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## General remarks

- Stochastic differential equation with gaussian noise

$$\partial_t \phi(x) = V(x, \phi) + f(x), \quad \langle f(x)f(x') \rangle = D(x, x') \quad x \equiv t, \mathbf{x}$$

- de Dominicis - Jansen action functional

$$S(\phi, \phi') = \frac{1}{2} \phi' D \phi' + \phi' [-\partial_t \phi + V(\phi)]$$

L.Ts. Adzhemyan, A.N. Vasil'ev, Yu. Pismak

*Renormalization-group approach in the theory of turbulence: The dimensions of composite operators*, TMF 57, 2 (1983)

C. De Dominicis, P.C. Martin, PRA 19 (1979)

# Turbulence as QFTM

- Stochastic NS equation

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = \mathbf{f}, \quad (\nabla \cdot \mathbf{v}) = 0$$

$$\langle \mathbf{f}(x) \cdot \mathbf{f}(x') \rangle = \delta(t - t') \mathbf{D}(\mathbf{x}, \mathbf{x}')$$

- Field-theoretic model

$$S(\mathbf{v}, \mathbf{v}') = \frac{1}{2} \mathbf{v}' \cdot \mathbf{D} \cdot \mathbf{v}' + \mathbf{v}' \cdot [-\partial_t \mathbf{v} + \nu_0 \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v}]$$

# master parameters, scales, renormalization

- energy dissipation rate  $\mathcal{E}$ , viscosity  $\nu_0$ , parameter  $\epsilon$ , scales  $l, L$  !

$$D_{ij}(\mathbf{x}, \mathbf{x}') = \frac{1}{(2\pi)^d} \int d\mathbf{k} \, d_f(k) e^{-i\mathbf{k}(\mathbf{x}-\mathbf{x}')} P_{ij}, \quad d_f(k) = D_0 k^{4-d-2\epsilon} h(kL)$$

$$\mathcal{W} = \frac{d-1}{2(2\pi)^d} \int d\mathbf{k} \, d_f(k), \quad \mathcal{E} \equiv \langle E(\mathbf{x}) \rangle = \frac{1}{2} \nu_0 \langle (\nabla_i \mathbf{v}_k + \nabla_k \mathbf{v}_i)^2 \rangle$$

- stationary turbulence:  $\mathcal{W} = \mathcal{E}$

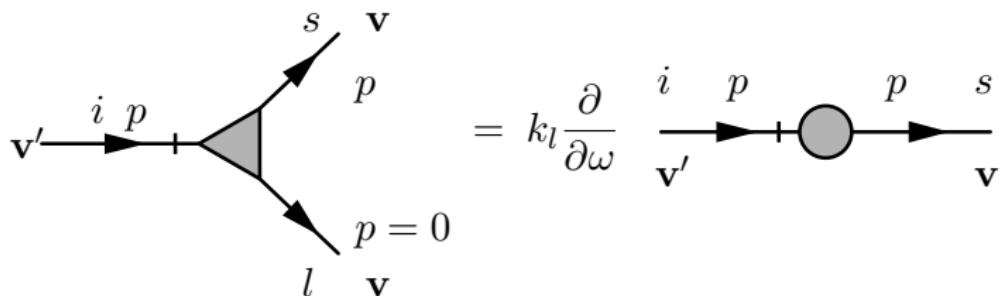
$$S_R(\mathbf{v}, \mathbf{v}') = \frac{1}{2} \mathbf{v}' \cdot \mathbf{D} \cdot \mathbf{v}' + \mathbf{v}' \cdot [-\partial_t \mathbf{v} + \nu Z_\nu \Delta \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v}]$$

$$\nu_0 = \nu Z_\nu, \quad g_0 = g \mu^{2\epsilon} Z_g, \quad Z_g = Z_\nu^{-3}, \quad D_0 = g_0 \nu_0^3 = g \nu^3$$

# Galilean invariance

- Ward identity

$$\Gamma_{is} l(p, p, 0) = k_l \frac{\partial}{\partial \omega} \Gamma_{is}(p)$$



# Renormalization group approach and scaling

- RG and scaling on large scales  $x, r \gg l$

$$\beta(g) = g(-2\epsilon + 3\gamma_\nu), \quad \beta(g_*) = 0, \quad \beta'(g_*) > 0, \quad \gamma_\nu(g_*) = \frac{2\epsilon}{3}$$

$$\left[ -x \frac{\partial}{\partial x} + \Delta_t \frac{\partial}{\partial t} - L \frac{\partial}{\partial L} - \Delta_n \right] W_n = 0$$

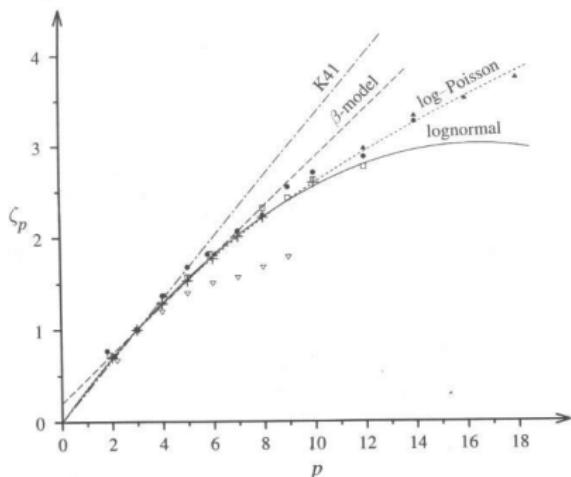
- Structure functions – equal-time two-point  $p$ -power correlations, second Kolmogorov hypothesis OK!

$$\mathcal{S}_p(r) = (\mathcal{E}r)^{\Delta_p} f_p(r/L), \quad \Delta_p = p/3, \quad (\Delta_p = p[\gamma_\nu(g_*) - 1])$$

- inertial interval  $l \gg r \gg L$ , first Kolmogorov hypothesis ?

$$\mathcal{S}_p(r) = C_p (\mathcal{E}r)^{p/3}, \quad \mathcal{S}_2(r) = C_k (\mathcal{E}r)^{2/3}$$

# Intermittency in turbulence



$$\mathcal{S}_p(r) = C_p \mathcal{E}^{p/3} r^{p/3} (r/L)^{-q_p}, \quad \langle E(x)E(y) \rangle = C_E \mathcal{E}^2 (r/L)^{-\mu},$$

$$q_p \geq 0, \quad 0.2 \leq \mu \leq 0.5$$

# Dangerous composite operators and anomalous scaling

- Wilson operator product expansion  $r/L \ll 1$ :

$$\mathcal{S}_p(r) = (\mathcal{E}r)^{p/3} f_p(r/L), \quad f_p(r/L) = \sum_{i=1}^{\infty} C_i^p (r/L)^{\Delta_i^p}$$

- Composite operators  $F(x)$

$$\mathbf{v}^4(x), \quad \mathbf{v}(x)\Delta\mathbf{v}(x), \quad \partial_t\mathbf{v}(x), \quad \Delta\mathbf{v}^2(x), \quad E(\mathbf{x})$$

$$\Delta^F = d_F + \gamma_\nu(g^*)d_F^\omega + \gamma_F(g^*)$$

L.Ts. Adzhemyan, A.N. Vasil'ev, M. Hnatic

*Renormalization group approach in the theory of turbulence: Renormalization and critical dimensions of the composite operators of energy-momentum tensor*, TMF 74, 2, (1988)

L.Ts. Adzhemyan, N.V. Antonov, A.N. Vasil'ev

*Quantum-field renormalization group in the theory of developed turbulence*, UFN 166, 12 (1996)

# Calculations, discussion, conclusion

- Calculation of critical dimensions  $\Delta_F$   
Ward identities, Swinger equations
- CO – powers of velocity field and its derivations  $\partial_t^m \mathbf{v}^n$ :  
 $\Delta = m(1 - \frac{2\epsilon}{3}) + n(1 - \frac{2\epsilon}{3})$  – exact
- CO  $E(\mathbf{x})$   $d_F = 4$ :  $\Delta_F = 4 - 2\epsilon$  – exact
- CO  $F(x)$   $d_F = 6$ :  $\Delta_{F_1} = 6 - 2\epsilon$  – exact,  $\Delta_{F_2} = 6 - \frac{8\epsilon}{7} + O(\epsilon^2)$
- CO  $F(x)$   $d_F = 8$ :  $\Delta_{F_1} = 8 - 8\epsilon$ ,  $\Delta_{F_2} = 8 - 8\epsilon$ ,  $\Delta_{F_3} = 8 - \frac{8\epsilon}{3}$  – exact
- powers of operator of energy dissipation rate  $F(x) = E^n(x)$ :  
 $\Delta_F \neq n(4 - 2\epsilon)$
- Conclusions: infinite number of CO with  $\Delta < 0$  in physical region  $\epsilon \geq 2$
- Dangerous operators enter into the operator product expansions in the form of infinite families with the spectrum of critical dimensions unbounded from below, and the analysis of the large  $L$  behaviour implies the summation of their contributions

# Toy models, passive scalar

- Advection of passive scalar field by a given turbulent flow

$$\partial_t \theta + (\mathbf{v} \cdot \nabla) \theta = \nu_0 \Delta \theta + f$$

$$\langle f(x)f(x') \rangle = \delta(t-t') C((\mathbf{x}-\mathbf{x}')/L)$$

$$\langle \mathbf{v}(x)\mathbf{v}(x') \rangle = D_{\mathbf{v}}(\mathbf{x}-\mathbf{x}')$$

$$\langle v_i(x)v_j(x') \rangle = D_0 \frac{\delta(t-t')}{(2\pi)^d} \int d\mathbf{k} P_{ij}(\mathbf{k}) (k^2 + L^{-2})^{-d/2-\epsilon/2} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}$$

- action functional

$$S(\theta, \theta', \mathbf{v}) = \theta' D_\theta \theta' / 2 + \theta' [-\partial_t + \nu_0 \Delta - (\mathbf{v} \cdot \nabla)] \theta - \mathbf{v} \cdot D_{\mathbf{v}}^{-1} \cdot \mathbf{v} / 2$$

# Structure functions

- Scaling in large scale region

$$\mathcal{S}_p(r) \sim r^{p(2-\epsilon)/2} f_p(r/L), \quad l \ll r$$

- Operator product expansion

$$f_p(r/L) = \sum_{k=0}^p C_k (r/L)^{\Delta_k} \sim (r/L)^{\Delta_p}, \quad r \ll L$$

- The leading composite operators

$$F_k = (\partial_i \theta \partial_i \theta)^k$$

- Two-loop calculations ( $d = 3$ ):

$$\Delta_k = \frac{2k(k-1)}{5} \left[ -\epsilon + \frac{\epsilon^2}{175} \left( 2k(255\pi\sqrt{3} - 1384) + 345\pi\sqrt{3} - 1884 \right) \right]$$

# Results and papers

L. Ts. Adzhemyan, N. V. Antonov, A. N. Vasil'ev

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N. V. Antonov, M. Hnatich, J. Honkonen, M. Jurčišin

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# Double $\epsilon$ -expansion

- Perturbative calculation of quantity  $A$  - anomalous exponents, amplitudes, fixed points

$$A(\epsilon, d) = \sum_{k=0}^{\infty} A_k(d) \epsilon^k$$

- $A_k(d)$  - singularities at dimension  $d = 2$  – Laurent series

$$A_k(d) = \sum_{l=0}^{\infty} a_{kl} \Delta^{l-k}, \quad (d-2) \equiv 2\Delta$$

- double expansion in  $\epsilon, \Delta$  at fixed  $\zeta$

$$A(\epsilon, d) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \zeta^k a_{kl} \Delta^l, \quad \zeta \equiv \epsilon/\Delta$$

J. Honkonen, M.Yu. Nalimov

*Two-parameter expansion in the renormalization group analysis of turbulence*  
Zeitschrift fur Physics B 99 (1995)

# Kolmogorov constant and skewness factor

$$S_2(r) = C_k \mathcal{E}^{2/3} r^{2/3}, \quad S_3(r) = -\frac{12}{d(d+2)} \mathcal{E} r$$

$$\mathcal{SF} = S_3(r)/S_2^{3/2}(r), \quad C_k = \left[ -\frac{12}{d(d+2)\mathcal{SF}} \right]^{2/3}$$

- One-loop calculations

$$C_k \approx 1.889, (1.47) \quad \text{exp.} \approx 2.01$$

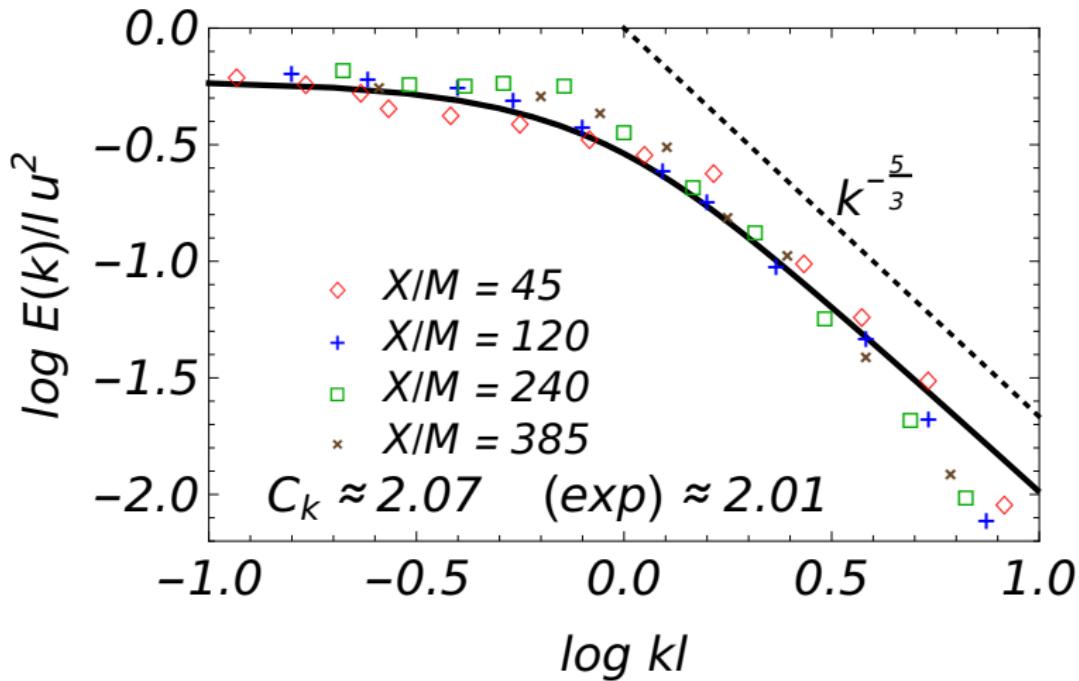
$$\mathcal{SF} \approx -0.308, (-0.45) \quad \text{exp.} \approx -0.28$$

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# Decaying turbulence



Obr.: Energy spectrum in energy-containing and inertial range

L.Ts.Adzhemyan, M.Hnatich, D.Horvath, M.Stehlik

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