Few-nucleon systems with relativistic separable kernel

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MQFT-2022, S.-Petersburg, Russia, 10-15 October 2022

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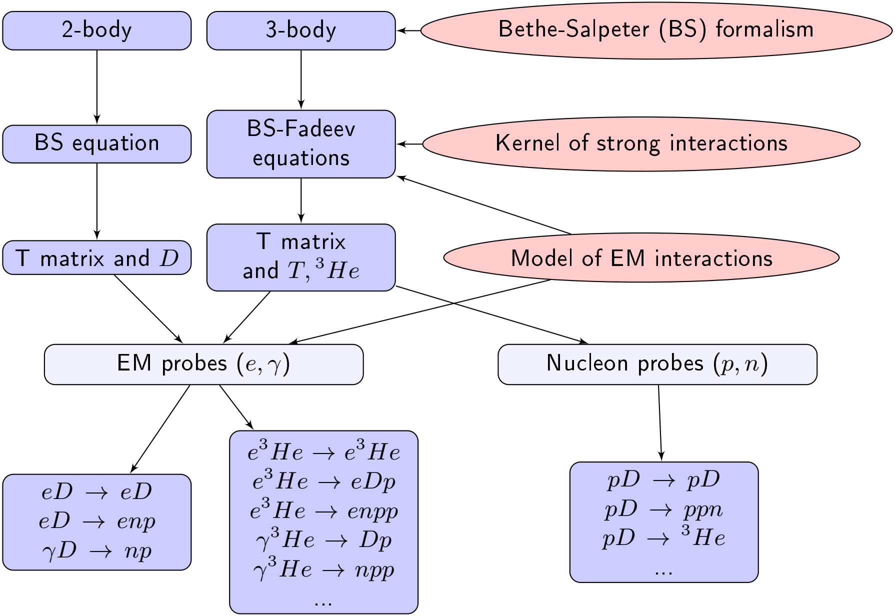

2-body
3-body
Bethe-Salpeter (BS) formalism
BS equation
BS-Fadeev equations
Kernel of strong interactions
T matrix and D
T matrix and $T, {}^3He$
Model of EM interactions
EM probes (e, γ)
Nucleon probes (p, n)
$eD \to eD$
$eD \to enp$
$\gamma D \to np$
$e^3He \to e^3He$
$e^3He \to eDp$
$e^3He \to enpp$
$\gamma^3He \to Dp$
$\gamma^3He \to npp$
...
$pD \to pD$
$pD \to ppn$
$pD \to {}^3He$
...

Bethe-Salpeter equation for the nucleon-nucleon T matrix

$$T(p,p';P) = V(p,p';P) + \frac{i}{(2\pi)^4}\int d^4k\, V(p,k;P)\, G(k;P)\, T(k,p';P)$$

p', p - the relative four-momenta
P - the total four-momentum

$T(p,p';P)$ – two-nucleon t matrix
$V(p,p';P)$ – kernel of nucleon-nucleon interaction
$G(p;P)$ – free scalar two-particle propagator

$$G^{-1}(p;P) = \left[(P/2+p)^2 - m_N^2 + i\epsilon\right]\left[(P/2-p)^2 - m_N^2 + i\epsilon\right]$$

Separable kernels of the NN interaction

The separable kernels of the nucleon-nucleon interaction are widely used in the calculations. The separable kernel as a *nonlocal* covariant interaction representing complex nature of the space-time continuum.

Separable rank-one *Ansatz* for the kernel

$$V_L(p_0^\prime,|\mathbf{p}^\prime|;p_0,|\mathbf{p}|;s) = \lambda^{[L]}(s) g^{[L]}(p_0^\prime,|\mathbf{p}^\prime|) g^{[L]}(p_0,|\mathbf{p}|)$$

Solution for the T matrix

$$T_L(p_0^\prime,|\mathbf{p}^\prime|;p_0,|\mathbf{p}|;s) = \tau(s)\, g^{[L]}(p_0^\prime,|\mathbf{p}^\prime|)\, g^{[L]}(p_0,|\mathbf{p}|)$$

with

$$\Big[\tau(s)\Big]^{-1} = \Big[\lambda^{[L]}(s)\Big]^{-1} + h(s),$$

$$h(s) = \sum_{coupled\, L} h_L(s) = -\frac{i}{4\pi^3} \int dk_0 \int |\mathbf{k}|^2\, d|\mathbf{k}| \sum_L [g^{[L]}(k_0,|\mathbf{k}|)]^2 S(k_0,|\mathbf{k}|;s)$$

$g^{[L]}$ - the model function, $\lambda^{[L^\prime L]}(s)$ - a model parameter.

The relativistic generalization of the NR Graz-II and Paris separable kernel:

- Graz-II: ${}^1S_0^+$ – rank 2, ${}^3S_1^+ - {}^3D_1$ – rank 3
- Paris-1,2: ${}^1S_0^+$ – rank 3, ${}^3S_1^+ - {}^3D_1$ – rank 4

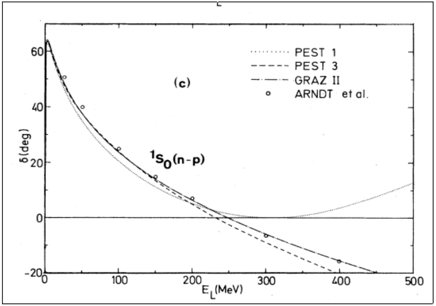
Results for ${}^1S_0^+$ channel

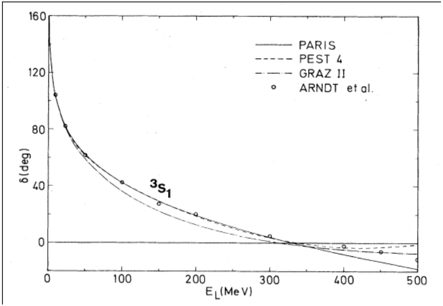
	Exp.	Graz-II	Paris-1	Paris-2
a (fm)	-23.748	-23.77	-23.72	-23.72
r_0 (fm)	2.75	2.683	2.810	2.817

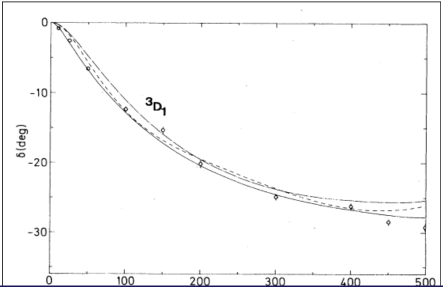
Results for ${}^3S_1^+ - {}^3D_1$ channels

	Exp.	Graz-II	Graz-II	Graz-II	Paris-1	Paris-2
p_d (%)		4	5	6	5.77	5.77
a (fm)	5.424	5.419	5.420	5.421	5.426	5.413
r_0 (fm)	1.759	1.780	1.779	1.778	1.775	1.765
E_d (MeV)	2.2246	2.2254	2.2254	2.2254	2.2246	2.2250

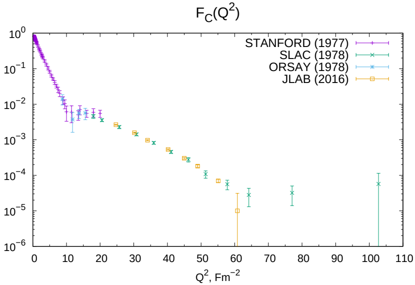
Phase shifts



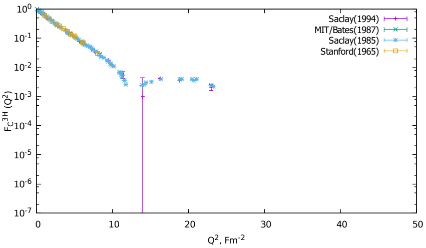


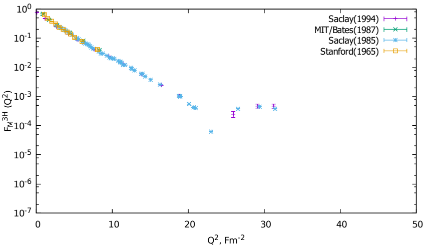


Experimental data for ^3He



Experimental data for ^3H





The relativistic three-particle equation for T matrix

is considered in the Fadeev form with the following assumptions:

- no three-particles interaction $V_{123} = \sum_{i \neq j} V_{ij}$
- two-particles interaction is separable
- nucleon propagators are chosen in a scalar form
- the only strong interactions are considered (not EM), so $^3He \equiv T$

Bethe-Salpeter-Fadeev equation

$$\left[\begin{array}{c} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{array}\right] = \left[\begin{array}{c} T_1 \\ T_2 \\ T_3 \end{array}\right] - \left[\begin{array}{ccc} 0 & T_1 G_1 & T_1 G_1 \\ T_2 G_2 & 0 & T_2 G_2 \\ T_3 G_3 & T_3 G_3 & 0 \end{array}\right] \left[\begin{array}{c} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{array}\right],$$

where full three-particles T matrix $T = \sum_i T^{(i)}$, G_i is the free two-particles (j and n) Green function (ijn is cyclic permutation of (1,2,3)):

$$G_i(k_j, k_n) = 1/(k_j^2 - m_N^2 + i\epsilon)/(k_n^2 - m_N^2 + i\epsilon),$$

and T_i is the two-particles T matrix which can be written as following

$$T_i(k_1, k_2, k_3; k_1', k_2', k_3') = (2\pi)^4 \delta^{(4)}(k_i - k_i') T_i(k_j, k_n; k_j', k_n').$$

with $s_i = (k_j + k_n)^2 = (k_j' + k_n')^2$.

Partial-wave three-nucleon functions

$$\Psi_{\lambda L}^{(a)}(p_0,|\mathbf{p}|,q_0,|\mathbf{q}|;s) = g^{(a)}(p_0,|\mathbf{p}|)\tau^{(a)}[(\frac{2}{3}\sqrt{s}+q_0)^2-\mathbf{q}^2]\,\Phi_{\lambda L}^{(a)}(q_0,|\mathbf{q}|;s)$$

System of the integral equations

$$\Phi_{\lambda L}^{(a)}(q_0,|\mathbf{q}|;s) = \frac{i}{4\pi^3}\sum_{a'\lambda'}\int_{-\infty}^{\infty}dq_0'\int_0^{\infty}\mathbf{q}'^2 d|\mathbf{q}'|\,Z_{\lambda\lambda'}^{(aa')}(q_0,q;q_0',|\mathbf{q}'|;s)$$

$$\frac{\tau^{(a')}[(\frac{2}{3}\sqrt{s}+q_0')^2-\mathbf{q}'^2]}{(\frac{1}{3}\sqrt{s}-q_0')^2-\mathbf{q}'^2-m^2+i\epsilon}\Phi_{\lambda' L}^{(a')}(q_0',|\mathbf{q}'|;s)$$

with effective kernels of equation

$$Z_{\lambda\lambda'}^{(aa')}(q_0,|\mathbf{q}|;q_0',|\mathbf{q}'|;s) = C_{(aa')}\int d\cos\vartheta_{\mathbf{q}\mathbf{q}'}\,K_{\lambda\lambda' L}^{(aa')}(|\mathbf{q}|,|\mathbf{q}'|,\cos\vartheta_{\mathbf{q}\mathbf{q}'})$$

$$\frac{g^{(a)}(-q_0/2-q_0',|\mathbf{q}/2+\mathbf{q}'|)g^{(a')}(q_0+q_0'/2,|\mathbf{q}+\mathbf{q}'/2|)}{(\frac{1}{3}\sqrt{s}+q_0+q_0')^2-(\mathbf{q}+\mathbf{q}')^2-m_N^2+i\epsilon}$$

Singularities

Poles from one-particle propagator

$$q^{0\prime}_{1,2} = \frac{1}{3}\sqrt{s} \mp [E_{|\mathbf{q}'|} - i\epsilon]$$

Poles from propagator in Z-function

$$q^{0\prime}_{3,4} = -\frac{1}{3}\sqrt{s} - q^0 \pm [E_{|\mathbf{q}'+\mathbf{q}|} - i\epsilon]$$

Poles from Yamaguchi-functions

$$q^{0\prime}_{5,6} = -2q^0 \pm 2[E_{|\frac{1}{2}\mathbf{q}'+\mathbf{q}|,\beta} - i\epsilon]$$

and

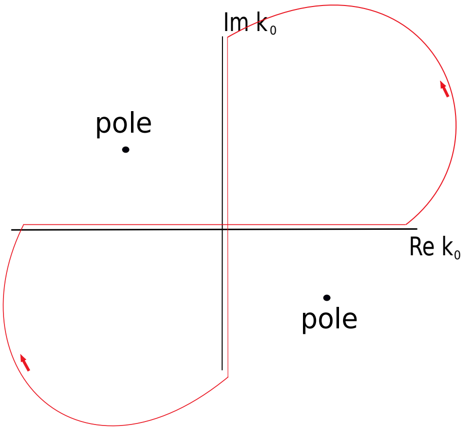
$$q^{0\prime}_{7,8} = -\frac{1}{2}q^0 \pm \frac{1}{2}[E_{|\mathbf{q}'+\frac{1}{2}\mathbf{q}|,\beta} - i\epsilon]$$

Cuts from two-particle propagator τ

$$q^{0\prime}_{9,10} = \pm\sqrt{q'^2 + 4m^2} - \frac{2}{3}\sqrt{s} \qquad \text{and} \qquad \pm\infty$$

Poles from two-particle propagator τ

$$q^{0\prime}_{11,12} = \pm\sqrt{q'^2 + 4M_d^2} - \frac{2}{3}\sqrt{s}$$


Im k_0
pole
Re k_0
pole

Method of solution

- Wick-rotation procedure: $q_0 \to iq_4$
- The Gaussian quadrature with $N_1 \times N_2[q_4 \times |\mathbf{q}|]$ grid

$$q_4 = (1+x)/(1-x)$$

$$|\mathbf{q}| = (1+y)/(1-y)$$

- Iteration method to obtain the triton binding energy

$$\lim_{n\to\infty} \frac{\Phi_n(s)}{\Phi_{n-1}(s)}\Big|_{s=M_B^2} = 1$$

Triton binding energy (MeV)

Graz-II 4	8.628
Graz-II 5	8.223
Graz-II 6	7.832
Paris-1	7.545
Exp.	8.48

Triton binding energy (MeV)

p_D	${}^1S_0 - {}^3S_1$	${}^3D_1$	${}^3P_0$	${}^1P_1$	${}^3P_1$
4	9.221	9.294	9.314	9.287	9.271
5	8.819	8.909	8.928	8.903	8.889
6	8.442	8.545	8.562	8.540	8.527
		Exp.	8.48		

- the main contribution is from S-states
- the D-state contribution is about 0.8 – 1.2 % depending on D-wave (pseudo)probability in deuteron
- the P-state contributions are alternating and give about -0.2%

Electromagnetic form factors of three-nucleon systems:

$$
\begin{aligned}
2F_{\rm C}(^3{\rm He}) &= (2F^p_{\rm C} + F^n_{\rm C})F_1 - \frac{2}{3}(F^p_{\rm C} - F^n_{\rm C})F_2,\\
F_C(^3{\rm H}) &= (2F^n_{\rm C} + F^p_{\rm C})F_1 + \frac{2}{3}(F^p_{\rm C} - F^n_{\rm C})F_2,\\
\mu(^3{\rm He})F_{\rm M}(^3{\rm He}) &= \mu_n F^n_{\rm M}F_1 + \frac{2}{3}(\mu_n F^n_{\rm M} + \mu_p F^p_{\rm M})F_2 + \frac{4}{3}(F^p_{\rm M} - F^n_{\rm M})F_3,\\
\mu(^3{\rm H})F_{\rm M}(^3{\rm H}) &= \mu_p F^p_{\rm M}F_1 + \frac{2}{3}(\mu_n F^n_{\rm M} + \mu_p F^p_{\rm M})F_2 + \frac{4}{3}(F^n_{\rm M} - F^p_{\rm M})F_3,
\end{aligned}
$$

Electric and magnetic form factors of the proton and neutron $F^{p,n}_{\rm C,M}$.

Impulse approximation:

$$F_i(\hat{Q}) = \int d^4\hat{p} \int d^4\hat{q} \;\; G_1'(\hat{k}_1')\, G_1(\hat{k}_1)\, G_2(\hat{k}_2)\, G_3(\hat{k}_3)\, f_i(\hat{p},\hat{q},\hat{q}';\hat{P},\hat{P}')$$

Nucleon propagators:

$$G_i(\hat{k}_1) = \left[\hat{k}_i^2 - m_N^2 + i\epsilon\right]^{-1},$$

$$G_1'(q_0',q') = \left[(\frac{1}{3}\sqrt{s} - q_0')^2 - \mathbf{q}'^2 - m_N^2 + i\epsilon\right]^{-1},$$

Three-nucleon vertex functions:

$$f_1 = \sum_{i=1}^{3} \Psi_i^*(\hat{p},\hat{q};\hat{P})\, \Psi_i(\hat{p},\hat{q}';\hat{P}')$$

$$f_2 = -3\Psi_1^*(\hat{p},\hat{q};\hat{P})\, \Psi_2(\hat{p},\hat{q}';\hat{P}')$$

$$f_3 = \Psi_3^*(\hat{p},\hat{q};\hat{P})\, \Psi_3(\hat{p},\hat{q}';\hat{P}')$$

Functions Ψ_i are the definite combinations of the partial state functions.

The Breit reference system

$$\hat{Q} = (0, \mathbf{Q}), \qquad \hat{P} = (E_B, -\frac{\mathbf{Q}}{2}), \qquad \hat{P}' = (E_B, \frac{\mathbf{Q}}{2}), \tag{1}$$

with $E_B = \sqrt{\mathbf{Q}^2/4 + s}$, $s = M_{3N}^2$.

$$\begin{array}{lll} \hat{P} = \mathrm{L}\hat{P}_{c.m.}, & \hat{p} = \mathrm{L}\hat{p}_{c.m.}, & \hat{q} = \mathrm{L}\hat{q}_{c.m.} \\ \hat{P}' = \mathrm{L}^{-1}\hat{P}'_{c.m.}, & \hat{p}' = \mathrm{L}^{-1}\hat{p}'_{c.m.}, & \hat{q}' = \mathrm{L}^{-1}\hat{q}'_{c.m.} \end{array}$$

The explicit form of the transformation L can be obtained by using (1). Let us assume the boost of the system to be along the Z axis:

$$\mathrm{L} = \begin{pmatrix} \sqrt{1+\eta} & 0 & 0 & -\sqrt{\eta} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sqrt{\eta} & 0 & 0 & \sqrt{1+\eta} \end{pmatrix}. \tag{2}$$

Relation of the arguments of initial and final $3N$ functions:

$$q_0' = (1+2\eta)\, q_0 - 2\sqrt{\eta}\sqrt{1+\eta}\, q_z + \frac{2}{3}\sqrt{\eta}\, Q, \tag{3}$$
$$q_x' = q_x \qquad q_y' = q_y$$
$$q_z' = (1+2\eta)\, q_z - 2\sqrt{\eta}\sqrt{1+\eta}\, q_0 - \frac{2}{3}\sqrt{1+\eta}\, Q,$$

here $q_z = q\cos\theta_{qQ}$ is the projection of momentum $\mathbf{q}$ onto the Z axis

Static approximation (SA):

$$q_0' = q_0, \qquad \mathbf{q}' = \mathbf{q} - \frac{2}{3}\mathbf{Q}$$

Propagator and final function:

$$G_1'(q_0', q') \to \left[(\frac{1}{3}\sqrt{s} - q_0)^2 - \mathbf{q}^2 - \frac{2}{3}\mathbf{q}\cdot\mathbf{Q} - \frac{4}{9}\mathbf{Q}^2 - m_N^2 + i\epsilon \right]^{-1}$$

$$\Psi_i(p_0, p, q_0', q') \to \Psi_i(p_0, p, q_0, |\mathbf{q} - \frac{2}{3}\mathbf{Q}|)$$

with $\mathbf{q}\cdot\mathbf{Q} = qQ\cos\theta_{qQ}$.
The poles of G_1' on q_0 do not cross the imaginary q_0 axis and always stay in the second and fourth quadrants. In this case, the Wick rotation procedure $q_0 \to iq_4$ can be applied.

Beyond the SA:

1. Exact propagator

$$G_1' = \left[q_0^2 + \frac{2}{3}\sqrt{s}(1+6\eta)q_0 + 4\sqrt{1+\eta}\sqrt{s}\sqrt{\eta}q_z - \frac{8}{3}\eta s + \frac{1}{9}s - \mathbf{q}^2 - m_N^2 + i\epsilon\right]^{-1}$$

$$\Psi_i(p_0, p, q_0', q') \to \Psi_i(p_0, p, q_0, |\mathbf{q} - \frac{2}{3}\mathbf{Q}|).$$

For any $t = -\hat{Q}^2 > -\hat{Q}^2_{min} = 2/3\sqrt{s}(3m_N - \sqrt{s})$ the pole of G_1' on q_0 crosses the imaginary q_0 axis and appears in the third quadrant.

Beyond the SA:

2. Additional term from residue inside the countour of integration

Using the Cauchy theorem, one can transform the integrals over p_0, q_0 as follows:

$$\int_{-\infty}^{\infty} dp_0 \int_{-\infty}^{\infty} dq_0 \int_{0}^{\infty} dq \int_{-1}^{1} dy \ldots f(p_0, q_0, p, q, x, y) = \tag{4}$$

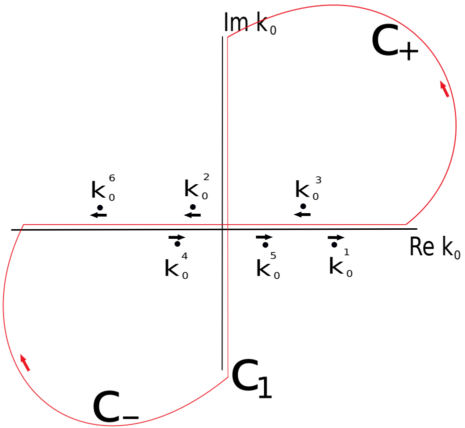
$$- \int_{-\infty}^{\infty} dp_4 \int_{-\infty}^{\infty} dq_4 \int_{0}^{\infty} dq \int_{-1}^{1} dy \ldots f(ip_4, iq_4, p, q, x, y)$$

$$+2\pi \operatorname*{Res}_{q_0=q_0^{(2)}} \int_{-\infty}^{\infty} dp_4 \int_{q_{min}}^{q_{max}} dq \int_{y_{min}}^{1} dy \ldots f(ip_4, q_0^{(2)}, p, q, x, y),$$

where (...) means the two-fold integral $\int_0^\infty dp \int_{-1}^1 dx$ and

$$q_0^{(1,2)} = \frac{\sqrt{s}}{3}(1+6\eta) \pm \sqrt{4\eta(1+\eta)s - 4\sqrt{s}\sqrt{\eta}\sqrt{1+\eta}qy + \mathbf{q}^2 + m_N^2} \tag{5}$$

are the simple poles of the propagator G_1'.



Im k0
C+
k0 6
k0 2
k0 3
Re k0
k0 4
k0 5
k0 1
C1
C−

Beyond the SA:

3. Final function arguments transformation

Remembering that the BSF solutions are known for real values of q_4 only, the following assumption was made:

$$\Psi(p_0,p,q_0',q') \to g(p_0,p)\,\tau[(\frac{2}{3}\sqrt{s}+q_0^{(2)})^2-\bar{\mathbf{q}}'^2]\,\Phi(0,\bar{q}'),$$

where value $\bar{q}'$ is obtained using (3) with $q_0=q_0^{(2)}$.

The expansion of the function $\Phi(q_4',q')$ up to the first order of the parameter η:

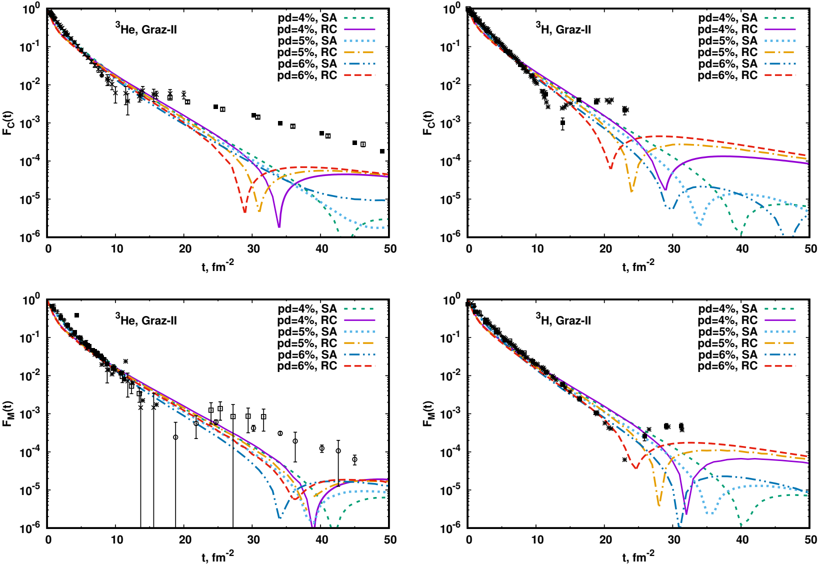
$$\Phi(iq_4',q')=\Phi(iq_4,|\mathbf{q}-\frac{2}{3}\mathbf{Q}|)+\left[C_{q_4}\frac{\partial}{\partial q_4}\Phi_j(iq_4,q)\right]_{q=|\mathbf{q}-\frac{2}{3}\mathbf{Q}|}$$
$$+\left[C_q\frac{\partial}{\partial q}\Phi_j(iq_4,q)\right]_{q=|\mathbf{q}-\frac{2}{3}\mathbf{Q}|},$$

where

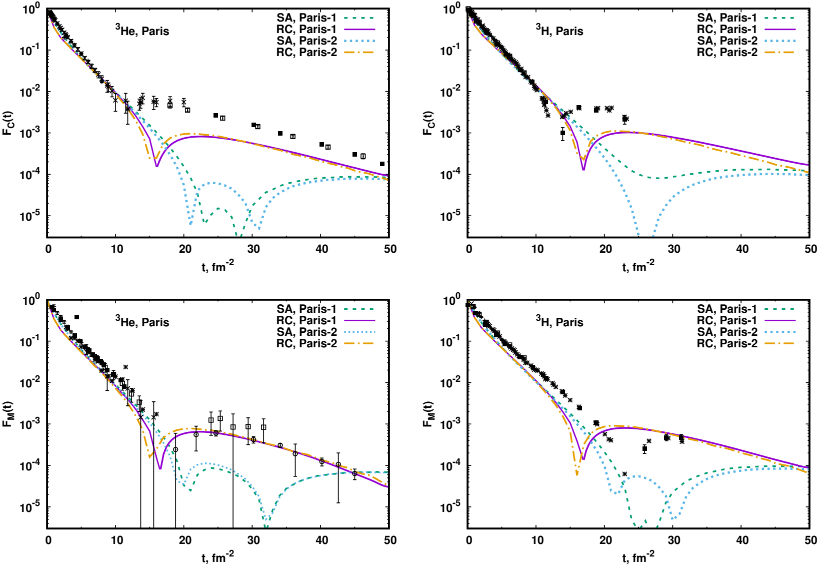
$$C_{q_4}=-i\left(2i\eta q_4-2\sqrt{\eta}\sqrt{1+\eta}q\cos\theta_{qQ}+\frac{2}{3}\sqrt{\eta}Q\right),$$

$$C_q=\left(2\eta q\cos\theta_{qQ}-2i\sqrt{\eta}\sqrt{1+\eta}q_4-\frac{2}{3}(\sqrt{1+\eta}-1)Q\right)\cos\theta_{qQ}.$$

Graz-II relativistic kernel



Paris relativistic kernel



Summary

- the relativistic three-nucleon vertex functions were founs solving the BSF systen of equations
- the charge and magnetic EM form factors of the 3N systems were calculated
- the static approximation and relativistic corrections were investigated
- the relativistic corrections were found to be significant in describing the experimental data

BACKUP SLIDES

The Bethe-Salpeter approach is a powerful tool to investigate few-body compounds such as the deuteron, unbound neutron-proton (np) system, three-nucleon systems, elastic and nonelastic scattering. We have a great experience in working within such approach.

- Bethe-Salpeter equation and its solution for the separable kernel of interaction
- Yamaguchi-type of kernel functions and Graz-II relativistic kernel
- elastic eD-scattering
- modified Yamaguchi functions and fitting of parameters
- wide dibaryon resonances
- conclusion and summary

Reactions

Solution of the BS (BSF) equations:
BS amplitude, T matrix

- Models of the EW/hadronic current: 1-body (impulse approximation), 2(3)-body;
- off-mass shell effects;
- form factors;
- so on

Why a relativistic approach?

- Elastic electron-deuteron scattering experiments
  "Large Momentum Transfer Measurements of the Deuteron Elastic Structure Function A(Q^2) at Jefferson Laboratory"
  JLab Hall A Collaboration, Phys.Rev.Lett.82:1374-1378,1999
  Q^2=0.7-6.0 (GeV/c)^2

  Lorentz transformation factor: $\eta_{LOR} = -Q^2/4M_d^2 \sim 0.43$,
  $\sqrt{1+\eta_{LOR}} \sim 1.19$, $\sqrt{\eta_{LOR}} \sim 0.65$
- Exclusive disintegration of the deuteron experiments
  JLab Hall C Deuteron Electro-Disintegration at Very High Missing Momenta (E12-10-003) proposal
  https://www.jlab.org/exp_prog/proposals/10/PR12-10-003.pdf:
  "We propose to measure the D(e,e'p)n cross section at $Q^2 = 4.25$ (GeV/c)^2 and xbj = 1.35 for missing momenta ranging from pm = 0.5 GeV/c to pm = 1.0 GeV/c expanding the range of missing momenta explored in the Hall A experiment (E01-020)"

  Lorentz transformation factor: $\eta_{LOR} = -Q^2/4s_{np} \sim 0.30$,
  $\sqrt{1+\eta_{LOR}} \sim 1.14$, $\sqrt{\eta_{LOR}} \sim 0.55$

What is a separable kernel?

The integral equations in the nuclear physics (Lippmann-Schwinger, Bethe-Salpeter) can be reduced to the Fredholm (first or second) type of equations. The separable kernel of the integral equation is the degenerated kernel.
Fredholm integral equation of the second type:

$$\phi(x) = f(x) + \lambda \int dy\ K(x,y)\phi(y)$$

Degenerated kernel of the equation:

$$K(x,y) = \sum_i a_i(x) b_i(y)$$

Solution of the equation:

$$\phi(x) = f(x) + \lambda \sum_i c_i a_i(x)$$

Constants c_i can be found by solving the system of linear equations

$$c_i - \lambda \sum_j k_{ij} c_j = f_i$$

Matrix k_{ij} and f_i are:

$$k_{ij} = \int dy\ b_i(y) a_j(y), \qquad f_i = \int dy\ f(y) b_i(y)$$

Separable kernel for Schrodinger equation with separable potential

Yoshio Yamaguchi "Two-Nucleon Problem When the Potential Is Nonlocal but Separable. I" Phys.Rev.95, 1628 (1954)
Yoshio Yamaguchi, Yoriko Yamaguchi "Two-Nucleon Problem When the Potential Is Nonlocal but Separable. II" Phys.Rev.95, 1635 (1954)

Nonlocal: $\langle \mathbf{r}|V|\mathbf{r}'\rangle \neq \delta^{(3)}(\mathbf{r}-\mathbf{r}')$
in configuration space

$$\langle \mathbf{r}|V|\mathbf{r}'\rangle = -(\lambda/m_N)v^*(\mathbf{r})v^*(\mathbf{r}')$$

in momentum space

$$\langle \mathbf{p}|V|\mathbf{p}'\rangle = (\lambda/m_N)g^*(\mathbf{p})g^*(\mathbf{p}')$$

for S-state: $g(p) = 1/(p^2+\beta^2)$
for D-state: $g(p) = p^2/(p^2+\beta^2)^2$
for the deuteron and scattering problem.

Separable nucleon-nucleon potential was widely uses for the two- and three-nucleon calculations in nonrelativistic nuclear physics

Willibald Plessas et al. Graz, Graz-II potentials, separable representation of the popular Bonn and Paris potentials

K. Schwarz, Willibald Plessas, L. Mathelitsch "Deuteron Form-factors And E D Polarization Observables For The Paris And Graz-II Potentials" Nuovo Cim. A76 (1983) 322-329.
J. Haidenbauer, Willibald Plessas "Separable Representation Of The Paris Nucleon Nucleon Potential" Phys.Rev. C30 (1984) 1822-1839.
Johann Haidenbauer, Y. Koike, Willibald Plessas "Separable representation of the Bonn nucleon-nucleon potential" Phys.Rev. C33 (1986) 439-446.

$$g(p) = \sum_n p^{2m}/(p^2 + \beta_n^2)^n,$$

m corresponds to angular momentum

Lippmann-Schwinger equation → Bethe-Salpeter equation

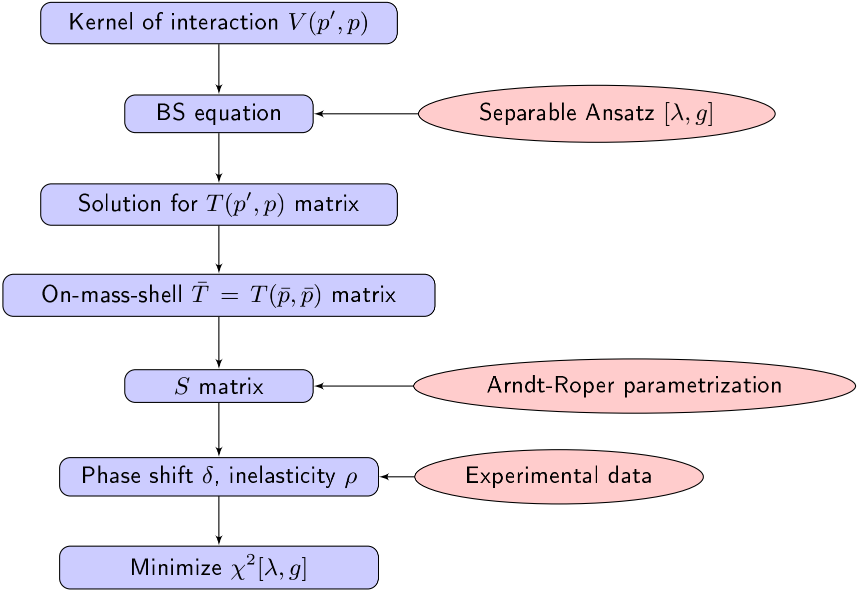
G. Rupp and J. A. Tjon "Relativistic contributions to the deuteron electromagnetic form factors" Phys. Rev. C41. 472 (1990)

$$\mathbf{p}^2 \to -p^2 = -p_0^2 + \mathbf{p}^2$$

$$g_p(p,P) = \frac{1}{-p^2+\beta^2} \xrightarrow{\text{c.m.}} \frac{1}{-p_0^2+\mathbf{p}^2+\beta^2+i\epsilon}$$

singularities: $p^0 = \pm\sqrt{\mathbf{p}^2+\beta^2} \mp i\epsilon$

This procedure works well for reactions with 2-body bound state but failed for unbound np-state


Kernel of interaction $V(p', p)$
BS equation
Separable Ansatz $[\lambda, g]$
Solution for $T(p', p)$ matrix
On-mass-shell $\bar{T} = T(\bar{p}, \bar{p})$ matrix
S matrix
Arndt-Roper parametrization
Phase shift δ, inelasticity ρ
Experimental data
Minimize $\chi^2[\lambda, g]$

S matrix (Arndt-Roper parametrization)

$$S = \frac{1 - K_i + iK_r}{1 + K_i - iK_r} = \eta \exp(2i\delta)$$

$$K = K_r + iK_i$$

$$K_r = \tan\delta, \quad K_i = \tan^2\rho$$

δ - the phase shift, ρ - the inelasticity parameter.

$$\eta^2 = \frac{1 + K^2 - 2K_i}{1 + K^2 + 2K_i} = |S|^2 \sim \sigma_{np}$$

$$K^2 = K_r^2 + K_i^2$$

$$\delta = \frac{1}{2}\{\tan^{-1}[K_r/(1 - K_i)] + \tan^{-1}[K_r/(1 + K_i)]\}$$

If there are no inelastic channels: $(\rho = 0)$, $\delta = \delta_e$, $\eta = 1$ and $S = S_e = \exp(2i\delta_e)$.

Procedure ($J=0-1$)

calculate the kernel parameters – $\lambda(s)$-matrix and parameter of the g-functions – to minimize the function χ^2:

$\chi^2 =$

$\sum_{i=1}^{n}(\delta^{\exp}(s_i)-\delta(s_i))^2/(\Delta\delta^{\exp}(s_i))^2$ – for all partial-wave states

$\sum_{i=1}^{n}(\rho^{\exp}(s_i)-\rho(s_i))^2/(\Delta\rho^{\exp}(s_i))^2$ – for all partial-wave states

$+(a_0^{\exp}-a_0)^2/(\Delta a_0^{\exp})^2$ – for the ${}^1S_0^+$ and ${}^3S_1^+$ partial-wave states

$+(E_d^{\exp}-E_d)^2/(\Delta E_d^{\exp})^2$ – for the ${}^3S_1^+$-${}^3D_1^+$ partial-wave states

$\{+...\}$

δ - the phase shifts, a_0, r_0 - the low-energy parameters (the scattering length, the effective range), E_d - the deuteron binding energy

Covariant generalization of the *Yamaguchi*-functions

functions for $g^{[L]}(p_0, p)$:

$$g^{[S]}(p_0, |\mathbf{p}|) = \frac{1}{p_0^2 - \mathbf{p}^2 - \beta_0^2 + i0}$$

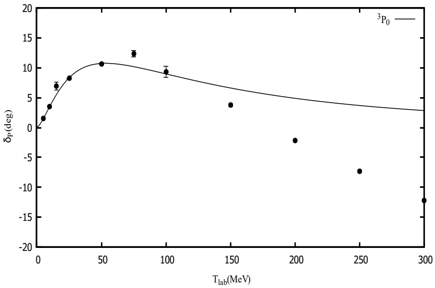
$$g^{[P]}(p_0, |\mathbf{p}|) = \frac{\sqrt{|-p_0^2 + \mathbf{p}^2|}}{(p_0^2 - \mathbf{p}^2 - \beta_1^2 + i0)^2}$$

$$g^{[D]}(p_0, |\mathbf{p}|) = \frac{C(p_0^2 - \mathbf{p}^2)}{(p_0^2 - \mathbf{p}^2 - \beta_2^2 + i0)^2}$$

Results for $^3P_0$, $^1P_1$ and $^3P_1$ channels

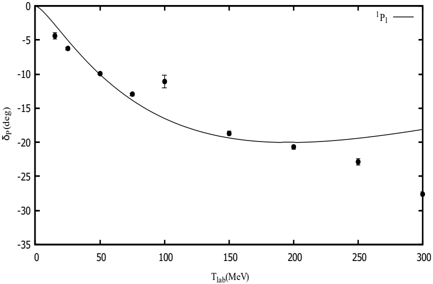
Table: Parameters

	$^3P_0$	$^3P_1$	$^1P_1$
λ (GeV^6)	0.0428572	-5.83051	-3.68029
β_1 (GeV)	0.19904	0.48273	0.44127

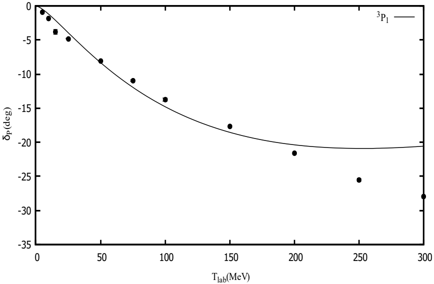


$^3P_0$ phase shifts

Results for ${}^3P_0$, ${}^1P_1$ and ${}^3P_1$ channels



${}^1P_1$ phase shifts

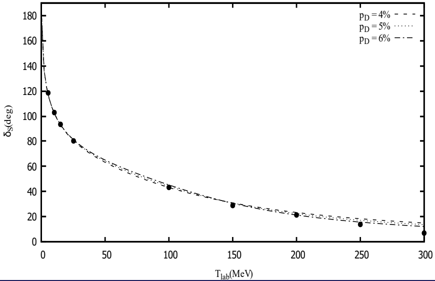


${}^3P_1$ phase shifts

Results for ${}^3S_1^+ - {}^3D_1^+$ channels

Table: Parameters

	Exp.	${}^3S_1 - {}^3D_1$ $(p_d = 4\%)$	${}^3S_1 - {}^3D_1$ $(p_d = 5\%)$	${}^3S_1 - {}^3D_1$ $(p_d = 6\%)$
λ (GeV^4)		-1.83756	-1.57495	-1.34207
β_0 (GeV)		0.251248	0.246713	0.242291
C_2		1.71475	2.52745	3.46353
β_2 (GeV)		0.294096	0.324494	0.350217
a_L (fm)	5.424	5.454	5.454	5.453
r_L (fm)	1.756	1.81	1.81	1.80



Bethe-Salpeter-Fadeev equation

Introducing the equal-mass Jacobi momenta

$$p_i = \frac{1}{2}(k_j - k_n), \quad q_i = \frac{1}{3}K - k_i, \quad K = k_1 + k_2 + k_3.$$

one can separate the conserved total momentum

$$T^{(i)}(k_1, k_2, k_3; k_1', k_2', k_3') = (2\pi)^4 \delta^{(4)}(K - K')T^{(i)}(p_i, q_i; p_i', q_i'; s),$$

with $s = K^2$

Amplitude of three-particle state as a projection of T matrix to the bound state:

$$\Psi^{(i)}(p_i, q_i; s) = \langle p_i, q_i | T^{(i)} | M_B \rangle,$$

with $\sqrt{s} = M_B = 3m_N - E_t$.

Bethe-Salpeter-Fadeev equation

Orbital momentum of triton

$$\boldsymbol{L} = \boldsymbol{l} + \boldsymbol{\lambda}$$

$\boldsymbol{l}$ – orbital momentum of NN-pair
$\boldsymbol{\lambda}$ – orbital momentum of 3d particle
Using separable Ansatz for two-particles T matrix one-rank

$$\Psi_{LM}(p,q;s) = \sum_{a\lambda} \Psi^{(a)}_{\lambda L}(p_0,|\mathbf{p}|,q_0,|\mathbf{q}|;s)\mathcal{Y}^{(a)}_{\lambda LM}(\hat{\mathbf{p}},\hat{\mathbf{q}})$$

$$\mathcal{Y}^{(a)}_{\lambda LM}(\hat{\mathbf{p}},\hat{\mathbf{q}}) = \sum_{m\mu} C^{LM}_{lm\lambda\mu} Y_{lm}(\hat{\mathbf{p}}) Y_{\lambda\mu}(\hat{\mathbf{q}}),$$

where $a \equiv {}^{2s+1}l_j$ is two-nucleon states of the NN-pair

Angular functions in general case

$$K^{(aa')}_{\lambda\lambda' L}(|\mathbf{q}|,|\mathbf{q}'|,\cos\vartheta_{\mathbf{q}\mathbf{q}'}) = (4\pi)^{3/2}\frac{\sqrt{2\lambda+1}}{2L+1}(-1)^{l'}$$

$$\sum_{mm'} C^{Lm}_{lm\lambda 0} C^{Lm}_{l'm'\lambda' m-m'} Y^*_{lm}(\vartheta,0) Y_{l'm'}(\vartheta',0) Y_{\lambda' m-m'}(\vartheta_{\mathbf{q}\mathbf{q}'},0)$$

where

$$\cos\vartheta = (\frac{|\mathbf{q}|}{2} + |\mathbf{q}'|\cos\vartheta_{\mathbf{q}\mathbf{q}'})/|\frac{\mathbf{q}}{2} + \mathbf{q}'|$$

$$\cos\vartheta' = (|\mathbf{q}| + \frac{|\mathbf{q}'|}{2}\cos\vartheta_{\mathbf{q}\mathbf{q}'})/|\mathbf{q} + \frac{\mathbf{q}'}{2}|$$

Consider the ground state of triton: $L = 0 \to \lambda = l$

Angular functions

$$K^{(aa')}_{ll'0} = (4\pi)^{3/2}(-1)^{l+l'} Y^*_{l0}(\vartheta, 0) A_{l'}(\vartheta', \vartheta_{\mathbf{qq}'})$$

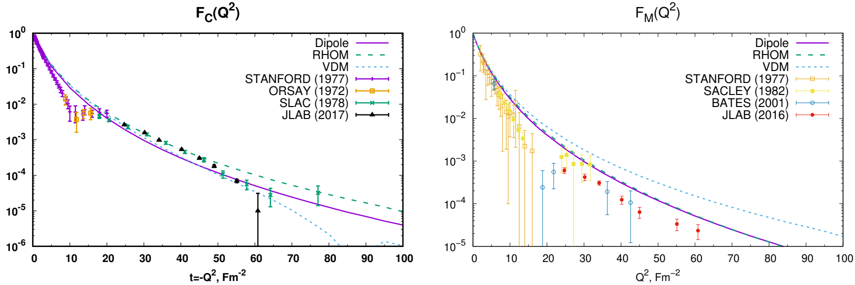
$$A_{l'}(\vartheta', \vartheta_{\mathbf{qq}'}) = \sum_{m'} C^{00}_{l'm'l'-m'} Y_{l'm'}(\vartheta', 0) Y_{l'-m'}(\vartheta_{\mathbf{qq}'}, 0)$$

Spin-isospin dependence

$[(a) =^1 S_0,^3 S_1,^3 D_1,^3 P_0,^1 P_1,^3 P_1]$

$$C_{(aa')} = \frac{1}{4} \begin{pmatrix} 1 & -3 & -3 & \sqrt{3} & -\sqrt{3} & \sqrt{3} \\ -3 & 1 & 1 & \sqrt{3} & -\sqrt{3} & \sqrt{3} \\ -3 & 1 & 1 & \sqrt{3} & -\sqrt{3} & \sqrt{3} \\ \sqrt{3} & \sqrt{3} & \sqrt{3} & -1 & -3 & -1 \\ -\sqrt{3} & -\sqrt{3} & -\sqrt{3} & -3 & -1 & -3 \\ \sqrt{3} & \sqrt{3} & \sqrt{3} & -1 & -3 & -1 \end{pmatrix}$$

One-rank relativistic kernel, static approximation, ^{3}He



Beyond the SA:

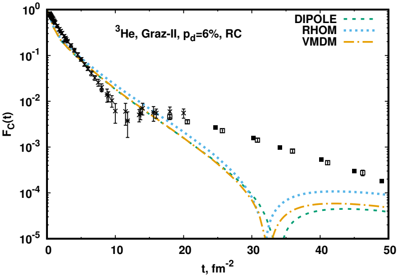
The first integral on the right-hand side of (4) is a six-fold integral. The second one is a five-fold integral with the limits of integration on q and y

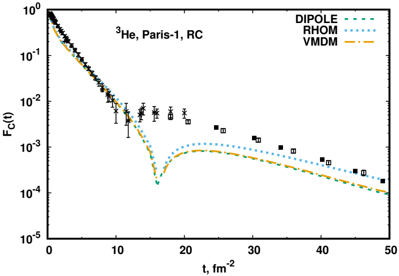
$$q_{min,max} = 2\sqrt{s}\sqrt{\eta}\sqrt{1+\eta} \mp \frac{1}{3}\sqrt{s + 12\eta s + 36\eta^2 s - 9m_N^2}, \qquad (6)$$

$$y_{min} = \frac{1}{36}\frac{24\eta s + 9m_N^2 + 9\mathbf{q}^2 - s}{q\sqrt{s}\sqrt{\eta}\sqrt{1+\eta}}, \qquad y_{max} = 1,$$

and the residue at the point $q_0 = q_0^{(2)}$ is calculated.

Nucleon form factor models





the dashed line - DIPOLE, the dotted line - RHOM, the dashed-dotted line - VMDM.