Lax representation of supersymmetric Calogero–Moser models

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based on the papers with Sergey Krivonos and Olaf Lechtenfeld
Integrability of supersymmetric Calogero-Moser models,
Phys. Lett. B **831** *(2022) 137184, arXiv:2204.02692,*
Supersymmetric Calogero–Moser models for B_n, C_n and D_n root systems and Lax representation (in preparation)

Plan

- Motivation
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- Superintegrability of $A_1 \oplus A_{n-1}$ Calogero–Moser models
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- Hamiltonian and Lax pair of the supersymmetric Calogero–Moser models for B_n, C_n, D_n root sysmtems
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Motivation

The original rational Calogero model of n interacting identical particles on a line is given by the classical Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j} \frac{g^2}{(x_i - x_j)^2}\,. \qquad (1)$$

This model receives much attention in different branches of physics such as high-energy and condensed-matter physics. From a mathematical point of view, the Calogero–Moser model, as well as its variants with special potential terms, belongs to an important class of integrable and even superintegrable systems (see, e.g.)

- M.A. Olshanetsky, A.M. Perelomov,
  Explicit solution of the Calogero model in the classical case and geodesic flows on symmetric spaces of zero curvature,
  Lett. Nuovo Cimento **16** (1976) 333.

Unsurprisingly, the Calogero–Moser model has often been the subject of "supersymmetrization", beginning with the $\mathcal{N}=2$ supersymmetric model of Freedman and Mende. However, all attempts to construct $\mathcal{N}=4$ supersymmetric extensions, despite the announced importance of such models

- G.W. Gibbons, P.K. Townsend,
  Black holes and Calogero models,
  Phys. Lett. B **454** (1999) 187, `arXiv:hep-th/9812034`.

were unsuccessful due to a barrier encountered in

- N. Wyllard,
  (Super)conformal many-body quantum mechanics with extended supersymmetry,
  J. Math. Phys. **41** (2000) 2826, `arXiv:hep-th/9910160`.
- S. Bellucci, A. Galajinsky, E. Latini,
  New insight into the Witten–Dijkgraaf–Verlinde–Verlinde equation,
  Phys. Rev. D **71** (2005) 044023, `arXiv:hep-th/9910160`.

To surmount this barrier new supersymmetric Calogero-like models have been proposed in

- S. Fedoruk, E. Ivanov, O. Lechtenfeld,
  Supersymmetric Calogero models by gauging,
  Phys. Rev. D **79** (2009) 105015, `arXiv:0812.4276[hep-th]`.
- S. Fedoruk, E. Ivanov, O. Lechtenfeld,
  OSp$(4|2)$ *superconformal mechanics,*
  JHEP **08** (2009) 081, `arXiv:0905.4951[hep-th]`.
- S. Fedoruk, E. Ivanov, O. Lechtenfeld,
  New D$(2,1;\alpha)$ *mechanics with spin variables,*
  JHEP **04** (2010) 129, `arXiv:0912.3508[hep-th]`.

Finally, a supersymmetric Calogero–Moser system with arbitrary $\mathcal{N}$-extended supersymmetry was constructed

- S. Krivonos, O. Lechtenfeld, A. Sutulin,
  $\mathcal{N}$*-extended supersymmetric Calogero model*,
  Phys. Lett. B **784** (2018) 137, `arXiv:1804.10825[hep-th]`
- S. Krivonos, O. Lechtenfeld, A. Provorov, A. Sutulin,
  Extended supersymmetric Calogero model,
  Phys. Lett. B **791** (2019) 385, `arXiv:1812.10168[hep-th]`

The main feature of the latter models is an increased number of fermionic coordinates, namely $\mathcal{N}n^2$ rather than the $\mathcal{N}n$ to be expected. It is therefore questionable whether they inherit the (super)integrability of the bosonic Calogero–Moser model. To settle this issue we must first determine how many conserved currents are required for Liouville or super-integrability in a supersymmetric model with $n_{\text{bos}}+n_{\text{fer}}$ degrees of freedom. Recall that, in the standard Lax description with a pair (L, A) of matrices subject to $\dot{L} = [A, L]$, the Liouville charges appear as the trace of powers of the L operator. In the supersymmetric extension, the Lax pair still produces all (bosonic and fermionic) equations of motion and n_{bos} Liouville currents as before, but it is unclear how additional n_{fer} conserved charges may arise and whether they should be commuting or anticommuting in nature. Of course, the problem extends to any additional (non-involutive) conserved charges, as required for superintegrability.

In the present talk we analyze the integrability of the $\mathcal{N}$-extended supersymmetric Calogero–Moser model. For convenience we add a confining supersymmetric oscillator potential. We explicitly construct the Lax pair for this system and demonstrate that it yields all (bosonic as well as fermionic) equations of motion. Employing the Olshanetsky–Perelomov approach, we solve the bosonic equations of motion. Their periodic trajectories prove the maximal superintegrability of this sector.

As a definition of (super)integrability for a supersymmetric system with $n_{\text{bos}}+n_{\text{fer}}$ degrees of freedom we adopt the formulation of Desrosiers, Lapointe and Mathieu

- P. Desrosiers, L. Lapointe, P. Mathieu,
Supersymmetric Calogero–Moser–Sutherland models and Jack superpolynomials,
Nucl. Phys. B **606** (2001) 547, `arXive:hep-th/0103178`.
- P. Desrosiers, L. Lapointe, P. Mathieu,
Supersymmetric Calogero–Moser–Sutherland models: superintegrability structure and eigenfunctions,
Proc. of the Workshop on Superintegrability in Classical and Quantum Systems, 16-22 Sept 2002, Montreal, Quebec, Canada, `arXive:hep-th/0210190`.

In these papers, the authors proposed the following formulation of the concept of integrability for supersymmetric models:

- integrability means the existence of $n_{\text{bos}}+n_{\text{fer}}$ Grassmannian-even conserved currents in involution,
- maximal superintegrability means the existence of $2(n_{\text{bos}}+n_{\text{fer}})-1$ Grassmannian-even conserved currents.

To visualize the structure of the conserved currents we construct all Liouville charges up to level 5 for the $\mathcal{N}=2$ Calogero–Moser model associated with A_n root system. This provides explicit expressions for a complete and functionally independent set in systems with $n_{\text{bos}} \leq 5$. We advocate a general procedure and hypothesize that it generates *all* Liouville currents for an *arbitrary* number of particles in that model. We can also construct the additional set of conserved currents required for maximal superintegrability of the considered $\mathcal{N}=2$ Calogero–Moser models. However, we skip the details of that consideration in the present talk.

Hamiltonian description of $\mathcal{N}$–supersymmetric Calogero–Moser models

In the Hamiltonian approach the construction of the n-particle rational Calogero–Moser model with $\mathcal{N}$-extended supersymmetry is based on the following set of components:

- n bosonic coordinates x_i and corresponding momenta p_i, $i = 1, \ldots, n$,
- $\mathcal{N}\, n^2$ fermions $\xi^a_{ij}, \bar{\xi}_{ij\,b}$, $\quad a, b = 1, 2, \ldots \mathcal{N}/2$.

The non-vanishing Poisson brackets have the standard form

$$\left\{x_i, p_j\right\} = \delta_{ij}\ , \qquad \left\{\xi^a_{ij}, \bar{\xi}_{km\,b}\right\} = -\mathrm{i}\delta^a_b \delta_{im}\delta_{jk}\ . \tag{2}$$

It is convenient to collect the bosonic coordinates in a diagonal matrix X with components $X_{ij} = \delta_{ij} x_j$. Basic to our construction are the fermionic bilinear objects

$$\Pi_{ij} = \sum_{a=1}^{\mathcal{N}/2}\sum_{k=1}^{n}\left(\xi^a_{ik}\bar{\xi}_{kj\,a} + \bar{\xi}_{ik\,a}\xi^a_{kj}\right) \qquad \text{and} \qquad \widetilde{\Pi}_{ij} = \sum_{a=1}^{\mathcal{N}/2}\sum_{k=1}^{n}\left(\xi^a_{ik}\bar{\xi}_{kj\,a} - \bar{\xi}_{ik\,a}\xi^a_{kj}\right)\ . \tag{3}$$

It is easily to check that they form an $s(u(n)\oplus u(n))$ algebra

$$\begin{aligned} &\left\{\Pi_{ij}, \Pi_{km}\right\} = \mathrm{i}\left(\delta_{im}\Pi_{kj} - \delta_{kj}\Pi_{im}\right), \quad \left\{\widetilde{\Pi}_{ij}, \widetilde{\Pi}_{km}\right\} = \mathrm{i}\left(\delta_{im}\Pi_{kj} - \delta_{kj}\Pi_{im}\right), \\ &\left\{\Pi_{ij}, \widetilde{\Pi}_{km}\right\} = \mathrm{i}\left(\delta_{im}\widetilde{\Pi}_{kj} - \delta_{kj}\widetilde{\Pi}_{im}\right), \quad \sum_i \Pi_{ii} = 0. \end{aligned} \tag{4}$$

The $\mathcal{N}$-extended supersymmetric $A_1 \oplus A_{n-1}$ rational Calogero model (with a harmonic confining potential) is described by supercharges

$$\mathbb{Q}^a = \sum_{i=1}^{n} \big(p_i + \mathrm{i}\omega x_i\big)\xi^a_{ii} \; - \; \mathrm{i}\sum_{i\neq j}^{n} \frac{\big(g + \Pi_{jj} - \Pi_{ij}\big)\xi^a_{ji}}{x_i - x_j} \;,$$

$$\overline{\mathbb{Q}}_a = \sum_{i=1}^{n} \big(p_i - \mathrm{i}\omega x_i\big)\bar{\xi}_{ii\,a} \; + \; \mathrm{i}\sum_{i\neq j}^{n} \frac{\big(g + \Pi_{ii} - \Pi_{ji}\big)\bar{\xi}_{ij\,a}}{x_i - x_j} \;, \tag{5}$$

and a Hamiltonian

$$\mathbb{H} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 \; + \; \frac{1}{2}\sum_{i\neq j}^{n} \frac{\big(g + \Pi_{jj} - \Pi_{ij}\big)\big(g + \Pi_{ii} - \Pi_{ji}\big)}{\big(x_i - x_j\big)^2} - \frac{2}{\mathcal{N}}\,\omega \sum_{a=1}^{\mathcal{N}/2}\sum_{i,j=1}^{n} \xi^a_{ij}\bar{\xi}_{ji\,a} + \frac{\omega^2}{2}\sum_{i=1}^{n} x_i^2 \;, \tag{6}$$

which form an $su(\frac{\mathcal{N}}{2}|1)$ super-algebra together with the R-symmetry generators

$$\mathbb{W}^a_b = \sum_{i,j=1}^{n} \xi^a_{ij}\bar{\xi}_{ji\,b} - \frac{2}{\mathcal{N}}\,\delta^a_b \sum_{c=1}^{\mathcal{N}/2}\sum_{i,j=1}^{n} \xi^c_{ij}\bar{\xi}_{ji\,c} \;. \tag{7}$$

The R-symmetry $su\left(\frac{\mathcal{N}}{2}\right)$ algebra reads

$$\left\{\mathbb{W}^a_b, \mathbb{W}^c_d\right\} = \mathrm{i}\delta^a_d \mathbb{W}^c_b - \mathrm{i}\delta^c_b \mathbb{W}^a_d \ , \tag{8}$$

and the remaining commutation relations of the $su\left(\frac{\mathcal{N}}{2}|1\right)$ superalgebra are given by

$$\begin{aligned}
&\left\{\mathbb{Q}^a, \overline{\mathbb{Q}}_b\right\} = -2\mathrm{i}\,\delta^a_b\,\mathbb{H} + 2\mathrm{i}\,\omega \mathbb{W}^a_b \ , && \left\{\mathbb{Q}^a, \mathbb{Q}^b\right\} = \left\{\overline{\mathbb{Q}}_a, \overline{\mathbb{Q}}_b\right\} = 0 \ , \\
&\left\{\mathbb{H}, \mathbb{Q}^a\right\} = -\mathrm{i}\,\omega\,\frac{\mathcal{N}-2}{\mathcal{N}}\,\mathbb{Q}^a \ , && \left\{\mathbb{H}, \overline{\mathbb{Q}}_a\right\} = \mathrm{i}\,\omega\,\frac{\mathcal{N}-2}{\mathcal{N}}\,\overline{\mathbb{Q}}_a \ , \\
&\left\{\mathbb{W}^a_b, \mathbb{Q}^c\right\} = -\mathrm{i}\delta^c_b\,\mathbb{Q}^a + \mathrm{i}\,\frac{2}{\mathcal{N}}\,\delta^a_b\,\mathbb{Q}^c \ , && \left\{\mathbb{W}^a_b, \overline{\mathbb{Q}}_c\right\} = \mathrm{i}\delta^a_c\,\overline{\mathbb{Q}}_b - \mathrm{i}\,\frac{2}{\mathcal{N}}\,\delta^a_b\,\overline{\mathbb{Q}}_c \ .
\end{aligned} \tag{9}$$

In the limit $\omega \to 0$ this turns into the $\mathcal{N}$-extended super-Poincaré algebra

$$\left\{Q^a, \overline{Q}_b\right\} = -2\mathrm{i}\,\delta^a_b\,H \qquad \text{and} \qquad \left\{Q^a, Q^b\right\} = \left\{\overline{Q}_a, \overline{Q}_b\right\} = 0 \tag{10}$$

for the unconfined charges

$$Q^a = \mathbb{Q}^a\Big|_{\omega=0} \ , \quad \overline{Q}^a = \overline{\mathbb{Q}}^a\Big|_{\omega=0} \qquad \text{and} \qquad H = \mathbb{H}\Big|_{\omega=0} \ . \tag{11}$$

Superintegrability of $A_1 \oplus A_{n-1}$ Calogero–Moser models

Based on the similarity of the Hamiltonian (11) and the Hamiltonian of the Euler–Calogero–Moser model in Wojciechowski paper

- G.W. Wojciechowski,
  An integrable marriage of the Euler equations with the Calogero–Moser system,
  Phys. Lett. A **111** (1985) 101.

and trying to represent the supercharges as

$$Q^a = \sum_{i,j=1}^{n} L_{ij}\xi_{ji}^a \qquad \text{and} \qquad \overline{Q}_a = \sum_{i,j=1}^{n} L_{ij}\bar{\xi}_{ji\,a}\,, \tag{12}$$

one may guess the Lax operator L with components

- S. Krivonos, O. Lechtenfeld, A. Sutulin,
  New $\mathcal{N}=2$ superspace Calogero model,
  JHEP **05** (2020) 132, `arXiv:1912.05989[hep-th]`.

$$L_{ij} = \delta_{ij}\,p_j - \mathrm{i}\big(1-\delta_{ij}\big)\frac{g+\Pi_{jj}-\Pi_{ij}}{x_i-x_j}. \tag{13}$$

It is indeed easily checked that

$$\tfrac{1}{2}\,\mathrm{Tr}\,L^2 = H \tag{14}$$

as it should be.

For a Lax-type equation we need an associated matrix A. By a simple computation its components are found as

$$A_{ij} = \mathrm{i}\,\delta_{ij} \sum_{k\neq i}^{n} \frac{g + \Pi_{kk} - \Pi_{ik}}{(x_i - x_k)^2} - \mathrm{i}\left(1 - \delta_{ij}\right) \frac{g + \Pi_{jj} - \Pi_{ij}}{(x_i - x_j)^2}\,. \tag{15}$$

With this, the Lax–type equations of motion related to the Hamiltonian (6) reads

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{ij} = \left\{L_{ij}, \mathbb{H}\right\} = \left[A, L\right]_{ij} - \omega^2 X_{ij}\,. \tag{16}$$

The equations of motion for the coordinate matrices $X_{ij} = \delta_{ij} x_j$ acquire the form

$$\frac{\mathrm{d}}{\mathrm{d}t} X_{ij} = \{X_{ij}, \mathbb{H}\} = [A, X]_{ij} + L_{ij} \qquad \text{and} \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\xi^a_{ij} = \left\{\xi^a_{ij}, \mathbb{H}\right\} = \left[A, \xi^a\right]_{ij} - 2\frac{\mathrm{i}\,\omega}{\mathcal{N}}\,\xi^a_{ij}\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\,\bar{\xi}_{ij\,a} = \left\{\bar{\xi}_{ij\,a}, \mathbb{H}\right\} = \left[A, \bar{\xi}_a\right]_{ij} + 2\frac{\mathrm{i}\,\omega}{\mathcal{N}}\,\bar{\xi}^a_{ij}\,. \tag{18}$$

As a corollary, the composite objects Π_{ij} and $\widetilde{\Pi}_{ij}$ (3) satisfy the following equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\Pi_{ij} \equiv \left\{\Pi_{ij}, \mathbb{H}\right\} = \left[A,\, \Pi\right]_{ij} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\,\widetilde{\Pi}_{ij} \equiv \left\{\widetilde{\Pi}_{ij}, \mathbb{H}\right\} = \left[A,\, \widetilde{\Pi}\right]_{ij}\,. \tag{19}$$

The equations of motion (16)–(18) are similar to those in

- G.W. Wojciechowski, *Superintegrability of the Calogero–Moser system,* Phys. Lett. A **95** (1983) 279.
- J. Gibbons, T. Hermsen, *A generalisation of the Calogero–Moser system,* Physica D **11** (1984) 337.

and can be solved by the Olshanetsky–Perelomov method. For this propose we need an invertible time-dependent matrix $U = (U_{ij})$ as the solution of the linear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} U_{ij} = \big(A\, U\big)_{ij} \qquad \text{with} \quad U_{ij}\big|_{t=0} = \delta_{ij}\ . \tag{20}$$

Using this matrix, one can pass to new tilded variables

$$\tilde{L}_{ij} = \big(U^{-1} L\, U\big)_{ij}\,, \quad \tilde{X}_{ij} = \big(U^{-1} X\, U\big)_{ij}\,, \quad \tilde{\xi}^a_{ij} = \big(U^{-1} \xi^a\, U\big)_{ij}\,, \quad \tilde{\bar{\xi}}_{ij\,a} = \big(U^{-1} \bar{\xi}_a\, U\big)_{ij}\,. \tag{21}$$

In terms of these, the A contribution of the equations (16)–(18) is removed, hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\tilde{L} = -\omega^2\tilde{X}\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}\,\tilde{X} = \tilde{L} \quad \Rightarrow \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\tilde{X} = -\omega^2\tilde{X}\,, \tag{22}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\,\tilde{\xi}^a_{ij} = -2\mathrm{i}\,\frac{\omega}{\mathcal{N}}\,\tilde{\xi}^a_{ij}\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}\,\tilde{\bar{\xi}}_{ij\,a} = 2\mathrm{i}\,\frac{\omega}{\mathcal{N}}\,\tilde{\bar{\xi}}_{ij\,a}\,.$$

The solutions to these equations are easily found as

$$\tilde{L}_{ij}(t) = \cos(\omega t)L_{ij}(0) - \omega\sin(\omega t)X_{ij}(0)\,, \quad \tilde{X}_{ij}(t) = \cos(\omega t)X_{ij}(0) + \omega^{-1}\sin(\omega t)L_{ij}(0)\,,$$
$$\tilde{\xi}^a_{ij}(t) = \exp(-2\tfrac{\mathrm{i}\omega t}{\mathcal{N}})\,\xi^a_{ij}(0) \qquad \text{and} \qquad \tilde{\bar{\xi}}_{ij\,a}(t) = \exp(2\tfrac{\mathrm{i}\omega t}{\mathcal{N}})\,\bar{\xi}_{ij\,a}(0)\,. \tag{23}$$

The matrix $U(t)$ then determines the time dependence of the original variables via

$$\begin{aligned}
L_{ij}(t) &= \cos(\omega t)\big(U(t)L(0)U^{-1}(t)\big)_{ij} - \omega\sin(\omega t)\big(U(t)X(0)U^{-1}(t)\big)_{ij}\,,\\
X_{ij}(t) &= \cos(\omega t)\big(U(t)X(0)U^{-1}(t)\big)_{ij} + \omega^{-1}\sin(\omega t)\big(U(t)L(0)U^{-1}(t)\big)_{ij}\,,\\
\xi^a_{ij}(t) &= \exp(-2\tfrac{\mathrm{i}\omega t}{\mathcal{N}})\,\big(U(t)\xi^a(0)U^{-1}(t)\big)_{ij}\,,\\
\bar{\xi}_{ij\,a}(t) &= \exp(2\tfrac{\mathrm{i}\omega t}{\mathcal{N}})\,\big(U(t)\bar{\xi}^a(0)U^{-1}(t)\big)_{ij}\,.
\end{aligned} \tag{24}$$

Of course, one still has to solve (20) to obtain the entire dynamics. However, for the time evolution of the bosonic coordinates x_i the U matrix is irrelevant because the eigenvalues of the matrix X are unchanged by the unitary transformation with U. Therefore, the motion $x_i(t)$ is periodic, and the bosonic trajectories are closed curves in phase space. Hence, the Hamiltonian (6) is completely degenerate with respect to its bosonic degrees of freedom, and so these provide $2n-1$ functionally independent constants of motion. This property is preserved in the unconfining limit $\omega \to 0$, implying that the $\mathcal{N}$–supersymmetric Calogero–Moser model is maximally superintegrable in its bosonic sector, just like the purely bosonic model is.

Conserved currents for $\mathcal{N}=2$ Calogero–Moser models

It is interesting to know the explicit form of the integrals of motion, especially in the present case where the number of the fermionic degrees of freedom is much larger that number of bosonic ones. Here, we will analyze the integrability of the simplest supersymmetric Calogero–Moser model with $\mathcal{N}=2$ supersymmetry.
From the equations of motion (16), (18), (19) it follows that any function $\hat{F}$ with a polynomial dependence on the matrices L, ξ^a or $\bar{\xi}_b$ obeys the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{F} = \left[A, \hat{F}\right] \tag{25}$$

and, therefore, the trace of such a function is conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Tr}\,\hat{F}(L, \xi^a, \bar{\xi}_b) = 0\,. \tag{26}$$

Let us consider the case with $\mathcal{N}=2$ supersymmetry and therefore omit the index '1′' in the fermions ξ^1_{ij} and $\bar{\xi}_{ij\,1}$. The whole system (16), (17), (18) has $2(n^2+n)$ dynamical variables, i.e. (x_i, p_i) and $(\xi_{ij}, \bar{\xi}_{ij})$. Thus the system requires $n^2 + n$ functionally independent integrals of motion in the involution to be integrable. Some of these integrals may be recovered from a spectral-parameter Lax representation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(L + \mu\Pi\right) = \left[A, L + \mu\Pi\right] . \tag{27}$$

The trace-powers

$$\mathrm{Tr}\left(L + \mu\Pi\right)^k \qquad \text{for} \quad k = 1, \ldots, n \tag{28}$$

are spectral-parameter dependent integrals of (27). We expand them in powers of μ and obtain a set C_n of conserved charges for n particles.
The number c_n of integrals (28) (the cardinality of C_n) is given recursively by

$$c_n = c_{n-1} + (n+1) . \tag{29}$$

Keeping in the mind that $\mathrm{Tr}\,(\Pi) = 0$ and, therefore, $c_1 = 1$, we conclude that

$$c_n = \tfrac{1}{2}n(n+1) + n - 1 . \tag{30}$$

It is instructive to visualize the structure of these integrals for a small number of particles:

$$
\begin{aligned}
c_1 = 1: &\quad C_1 = \{ \mathrm{Tr}\,(L) \} \ , \\
c_2 = 4: &\quad C_2 = C_1 \cup \Big\{ \mathrm{Tr}\,(L^2),\ \mathrm{Tr}\,(L\Pi),\ \mathrm{Tr}\,(\Pi^2) \Big\} \ , \\
c_3 = 8: &\quad C_3 = C_2 \cup \Big\{ \mathrm{Tr}\,(L^3),\ \mathrm{Tr}\,(L^2\Pi),\ \mathrm{Tr}\,(L\Pi^2),\ \mathrm{Tr}\,(\Pi^3) \Big\} \ , \\
c_4 = 13: &\quad C_4 = C_3 \cup \Big\{ \mathrm{Tr}\,(L^4),\ \mathrm{Tr}\,(L^3\Pi),\ \mathrm{Tr}\,(2L^2\Pi^2 + L\Pi L\Pi),\ \mathrm{Tr}\,(L\Pi^3),\ \mathrm{Tr}\,(\Pi^4) \Big\} \ , \\
c_5 = 19: &\quad C_5 = C_4 \cup \Big\{ \mathrm{Tr}\,(L^5),\ \mathrm{Tr}\,(L^4\Pi),\ \mathrm{Tr}\,(L^3\Pi^2 + L^2\Pi L\Pi), \\
&\qquad\qquad \mathrm{Tr}\,(L^2\Pi^3 + L\Pi^2 L\Pi),\ \mathrm{Tr}\,(L\Pi^4),\ \mathrm{Tr}\,(\Pi^5) \Big\}
\end{aligned}
\tag{31}
$$

and so on. However, the number (30) of conserved currents is less that we need for the integrability, i.e. n^2+n.

To construct more currents one may try to use a different spectral Lax representation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(L+\mu\widetilde{\Pi}\big) = \big[A, L+\mu\widetilde{\Pi}\big] \tag{32}$$

with $\widetilde{\Pi}$ defined in (3). However, the integrals in the expressions

$$\mathrm{Tr}\,\big(L+\mu\widetilde{\Pi}\big)^k \qquad \text{for} \quad k=1,\ldots,n \tag{33}$$

obtained from expanding in powers of μ do not commute with the currents (28). Thus, the remaining possibility for additional conserved currents of Liouville type resides in the expression

$$\mathrm{Tr}\,\big(\Pi+\mu\widetilde{\Pi}\big)^k \qquad \text{for} \quad k=1,\ldots,n\ . \tag{34}$$

We have no rigorous proof, but we checked for a small number of particles that the currents in (34) perfectly commute with the currents in (28). Observing that the $\mu{=}0$ currents in (34) already are contained in the currents (28), one evaluates the number of new currents in (34) to be

$$\widetilde{c}_n = \tfrac{1}{2}n(n+1)\ . \tag{35}$$

Thus, the total number of Liouville currents from (28) and (34) is $n^2{+}2n{-}1$, while the system has $n^2{+}n$ degrees of freedom. Hence, there must exist $n{-}1$ constraints among the currents (34).

One may check that these constraints read: $2\,\chi_k \;\equiv\; \mathrm{Tr}\,(\Pi+\widetilde{\Pi})^k + \mathrm{Tr}\,(\Pi-\widetilde{\Pi})^k \;=\; 0$
Until level 5 the constraints look as follows,

$$\begin{aligned}
\chi_1 &= \mathrm{Tr}\,(\Pi)\,, \\
\chi_2 &= \mathrm{Tr}\,(\Pi^2) + \mathrm{Tr}\,(\widetilde{\Pi}^2)\,, \\
\chi_3 &= \mathrm{Tr}\,(\Pi^3) + 3\,\mathrm{Tr}\,(\Pi\widetilde{\Pi}^2)\,, \\
\chi_4 &= \mathrm{Tr}\,(\Pi^4) + 4\,\mathrm{Tr}\,(\Pi^2\widetilde{\Pi}^2) + 2\,\mathrm{Tr}\,(\Pi\widetilde{\Pi}\Pi\widetilde{\Pi}) + \mathrm{Tr}\,(\widetilde{\Pi}^4)\,, \\
\chi_5 &= \mathrm{Tr}\,(\Pi^5) + 5\,\mathrm{Tr}\,(\Pi^3\widetilde{\Pi}^2) + 5\,\mathrm{Tr}\,(\Pi^2\widetilde{\Pi}\Pi\widetilde{\Pi}) + 5\,\mathrm{Tr}\,(\Pi\widetilde{\Pi}^4)\,.
\end{aligned} \tag{36}$$

A set of $\widetilde{c}_k$ independent additional Liouville integrals up to this level is

$$\begin{aligned}
\widetilde{c}_1 = 1:&\quad \widetilde{C}_1 = \{\,\mathrm{Tr}\,(\widetilde{\Pi})\}\,, \\
\widetilde{c}_2 = 2:&\quad \widetilde{C}_2 = \widetilde{C}_1 \cup \{\,\mathrm{Tr}\,(\Pi\widetilde{\Pi})\}\,, \\
\widetilde{c}_3 = 4:&\quad \widetilde{C}_3 = \widetilde{C}_2 \cup \{\,\mathrm{Tr}\,(\widetilde{\Pi}^3),\ \mathrm{Tr}\,(\widetilde{\Pi}\Pi^2)\}\,, \\
\widetilde{c}_4 = 7:&\quad \widetilde{C}_4 = \widetilde{C}_3 \cup \{\,\mathrm{Tr}\,(\widetilde{\Pi}^3\Pi),\ \mathrm{Tr}\,(2\widetilde{\Pi}^2\Pi^2+\widetilde{\Pi}\Pi\widetilde{\Pi}\Pi),\ \mathrm{Tr}\,(\widetilde{\Pi}\Pi^3)\}\,, \\
\widetilde{c}_5 = 11:&\quad \widetilde{C}_5 = \widetilde{C}_4 \cup \{\,\mathrm{Tr}\,(\widetilde{\Pi}^5),\ \mathrm{Tr}\,(\widetilde{\Pi}^3\Pi^2+\widetilde{\Pi}^2\Pi\widetilde{\Pi}\Pi),\ \mathrm{Tr}\,(\widetilde{\Pi}^2\Pi^3+\widetilde{\Pi}\Pi^2\widetilde{\Pi}\Pi),\ \mathrm{Tr}\,(\widetilde{\Pi}\Pi^4)\}\,.
\end{aligned} \tag{37}$$

We do not discuss here the construction of the non-involutive conserved currents needed for superintegrability (see, for datails, our paper in PLB).

Hamiltonian and Lax pair of the supersymmetric Calogero–Moser models for B_n, C_n, D_n root sysmtems

For the B_n, C_n and D_n-type models, the supercharges take a more complicated generic form (including $\widetilde{\Pi}_{ij}$)

$$
\begin{aligned}
\mathcal{Q}^a &= \sum_i^n p_i \xi_{ii}^a - \mathrm{i} \sum_{i\neq j}^n \frac{(\Pi_{jj} - \Pi_{ij})\,\xi_{ji}^a}{x_i - x_j} + \mathrm{i} \sum_{i,j}^n \frac{\left(\Pi_{jj} + \widetilde{\Pi}_{ij}\right)\xi_{ji}^a}{x_i + x_j} \\
&+ \mathrm{i}\,\frac{g}{2} \sum_{i\neq j}^n \left(\frac{\xi_{ij}^a - \xi_{ji}^a}{x_i - x_j} + \frac{\xi_{ij}^a + \xi_{ji}^a}{x_i + x_j} \right) + \mathrm{i}\,g' \sum_i^n \frac{\xi_{ii}^a}{x_i} \qquad \text{and} \\
\overline{\mathcal{Q}}_a &= \sum_i^n p_i \bar{\xi}_{ii\,a} - \mathrm{i} \sum_{i\neq j}^n \frac{(\Pi_{jj} - \Pi_{ij})\,\bar{\xi}_{ji\,a}}{x_i - x_j} - \mathrm{i} \sum_{i,j}^n \frac{\left(\Pi_{jj} + \widetilde{\Pi}_{ij}\right)\bar{\xi}_{ji\,a}}{x_i + x_j} \\
&+ \mathrm{i}\,\frac{g}{2} \sum_{i\neq j}^n \left(\frac{\bar{\xi}_{ij\,a} - \bar{\xi}_{ji\,a}}{x_i - x_j} - \frac{\bar{\xi}_{ij\,a} + \bar{\xi}_{ji\,a}}{x_i + x_j} \right) - \mathrm{i}\,g' \sum_i^n \frac{\bar{\xi}_{ii\,a}}{x_i}
\end{aligned}
\tag{38}
$$

which, together with the Hamiltonian

$$
\begin{aligned}
\mathcal{H} \quad = \quad & \frac{1}{2}\sum_i^n p_i^2 + \frac{1}{2}\sum_{i\neq j}^n \left[\frac{\left(g+\Pi_{jj}-\Pi_{ij}\right)\left(g+\Pi_{ii}-\Pi_{ji}\right)}{(x_i-x_j)^2} \right. \\
+ \quad & \left. \frac{\left(g+\Pi_{jj}+\widetilde{\Pi}_{ij}\right)\left(g+\Pi_{ii}+\widetilde{\Pi}_{ji}\right)}{(x_i+x_j)^2} \right] \\
+ \quad & \frac{1}{8}\sum_i^n \frac{\left(2g'+\Pi_{ii}+\widetilde{\Pi}_{ii}\right)\left(2g'+\Pi_{ii}+\widetilde{\Pi}_{ii}\right)}{x_i^2}\,,
\end{aligned}
\tag{39}
$$

generate an $\mathcal{N}$-extended super-Poincaré algebra. The bosonic part of this Hamiltonian has the standard form for the B_n, C_n and D_n (with $g'=0$ in this case) Calogero–Moser models,

$$
\mathcal{H}_{\text{bos}} = \frac{1}{2}\sum_i^n p_i^2 + \frac{g^2}{2}\sum_{i\neq j}^n \left[\frac{1}{(x_i-x_j)^2} + \frac{1}{(x_i+x_j)^2}\right] + \frac{g'^2}{2}\sum_i^n \frac{1}{x_i^2}. \tag{40}
$$

The equations of motion for all variables can be written in the form of the Lax equations. To represent this, we introduce the following set of matrices

$$\mathcal{L} = \begin{pmatrix} L_1 & L_2 \\ -L_2 & -L_1 \end{pmatrix} \qquad \mathcal{A} = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix} \qquad \mathcal{X} = \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \tag{41}$$

$$\mathcal{K} = \begin{pmatrix} \Pi & \widetilde{\Pi} \\ \widetilde{\Pi} & \Pi \end{pmatrix} \quad \widetilde{\mathcal{K}} = \begin{pmatrix} \widetilde{\Pi} & \Pi \\ \Pi & \widetilde{\Pi} \end{pmatrix} \quad \Xi^a = \begin{pmatrix} \xi^a_{ij} & -\xi^a_{ij} \\ \xi^a_{ij} & -\xi^a_{ij} \end{pmatrix} \quad \Xi_a = \begin{pmatrix} \bar{\xi}_{ij\,a} & \bar{\xi}_{ij\,a} \\ -\bar{\xi}_{ij\,a} & -\bar{\xi}_{ij\,a} \end{pmatrix}$$

The bosonic coordinates is collected in a diagonal matrix X with components

$$X_{ij} = \delta_{ij}\, x_j\,. \tag{42}$$

Note that each of the components of all introduced matrices is a square n-dimensional matrix.

An explicit realization of matrices that form the Lax pair reads

$$
\begin{aligned}
L_{1\,ij} &= \delta_{ij}\, p_j - \mathrm{i}\left(1-\delta_{ij}\right)\frac{g+\Pi_{jj}-\Pi_{ij}}{x_i-x_j}\,,\\
L_{2\,ij} &= \mathrm{i}\left(1-\delta_{ij}\right)\frac{g+\Pi_{jj}+\widetilde{\Pi}_{ij}}{x_i+x_j} + \mathrm{i}\,\delta_{ij}\,\frac{g'+\Pi_{jj}+\widetilde{\Pi}_{jj}}{2x_j}\,,
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
A_{1\,ij} &= \mathrm{i}\,\delta_{ij}\sum_{k\neq j}^{n}\left(1-\delta_{jk}\right)\left(\frac{g+\Pi_{kk}-\Pi_{jk}}{(x_j-x_k)^2}+\frac{g+\Pi_{kk}+\widetilde{\Pi}_{jk}}{(x_j+x_k)^2}\right)\\
&-\mathrm{i}\left(1-\delta_{ij}\right)\frac{g+\Pi_{jj}-\Pi_{ij}}{(x_i-x_j)^2} + \mathrm{i}\,\delta_{ij}\,\frac{g'+\Pi_{jj}+\widetilde{\Pi}_{ij}}{(2x_j)^2}\,,\\
A_{2\,ij} &= \mathrm{i}\left(1-\delta_{ij}\right)\frac{g+\Pi_{jj}+\widetilde{\Pi}_{ij}}{(x_i+x_j)^2} + \mathrm{i}\,\delta_{ij}\,\frac{g'+\Pi_{jj}+\widetilde{\Pi}_{jj}}{(2x_j)^2}\,.
\end{aligned}
\tag{44}
$$

The Lax–type equations of motion related to the Hamiltonian of supersymmetric B_n, C_n and D_n Calogero–Moser models read

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{L} = \left\{\mathcal{L}, H\right\} = \left[\mathcal{A}, \mathcal{L}\right] . \tag{45}$$

The equations of motion for the coordinate matrices acquire the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{X} = \{\mathcal{X}, H\} = [\mathcal{A}, \mathcal{X}] \;, \tag{46}$$

while the equations for fermions become

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\Xi^a = \left\{\Xi^a, H\right\} = \left[\mathcal{A}, \Xi^a\right] , \qquad \frac{\mathrm{d}}{\mathrm{d}t}\,\bar{\Xi}_a = \left\{\bar{\Xi}_a, H\right\} = \left[\mathcal{A}, \bar{\Xi}_a\right] . \tag{47}$$

As a corollary, the composite objects $\mathcal{K}$ satisfy the following equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{K} \equiv \left\{\mathcal{K}, H\right\} = \left[\mathcal{A},\, \mathcal{K}\right] . \tag{48}$$

Now, repeating the arguments for the previous case, one can prove the integrability of the system under consideration. Moreover, for the supersymmetric $\mathcal{N} = 2$ Calogero–Moser model, one can explicitly construct a complete set of conserved Liouville currents. However, we will not present them here.

Conclusion

We have analyzed the integrability of the $\mathcal{N}$-extended supersymmetric Calogero–Moser models. We explicitly constructed the Lax pair for these systems and proved (maximal) superintegrability, at least for the bosonic sector. A procedure has been proposed for constructing sufficiently many functionally independent conserved currents, including but extending the Liouville charges. We have demonstrated this by explicit expressions up to level five.
One may also try to repeat this analysis for the $\mathcal{N}$-extended supersymmetric Euler–Calogero–Moser

- S. Krivonos, O. Lechtenfeld, A. Sutulin, *Supersymmetric many-body Euler–Calogero–Moser model,* Phys. Lett. **B** 790 (2019) 191, `arXiv:1812.03530[hep-th]`.

and Calogero–Moser–Sutherland

- S. Krivonos, O. Lechtenfeld, $\mathcal{N} = 4$ *supersymmetric Calogero–Sutherland models,* Phys. Rev. **D** 101 (2020) 086010, `arXiv:2002.03929[hep-th]`.

models, although this is less straightforward.