Anomaly inflow for local boundary conditions

Dmitri Vassilevich

UFABC

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What is anomaly inflow?

Consider a manifold $\mathcal{M}$ with fermions described by $\not{D}$ and an effective theory on $\partial\mathcal{M}$ with a Dirac operator $\mathcal{D}$.

Anomaly inflow:

Anomalies of $\not{D}$ are expressed through anomalies of $\mathcal{D}$.

Anomalies: (i) the parity anomaly (spectral asymmetry), (ii) global chiral anomaly (the index).

Applications:

- Calculation of the anomalies.
- Construction of anomaly-free theories (Witten & Yonekura (2019)).
- Cobordism invariants.

The setup

The operator:

$$\not{D} = i\gamma^\mu \nabla_\mu,$$

where ∇ is a gauge and Riemannian covariant derivative.

The manifold: smooth, compact, having a product structure near the boundary.

Spectral η function: let λ be eigenvalues of $\not{D}$

$$\eta(s, \not{D}) = \sum_{\lambda>0} \lambda^{-s} - \sum_{\lambda<0} (-\lambda)^{-s}$$

Parity anomaly/spectral asymmetry/phase of the partition function: $\eta(0, \not{D})$. This quantity jumps whenever an eigenvalue crosses the origin. A smoothed version is the exponentiated η invariant $\mathcal{E}_{\not{D}} = e^{-i\pi(\eta(0,\not{D})+\dim \mathrm{Ker}\, \not{D})}$.

Boundary conditions

$$\Pi_{-}\psi|_{\partial\mathcal{M}} = 0,$$

where Π_{-} is a local rank 1/2 projector.
Auxiliary boundary value problem: (i) neglect all non-derivative terms in $\not{D}$, (ii) replace the coefficients in front of derivatives by their values on the boundary, (iii) replace tangential derivatives by their Fourier transforms, (iv) keep the normal derivative as it is.
The boundary conditions are called strongly elliptic if the auxiliary boundary value problem on $\mathbb{R}_{+}$ has no integrable eigenfunctions with eigenvalues in some "wrong part" of the complex plane.
For strongly elliptic boundary conditions the variations of (the smooth part of) $\eta(0, \not{D})$ are local which allows to separated bulk and boundary contributions.

Euclidean bag boundary conditions

Let $n = \dim \mathcal{M}$ be even. $\partial\mathcal{M} = \cup_\alpha \partial\mathcal{M}_\alpha$.

$$\Pi_- = \tfrac{1}{2}(1 - i\varepsilon_\alpha \gamma_* \gamma^n),$$

where γ_* is the chirality matrix, γ^n is the normal γ matrix, $\varepsilon_\alpha = \pm 1$. These boundary conditions are hermitian and strongly elliptic. In a suitable basis,

$$\not{D} = \begin{pmatrix} \mathcal{D} & i\partial_n \\ i\partial_n & -\mathcal{D} \end{pmatrix},$$

The restriction of $\not{D}$ on the space of boundary values of ψ is $\mathcal{D}_\alpha := -\varepsilon_\alpha \mathcal{D}$. One can argue that

$$\eta(0, \not{D}) = \sum_\alpha f(\mathcal{D}_\alpha)$$

with some universal function f.

To define f we use a product case example [Gilkey & Smith, 1983]: $\mathcal{M} = \mathcal{N} \times [0, \pi]$. This case is exactly solvable for any values of ε_α. For a non-zero momentum along $x^n \in [0, \pi]$, the spectrum is symmetric and thus does not contribute to $\eta(0, \not{D})$. For zero momentum, the eigenvalues of $\not{D}$ coincide with the eigenvalues of $\mathcal{D}_\alpha$. Thus, since there are two components of the boundary,

$$f(\mathcal{D}_\alpha) = \frac{1}{2}\eta(0, \mathcal{D}_\alpha)$$

The parity anomaly for $\not{D}$ has been expressed through parity anomalies of boundary Dirac operators.

Witten–Yonekura boundary conditions

$$\Pi_- = \frac{1}{2}\left(1 - \varepsilon_\alpha \gamma^n\right)$$

These boundary conditions

- can be formulated in both even and odd number of dimensions.
- are not strongly elliptic (though very close to strong ellipticity).

Our methods do not work.

Odd n

Let us double the fermions.

$$\gamma^\mu = \Gamma^\mu \otimes \sigma_2$$

The most general hermitian strongly elliptic local rotation invariant boundary conditions are given by

$$\Pi_- = \tfrac{1}{2}(1 - \Gamma^n \otimes (\cos(2\pi\vartheta_\alpha)\sigma_3 + \sin(2\pi\vartheta_\alpha)\sigma_1)).$$

One can show

$$\mathcal{E}(\not{D}) = \exp\left(-\pi i \mathcal{N}_0 + 2\pi i \sum_\alpha \vartheta_\alpha \,\mathrm{ind}\,(\mathcal{D}_\alpha, \Gamma^n)\right)$$

where $\mathcal{N}_0$ is the number of zero modes of $\not{D}$, when all $\vartheta_\alpha = 0$.