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Landau-Khalatnikov-Fradkin transformation in quenched QED_3

OUTLINE

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Abstract

The Landau-Khalatnikov-Fradkin (LKF) transformation is a powerful and elegant transformation allowing to study the gauge dependence of the propagator of charged particles interacting with gauge fields.

With the help of this transformation, we derive a non-perturbative identity between massless propagators in two different gauges.

in quenched QED3

assuming the finiteness of the perturbative expansion, we state that, exactly in $d = 3$, all odd perturbative coefficients, starting with the third order one, should be zero in any gauge. To check the result, we calculate the three- and four-loop corrections to the massless fermion propagator. The three-loop correction is finite and gauge invariant but, however, the four-loop one has singularities except in the Feynman gauge where it is also finite. These results explicitly show an absence of the finiteness of the perturbative expansion in quenched three-dimensional QED. Moreover, up to four loops, gauge-dependent terms are completely determined by lower order ones in agreement with the LKF transformation.

0. Introduction

Gauge invariance governs the dynamics of systems of charged particles with deep consequences in elementary particle physics and beyond. Through the gauge principle, it gives rise to gauge field theories the prototype of which is quantum electrodynamics (QED).

While physical quantities should not depend on this parameter, precious information can be obtained by studying the ξ-dependence of various correlation functions.

Such a task can be carried out with the help of the Landau-Khalatnikov-Fradkin (LKF) transformation (Landau, Khalatnikov: 1956), (Fradkin:1956) that elegantly relates the QED fermion propagator $S_F(p,\xi)$ and $S_F(p,\eta)$ in two different ξ and η gauges. In dimensional regularization, it reads:

$$S_F(x,\xi) = S_F(x,\eta)\, e^{\mathrm{i}\,(D(x)-D(0))}\,,$$

where

$$D(x) = -\mathrm{i}\,\Delta\, e^2\, \mu^{4-d} \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, \frac{e^{-\mathrm{i}px}}{p^4}, \quad \Delta = \xi - \eta\,.$$

Let us show basic steps of (Landau, Khalatnikov: 1956).

Gauge invariance arises in the field theory of charged particles interacting with an electromagnetic field. Given a gauge transformation of the potential of electromagnetic field

$$A_\mu \to A_\mu + \frac{\partial \varphi(x)}{\partial x_\mu},$$

where $\varphi(x)$ is an arbitrary operator function.

The Ψ-function of particle is transformed as follows:

$$\Psi(x) \to \Psi(x) e^{ie\varphi(x)}$$

Question: how the Green's function $S_F(x)$ for the particles will change under such a gauge transformation.

We would like to note that Fourier components of the Green's function $G_{\mu\nu}(x)$ for photons can be written in the general case in the form

$$G_{\mu\nu}(k) \sim \frac{d_t(k)}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + d_l(k)\,\frac{k_\mu k_\nu}{(k^2)^2},$$

where the terms containing $d_t(k)$ and $d_l(k)$ represent respectively the transverse and longitudinal parts of the function $G_{\mu\nu}$.

Moreover, the longitudinal part does not depend upon interaction with the field.

The Green's function $D_\varphi(x)$ for the $\varphi(x)$ field is connected with the longitudinal part $G^l_{\mu\nu}(x)$ of the Green's function for photons:

$$G^l_{\mu\nu}(x) = \frac{\partial^2 D_\varphi(x)}{\partial x_\mu \partial x_\nu}\,.$$

So, Fourier components of the Green's function $D_\varphi(x)$ for the $\varphi(x)$ field can be written in the form

$$D_\varphi(k) \sim \frac{d_l(k)}{k^4}$$

with $d_l(k) \sim 1$. It is very unusual Green's function.

Taking the above transformation for Ψ-function of particle and using the fact that he operators $\varphi(x)$ represent a free field, Landau, and Khalatnikov found the gauge transformation for the Green's function $S_F(x)$ as

$$S_F(x) = S_F^t(x) \times \; e^{ie^2(D_\varphi(0) - D_\varphi(x))}$$

where $S_F^t(x)$ is the Green's function in the Landau gauge.

The most important applications of the LKF transformation (Kotikov, Teber: 2019)

From this identity, we find that the corresponding perturbative series can be exactly expressed in terms of a hatted transcendental basis that eliminates all even ζ-values. Our construction further allows us to derive an exact formula relating hatted and standard ζ-values to all orders of perturbation theory.

G-scheme: (Broadhurst: 1999), (Baikov, Chetyrkin: 2010, 2018, 2019)

$$\hat{\zeta}_{2s-1} \equiv \zeta_{2s-1} + \varepsilon\, K_{2s-1,2s}\, \zeta_{2s} + \varepsilon^3\, K_{2s-1,2s+2}\, \zeta_{2s+2} + \ldots, \quad (s \geq 2)$$

(Kotikov, Teber: 2019)

$$K_{2s-1,2k} = b_{2k-2s+1}\, \frac{(2k-1)!}{(2s-2)!\,(2k-2s+1)!}\,,$$

where $b_{2k-2s+1}$ are Bernoulli numbers.

(Curtis, Pennington: 1990), (Dong, Munczek, Roberts: 1994, 1996), (Bashir, Kizilersu, Pennington: 1998, 2000), (Burden, Tjiang: 1998), (Jia, Pennington: 2016, 2017)
are related to the study of the gauge covariance of QED Schwinger-Dyson equations and their solutions. This allows, $e.g.$, to construct a charged-particle-photon vertex ansatz both in scalar (Fernandez-Rangel, Bashir, Gutierrez-Guerrero, Concha-Sanchez: 2016), (Ahmadiniaz, Bashir, Schubert: 2016)
and spinor QED (Kizilersu, Pennington: 2009).

Other applications

(Bashir, Raya: 2002), (Jia, Pennington: 2017)

are focused on estimating large orders of perturbation theory. Indeed, the non-perturbative nature of the LKF transformation allows to fix some of the coefficients of the all-order expansion of the fermion propagator. Starting with a perturbative propagator in some fixed gauge, say η, all the coefficients depending on the difference between the gauge fixing parameters of the two propagators, $\xi - \eta$, get fixed by a weak coupling expansion of the LKF-transformed initial one. Such estimations have been carried out for QED in various dimensions (see (Bashir, Raya: 2002), (Jia, Pennington: 2017)), for generalizations to brane worlds (Ahmad, Cobos-Martinez, Concha-Sanchez, Raya: 2016), (James, A.V.K., Teber: 2020) and for more general SU(N) gauge theories (Meerleer, Dudal, Sorella, Dall'Olio, Bashir: 2018).

1. LKF transformation

In the following, we shall consider QED in an Euclidean space of dimension d ($d = 4 - 2\varepsilon$). The general form of the fermion propagator $S_F(p, \xi)$ in some gauge ξ reads:

$$S_F(p, \xi) = \frac{i}{\hat{p}} P(p, \xi) \,,$$

where the factor $\hat{p}$ containing Dirac γ-matrices, has been extracted. It is also convenient to introduce the x-space representation $S_F(x, \xi)$ of the fermion propagator as:

$$S_F(x, \xi) = \hat{x} \, X(x, \xi) \,.$$

The two representations, $S_F(x,\xi)$ and $S_F(p,\xi)$, are related by the Fourier transform which is defined as:

$$S_F(p,\xi) = \int \frac{\mathrm{d}^d x}{(2\pi)^{d/2}}\, e^{\mathrm{i}px}\, S_F(x,\xi)\,,$$

$$S_F(x,\xi) = \int \frac{\mathrm{d}^d p}{(2\pi)^{d/2}}\, e^{-\mathrm{i}px}\, S_F(p,\xi)\,.$$

The famous LKF transformation connects in a very simple way the fermion propagator in two different gauges, $e.g.$, ξ and η. In dimensional regularization, it reads:

$$S_F(x,\xi) = S_F(x,\eta)\, e^{\mathrm{i}\,(D(x)-D(0))}\,,$$

where

$$D(x) = -\mathrm{i}\,\Delta\, e^2\, \mu^{4-d} \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, \frac{e^{-\mathrm{i}px}}{p^4}, \quad \Delta = \xi - \eta\,.$$

Note that, in dimensional regularization, the term $D(0)$ is proportional to the massless tadpole T_2, the massive counterpart of which is defined as:

$$T_\alpha(m^2) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{(p^2+m^2)^\alpha} \,.$$

The tadpole $T_\alpha(m^2) \sim \delta(\alpha - d/2)$ in the massless limit and, thus, $D(0) = 0$ in the framework of dimensional regularization. So, the LKF transformation can be simplified as follows:

$$S_F(x,\xi) = S_F(x,\eta)\, e^{\mathrm{i}D(x)} \,.$$

We may now proceed in calculating $D(x)$ using the Fourier transforms

$$\int \mathrm{d}^d x \, \frac{e^{\mathrm{i}px}}{x^{2\alpha}} = \frac{2^{2\tilde{\alpha}} \pi^{d/2} a(\alpha)}{p^{2\tilde{\alpha}}}, \quad a(\alpha) = \frac{\Gamma(\tilde{\alpha})}{\Gamma(\alpha)}, \quad \tilde{\alpha} = \frac{d}{2} - \alpha \,,$$

$$\int \mathrm{d}^d p \, \frac{e^{-\mathrm{i}px}}{p^{2\alpha}} = \frac{2^{2\tilde{\alpha}} \pi^{d/2} a(\alpha)}{x^{2\tilde{\alpha}}} \,.$$

This yields:

$$D(x) = -\mathrm{i} \, \Delta \, e^2 \, (\mu^2 x^2)^{2-d/2} \, \frac{\Gamma(d/2-2)}{2^4 (\pi)^{d/2}} \,,$$

or, equivalently, with the parameter ε made explicit:

$$D(x) = \frac{\mathrm{i} \, \Delta \, A}{\varepsilon} \, \Gamma(1-\varepsilon) \, (\pi \mu^2 x^2)^{\varepsilon}, \quad A = \frac{\alpha_{\mathrm{em}}}{4\pi} = \frac{e^2}{(4\pi)^2} \,.$$

We see that $D(x)$ contributes with a common factor ΔA accompanied by the singularity ε^{-1}.

1.2. LKF transformation in momentum space

Let's assume that, for some gauge fixing parameter η, the fermion propagator $S_F(p,\eta)$ with external momentum p has the form

$$S_F(p,\eta) = \frac{1}{i\hat{p}}\, P(p,\eta)\,, \quad P(p,\eta) = \sum_{m=0}^{\infty} a_m(\eta)\, A^m \left(\frac{\tilde{\mu}^2}{p^2}\right)^{m\varepsilon} .$$

The $a_m(\eta)$ are coefficients of the loop expansion of the propagator and $\tilde{\mu}$ is the renormalization scale:

$$\tilde{\mu}^2 = 4\pi\mu^2\,,$$

which lies somehow between the MS-scale μ and the $\overline{\mathrm{MS}}$-scale $\overline{\mu}$.

Then, using Fourier transforms , we obtain that:

$$S_F(x,\eta) = \frac{2^{d-1}\,\hat{x}}{(4\pi\,x^2)^{d/2}} \sum_{m=0}^{\infty} b_m(\eta)\, A^m \left(\pi\mu^2 x^2\right)^{m\varepsilon},$$

$$b_m(\eta) = a_m(\eta)\,\frac{\Gamma(d/2 - m\varepsilon)}{\Gamma(1+m\varepsilon)}\,.$$

With the help of an expansion of the LKF exponent, we have

$$S_F(x,\xi) = S_F(x,\eta)e^{D(x)} = \frac{2^{d-1}\,\hat{x}}{(4\pi\,x^2)^{d/2}} \sum_{m=0}^{\infty} b_m(\eta)\, A^m \left(\pi\mu^2 x^2\right)^{m\varepsilon}$$

$$\times \sum_{l=0}^{\infty} \left(-\frac{A^m\,\Delta}{\varepsilon}\right)^l \frac{\Gamma^l(1-\varepsilon)}{l!}\,(\pi\mu^2 x^2)^{l\varepsilon}\,.$$

Factorizing all x-dependence yields:

$$S_F(x,\xi) = \frac{2^{d-1}\,\hat{x}}{(4\pi\,x^2)^{d/2}} \sum_{p=0}^{\infty} b_p(\xi)\, A^m \left(\pi\mu^2 x^2\right)^{p\varepsilon},$$

$$b_p(\xi) = \sum_{m=0}^{p} \frac{b_m(\eta)}{(p-m)!} \left(-\frac{\Delta}{\varepsilon}\right)^{p-m} \Gamma^{p-m}(1-\varepsilon)\,.$$

Hence, taking the correspondence between the results for propagators $P(p,\eta)$ and $S_F(x,\eta)$, respectively, together with the result for $S_F(x,\xi)$, we have for $P(p,\xi)$:

$$P(p,\xi) = \sum_{m=0}^{\infty} a_m(\xi)\, A^m \left(\frac{\tilde{\mu}^2}{p^2}\right)^{m\varepsilon} ,$$

where

$$\begin{aligned} a_m(\xi) &= b_m(\xi)\, \frac{\Gamma(1+m\varepsilon)}{\Gamma(d/2-m\varepsilon)} \\ &= \sum_{l=0}^{m} \frac{a_l(\eta)}{(m-l)!}\, \frac{\Gamma(d/2-l\varepsilon)\Gamma(1+m\varepsilon)}{\Gamma(1+l\varepsilon)\Gamma(d/2-m\varepsilon)} \left(-\frac{\Delta}{\varepsilon}\right)^{m-l} \Gamma^{m-l}(1-\varepsilon)\,. \end{aligned}$$

In this way, we have derived the expression of $a_m(\xi)$ using a simple expansion of the LKF exponent in x-space. From this representation of the LKF transformation, we see that the magnitude $a_m(\xi)$ is determined by $a_l(\eta)$ with $0 \le l \le m$.

The corresponding result for the p- and Δ-dependencies of $\hat{a}_m(\xi, p)$ can be obtained by interchanging the order in the sums in the results for $P(p, \xi)$. So, we have

$$P(p, \xi) = \sum_{m=0}^{\infty} \hat{a}_m(\xi, p)\, A^m \left(\frac{\tilde{\mu}^2}{p^2}\right)^{m\varepsilon} ,$$

where

$$\hat{a}_m(\xi, p) = a_m(\eta) \sum_{l=0}^{\infty} \frac{\Gamma(d/2 - m\varepsilon)\Gamma(1 + (l+m)\varepsilon}{\Gamma(1 + m\varepsilon)\Gamma(d/2 - (l+m)\varepsilon)} \left(-\frac{A^m\, \Delta}{\varepsilon}\right)^l \times \frac{\Gamma^l(1-\varepsilon)}{l!} \left(\frac{\tilde{\mu}^2}{p^2}\right)^{l\varepsilon} .$$

2. QED3

We would like to note that all of the above results may be expressed in $d = 3-2\varepsilon$ with the help of the substitutions $\varepsilon \to 1/2+\varepsilon$ and $e^2_{d=4}\,\mu \to e^2$. The last replacement can also be expressed as $A\,\mu = \alpha/(4\pi)$, with the dimensionful $\alpha = e^2/(4\pi)$.

Let we have initially

$$P(p,\eta) = \sum_{m=0}^{\infty} a_m(\eta) \left(\frac{\alpha}{2\sqrt{\pi}\,p}\right)^m \left(\frac{\tilde{\mu}^2}{p^2}\right)^{m\varepsilon},$$

where $a_m(\eta)$ are coefficients of the loop expansion of the propagator and $\tilde{\mu}$ is the scale

$$\tilde{\mu}^2 = 4\pi\mu^2\,,$$

So, we have

$$P(p,\xi) = \sum_{k=0}^{\infty} a_k(\xi) \left(\frac{\alpha}{2\sqrt{\pi}\,p}\right)^k \left(\frac{\tilde{\mu}^2}{p^2}\right)^{k\varepsilon},$$

where

$$a_k(\xi) = \sum_{m=0}^{k} (-2\Delta)^{k-m}\, a_m(\eta)\, \hat{\Phi}(m,k,\varepsilon)\ \phi(k-m,\varepsilon)$$

and

$$\hat{\Phi}(m,k,\varepsilon) = \frac{\Gamma(3/2 - m/2 - (m+1)\varepsilon)\Gamma(1 + k/2 + k\varepsilon)}{\Gamma(1 + m/2 + m\varepsilon)\Gamma(3/2 - k/2 - (k+1)\varepsilon)}\,.$$

In this way, we have derived the expression of $a_k(\xi)$ using a simple expansion of the LKF exponent in x-space. From this representation of the LKF transformation, we see that the magnitude $a_k(\xi)$ is determined by $a_m(\eta)$ with $0 < m < k$.

Very often, however, the subject of the study is not the magnitude $a_m(\xi)$ but the p- and Δ-dependencies of each magnitude $a_l(\eta)$ as it evolves from the η to the ξ gauge. The corresponding result for the p- and Δ-dependencies of $\hat{a}_m(\xi, p)$ can be obtained interchanging the order of the sums. Performing such interchange yields:

$$P(p,\xi) = \sum_{m=0}^{\infty} \hat{a}_m(\xi,p) \left(\frac{\alpha}{2\sqrt{\pi}\,p}\right)^m \left(\frac{\tilde{\mu}^2}{p^2}\right)^{m\varepsilon} ,$$

where, now, the coefficients transform as

$$\hat{a}_m(\xi,p) = a_m(\eta) \sum_{l=0}^{\infty} \tilde{\Phi}(m,l,\varepsilon)\,\phi(l,\varepsilon) \left(-\frac{\alpha\Delta}{\sqrt{\pi}p}\right)^l \left(\frac{\tilde{\mu}^2}{p^2}\right)^{l\varepsilon} ,$$

with

$$\begin{aligned} &\tilde{\Phi}(m,l,\varepsilon) = \hat{\Phi}(m,m+l,\varepsilon) \\ &= \frac{\Gamma(3/2 - m/2 - (m+1)\varepsilon)\Gamma(1 + (m+l)/2 + (m+l)\varepsilon)}{\Gamma(1 + m/2 + m\varepsilon)\Gamma(3/2 - (m+l)/2 - (m+l+1)\varepsilon)} . \end{aligned}$$

2.1 Coefficients $a_k(\xi)$ at $\varepsilon \to 0$

The analysis of the coefficients $a_k(\xi)$ requires considering the cases of even and odd values of k separately.

1. In the case of even k values, $i.e.$, $k = 2r$, the final results for $a_{2r}(\xi)$ can be expressed as a sum of the contributions $a_{2r}^{(i)}(\xi)$ with $i = 1, 2$ and 3, $i.e.$,

$$a_{2r}(\xi) = a_{2r}^{(1)}(\xi) + a_{2r}^{(2)}(\xi) + a_{2r}^{(3)}(\xi)\,. \tag{1}$$

The latter come in-turn from the corresponding contributions of the initial amplitudes $a_{2s}(\eta)$, $a_1(\eta)$ and $a_{2s+1}(\eta)$ as

$$a_{2r}^{(1)}(\xi) = \sum_{s=0}^{r} a_{2s}(\eta)$$
$$\times \frac{\Gamma(r-1/2)\Gamma(1+r)}{\Gamma(1+s))\Gamma(s-1/2)} \frac{\Gamma(1/2)\,(-\delta^2)^{r-s}}{\Gamma(r-s+1/2)(r-s)!}\,;$$
$$a_{2r}^{(2)}(\xi) = \frac{2}{\pi}\frac{r}{r-1/2}\frac{(-\delta^2)^r}{\delta}\,a_1(\eta)\,;$$
$$a_{2r}^{(3)}(\xi) = \sum_{s=1}^{r-1} a_{2s+1}(\eta)\,\frac{(-1)^{r+s+1}}{2(s+1)\pi\varepsilon}$$
$$\times \frac{\Gamma(r-1/2)\Gamma(1+r)}{\Gamma(s)\Gamma(s+3/2)} \frac{\Gamma(1/2)\,(-\delta)^{2r-2s-1}}{\Gamma(r-s)\Gamma(r-s+1/2)}\,,$$

where $\delta = \sqrt{\pi}\Delta$.

2. In the case of odd k values, *i.e.*, $k = 2r + 1$, we should consider the cases $r = 0$ and $r \geq 1$ separately.

In the case $k = 1$, we have the following result:

$$a_1^{(1)}(\xi) = a_1(\eta) - \frac{\pi}{2}\,\delta\, a_0(\eta)\,.$$

The final result for $a_{2r+1}(\xi)$ (for $r \geq 1$) can be expressed as a sum of the contributions $a_{2r+1}^{(i)}(\xi)$ with $i = 1$, 2 and 3, *i.e.*,

$$a_{2r+1}(\xi) = a_{2r+1}^{(1)}(\xi) + a_{2r+1}^{(2)}(\xi) + a_{2r+1}^{(3)}(\xi)\,.$$

The latter come in-turn from the corresponding contributions of the initial amplitudes $a_{2s}(\eta)$, $a_1(\eta)$ and $a_{2s+1}(\eta)$ as

$$a^{(1)}_{2r+1}(\xi) = [2\pi(r+1)\varepsilon] \sum_{s=0}^{r} a_{2s}(\eta)\,(-1)^{r+s+1}$$
$$\times \frac{\Gamma(r+3/2)\Gamma(r)}{\Gamma(1+s))\Gamma(s-1/2)} \frac{\Gamma(1/2)\,(-\delta)^{2r-2s+1}}{\Gamma(r-s+1)\Gamma(r-s+3/2)}\,;$$
$$a^{(2)}_{2r+1}(\xi) = [4(r+1)\varepsilon] \frac{r+1/2}{r} (-\delta^2)^r\, a_1(\eta)\,;$$
$$a^{(3)}_{2r+1}(\xi) = \sum_{s=1}^{r} a_{2s+1}(\eta) \frac{(r+1)}{(s+1)} \frac{\Gamma(r+3/2)\Gamma(r)}{\Gamma(s)\Gamma(s+3/2)}$$
$$\times \frac{\Gamma(1/2)\,(-\delta^2)^{r-s}}{\Gamma(r-s+1)\Gamma(r-s+1/2)}\,.$$

We note that, these contributions correspond to the first terms of the ε-expansion, which is sufficient to analyze the self-consistency given in the next subsection.

2.2 Self-consistency

Consider $a_m(\xi)$ with $m \leq 6$. Using the results of the previous subsection, we have:

$$a_0(\xi) = a_0(\eta)\,, \quad a_1(\xi) = a_1(\eta) - \frac{\pi}{2}\,\delta\, a_0(\eta)\,,$$

$$a_2(\xi) = a_2(\eta) - \frac{4}{\pi}\,\delta\, a_1(\eta) + \delta^2\, a_0(\eta)\,,$$

$$a_3(\xi) = a_3(\eta) + 6\pi\varepsilon\,\delta\, a_2(\eta) - 12\varepsilon\,\delta^2\, a_1(\eta) + 2\pi\varepsilon\,\delta^3\, a_0(\eta)\,,$$

$$a_4(\xi) = a_4(\eta) - \frac{2\delta}{3\pi\varepsilon}\, a_3(\eta) - 2\delta^2\, a_2(\eta) + \frac{8\delta^3}{3\pi}\, a_1(\eta) - \frac{\delta^4}{3}\, a_0(\eta)\,,$$

$$a_5(\xi) = a_5(\eta) + \frac{45}{2}\pi\varepsilon\,\delta\, a_4(\eta) - \frac{15}{2}\,\delta^2\, a_3(\eta)$$
$$-15\pi\varepsilon\,\delta^3\, a_2(\eta) + 15\varepsilon\,\delta^4\, a_1(\eta) - \frac{3}{2}\pi\varepsilon\,\delta^5\, a_0(\eta)\,,$$

$$a_6(\xi) = a_6(\eta) + \frac{4\delta}{5\pi\varepsilon}\, a_5(\eta) - 9\delta^2\, a_4(\eta) + \frac{2\delta^3}{\pi\varepsilon}\, a_3(\eta)$$
$$+ 3\delta^4\, a_2(\eta) - \frac{12\delta^5}{5\pi}\, a_1(\eta) + \frac{\delta^6}{5}\, a_0(\eta)\,.$$

Remarkably, these equations are self-consistent. For example, if we would like to obtain the expression of $a_m(\xi_1)$ in some gauge with parameter ξ_1 ($i.e.$, the ξ_1-gauge), we can derive it directly from the η-gauge and then proceed in two steps: from the η-gauge to the ξ-gauge and later from the ξ-gauge to the ξ_1-gauge.

Let's show explicitly this self-consistency in the case of $a_m(\xi_1)$ with $m = 0, 1, 2$. The coefficient $a_0(\xi_1)$ does not change, $i.e.$,

$$a_0(\xi_1) = a_0(\xi) = a_0(\eta)\,.$$

For the coefficient $a_1(\xi_1)$, we have (hereafter $\overline{\delta}_1 = \sqrt{\pi}(\xi_1 - \xi)$, $\delta_1 = \sqrt{\pi}(\xi_1 - \eta)$):

$$\begin{aligned} a_1(\xi_1) &= a_1(\xi) - \frac{\pi}{2}\,\overline{\delta}_1\, a_0(\xi) \\ &= (a_1(\eta) - \frac{\pi}{2}\,\delta\, a_0(\eta)) - \frac{\pi}{2}\,\overline{\delta}_1\, a_0(\eta) = a_1(\eta) - \frac{\pi}{2}\,\delta_1\, a_0(\eta)\,, \end{aligned}$$

because

$$\delta_1 = \delta + \overline{\delta}_1\,.$$

So, we obtain the expression of $a_1(\xi_1)$ and it coincides with the one obtained directly from the η-gauge.

Similarly, the coefficient $a_2(\xi_1)$ changes as:

$$a_2(\xi_1) = a_2(\xi) - \frac{4}{\pi}\,\overline{\delta}_1\, a_1(\xi) + \overline{\delta}_1^2\, a_1(\xi)$$
$$= (a_2(\eta) - \frac{4}{\pi}\,\delta\, a_1(\eta) + \delta^2\, a_0(\eta)) - \frac{4}{\pi}\,(a_1(\eta) - \frac{\pi}{2}\,\delta\, a_0(\eta)) + \overline{\delta}_1^2\, a_1(\eta)\,.$$

The term in factor of $a_1(\eta)$ corresponds to:

$$-\frac{4}{\pi}\,\overline{\delta}_1 - \frac{4}{\pi}\,\delta = -\frac{4}{\pi}\,\delta_1\,.$$

The term in factor of $a_2(\eta)$ corresponds to:

$$\overline{\delta}_1^2 + 2\overline{\delta}_1\delta + \delta^2 = (\overline{\delta}_1 + \delta)^2 = \delta_1^2\,.$$

Taking all the results together, we have:

$$a_2(\xi_1) = a_2(\eta) - \frac{4}{\pi}\,\delta_1\, a_1(\eta) + \delta_1^2\, a_0(\eta)\,.$$

Thus, we derive the expression of $a_2(\xi_1)$ and it coincides with the one obtained directly from the η-gauge.

Similar transformations can also be performed for the other coefficients $a_i(\xi_1)$ $(i \geq 2)$ in a similar way. So, we can obtain a full agreement between the transformation and the results for $a_i(\xi_1)$ obtained directly from the η-gauge.

A central result to the present study that can be derived from above equations is that, excepting the case of $a_1(\xi)$, all $a_{2m+1}(\xi)$ can be excluded.

Indeed, assuming that quenched QED is both UV and IR finite, (see, for example, (R.Jackiw, S.Templeton; 1981), (O.M. Del Cima, D.H.T.Franco, O.Piguet; 2014), (N.Karthik, R.Narayanan; 2017). and discussion therein)

setting $\varepsilon = 0$ enforces $a_{2m+1}(\xi) = 0$ for $(m \geq 1)$.

It follows then that simpler expressions are obtained for $a_{2i}(\xi)$ with $(i \geq 2)$:

$$a_4(\xi) = a_4(\eta) - 2\delta^2\, a_2(\eta) + \frac{8\delta^3}{3\pi}\, a_1(\eta) - \frac{\delta^4}{3}\, a_0(\eta)\,,$$

$$a_6(\xi) = a_6(\eta) - 9\delta^2\, a_4(\eta) + 3\delta^4\, a_2(\eta) - \frac{12\delta^5}{5\pi}\, a_1(\eta) + \frac{\delta^6}{5}\, a_0(\eta)\,.$$

Importantly, the coefficient $a_1(\xi)$ cannot be excluded. In a sense, the coefficient $a_1(\xi)$ (really, $a_1(\xi)/\pi$) behaves in a similar way to the even coefficients $a_{2r}(\xi)$.

With the purpose of checking the statement $a_{2m+1}(\xi) = 0$ for $(m \geq 1)$ for $d = 3$, we plan to perform a direct calculation of the coefficient a_3.

3. Summary I

We have studied the LKF transformation for the massless fermion propagator of three-dimensional QED in the quenched approximation to all orders in the coupling α. Our investigations were performed in dimensional regularization in $d = 3-2\varepsilon$ Euclidean space.

The transformation $a_m(\eta) \to a_m(\xi)$ relates the magnitudes $a_m(\xi)$ in ξ-gauge to a combination of initial magnitudes $a_l(\eta)$, where $0 \leq l \leq m$. Studying this relation in dimensional regularization, we observed that the contributions of odd magnitudes $a_{2t+1}(\eta)$ $(1 \geq t \geq s-1)$ to even magnitudes $a_{2s}(\xi)$ are accompanied by singularities which look like ε^{-1} in dimensional regularization. In turn, the even magnitudes $a_{2s}(\eta)$ produce contributions to odd magnitudes $a_{2t+1}(\xi)$ $(t \geq s) \sim \varepsilon$ if $t \geq 1$.

There are arguments in favor of ultraviolet and infrared perturbative finiteness of massless quenched QED$_3$ (R.Jackiw, S.Templeton; 1981), (O.M. Del Cima, D.H.T.Franco, O.Piguet; 2014), (N.Karthik, R.Narayanan; 2017).
Hence, assuming the existence of a finite limit as $\varepsilon \to 0$, we find that, exactly in $d = 3$, all odd terms $a_{2t+1}(\xi)$ in perturbation theory, except a_1, should be exactly zero in any gauge.

This statement is very strong and needs a further check. At the order α^2, analytical expressions for the fermion self-energy diagrams are well known. However, such results are absent at three-loop order. We plan to study the a_3 term, $i.e.$, three-loop diagrams, directly in the framework of perturbation theory in our future investigations.

4. Fermion propagator: three- and four-loop coefficients

As it was before, we consider a Euclidean space of dimension $d = 3 - 2\varepsilon$. The general form of the fermion propagator $S_F(p, \xi)$ in some gauge ξ reads:

$$S_F(p, \xi) = \frac{\mathrm{i}}{\hat{p}} P(p, \xi) \,.$$

It is convenient to first express $P(p, \xi)$ as

$$P(p, \xi) = \frac{1}{1 - \sigma(p, \xi)} \tag{2}$$

where the 1-particle-irreducible (1PI) part, $\sigma(p, \xi)$, can be represented as

$$\sigma(p, \xi) = \sum_{m=1}^{\infty} \sigma_m(\xi) \left(\frac{\alpha}{2\sqrt{\pi}\, p} \right)^m \left(\frac{\bar{\mu}^2}{p^2} \right)^{m\varepsilon} .$$

Here, $\sigma_m(\xi)$ are the coefficients of the loop expansion of the fermion self-energy, $\alpha = e^2/(4\pi)$ is the dimensionful coupling constant and $\bar{\mu}$ is the $\overline{\mathrm{MS}}$-scale.

Following previous sections, the fermion propagator can be equivalently represented as

$$P(p,\xi) = \sum_{m=0}^{\infty} a_m(\xi) \left(\frac{\alpha}{2\sqrt{\pi}\,p}\right)^m \left(\frac{\bar{\mu}^2}{p^2}\right)^{m\varepsilon},$$

where $a_m(\xi)$ are now the coefficients of the loop expansion of $P(p,\xi)$. As it was already shown, this form is convenient to study the properties of the propagator under the LKF transformation.

Up to four loops, the coefficients $a_m(\xi)$ and $\sigma_m(\xi)$ are related to each other as

$$a_1 = \sigma_1, \quad a_2 = \sigma_2 + \sigma_1^2, \quad a_3 = \sigma_3 + 2\sigma_2\sigma_1 + \sigma_1^3,$$
$$a_4 = \sigma_4 + 2\sigma_3\sigma_1 + \sigma_2^2 + 3\sigma_2\sigma_1^2 + \sigma_1^4\,.$$

4.1 Calculational details

In quenched QED at 1-, 2-, 3- and 4-loops we encountered 1, 2, 10 and 74 fermion self-energy diagrams, respectively.

Let's note that the two-loop diagrams of QED_3 were considered earlier in (R.Jackiw, S.Templeton; 1981), (E.I.Guendelman, Z.M.Radulovic, O.Piguet; 1983,1984).
These papers mainly focused on the IR divergent two-loop diagram (with a fermion loop insertion) which is absent in the quenched case.

The two-loop quenched QED_3 fermion propagator was calculated in (A.Bashir, A.Kizilersu, M.R.Pennington; 2000), (A.Bashir; 2000), (A.Bashir, A.Raya; 2002).

However, three- and four-loop corrections to the quenched QED_3 fermion propagator have not been previously computed. As will be shown in the next subsections, the three-loop correction is finite but IR singular diagrams do appear at 4-loops in the quenched case and there are 42 of them, the sum of which will be analyzed in the following.

In order to compute all of these diagrams and extract from them the unrenormalized fermion self-energy of QED_3 up to four loops, we first considered the corresponding results for the unrenormalized QCD quark propagator. The exact expression for the latter, written in terms of a set of master integrals and valid for arbitrary space-time dimension d and arbitrary gauge-fixing parameter ξ, is available up to four loops from (B.Ruijl et al.; 2017), and also shipped with the `FORCER` package (B.Ruijl, T.Ueda, J.A.M.Vernaseren; 2017)
designed for the reduction of four-loop massless propagator-type integrals.

The fermion propagator of QED$_d$ is obtained from this QCD$_d$ result upon performing the following substitutions:

$$C_A = d_A^{abcd} d_A^{abcd} = d_A^{abcd} d_F^{abcd} = 0, \quad C_F = d_F^{abcd} d_F^{abcd} = T_F = 1\,.$$

After that, the QED$_d$ quenched limit is obtained by setting $n_f = 0$ which discards all diagrams with closed fermion loops.

The main remaining task was then to compute all required propagator-type master integrals in an ε-expansion around $d = 3$. This could be achieved with the help of the Dimensional Recurrence and Analyticity (DRA) method (R.N.Lee; 2010)
which expresses the integrals in the form of fast convergent sums. The latter are then evaluated with high-precision numerical values. This in turn allows to reconstruct the analytic expression of master integrals (in any space-time dimension) with the help of the `PSLQ` algorithm (H.Ferguson, D.Bailey, S.Arno; 1999)
once an adequate basis of transcendental constants is defined.

We note that near $d = 4$, such calculations yield the expansions of all needed masters (R.N.Lee, A.V.Smirnov,V.A.Smirnov; 2012).

The case $d = 3 - 2\varepsilon$ is less well known and was considered in (R.N.Lee, K.T.Mingulov; 2016)
from which the ε-expansion of most of the needed master integrals for the current calculation is available. The successful reconstructions around $d = 3$, were carried out using a basis of transcendental constants consisting only of multiple zeta values (MZV) and alternating MZVs. As remarked already in the paper, such a basis is too restrictive to enable the representation of all of the masters and some of them were left unreconstructed.

In our work we successfully reconstructed all the needed integrals and found agreement with results of (R.N.Lee, A.V.Smirnov,V.A.Smirnov; 2012) using a basis consisting of MZV and alternating MZVs. On top of that, we encountered one of the constants left unknown in (R.N.Lee, K.T.Mingulov; 2016).

By a careful analysis of the representation of one such integrals with known closed form expressions in the form of the ${}_3F_2$-functions (A.V.Kotikov, S.Teber; 2014)

we found that elements of its ε-expansion belong to the set of generalized polylogarithms (GPLs) with fourth-root of unity alphabet.

Extending our PSLQ basis to include the full set of GPLs with fourth root of unity arguments we have

$$G(p;1,1/2,1,1/2,1) = \frac{1}{(4\pi)^d}\frac{8}{3\pi}(C_1 + O(\varepsilon^1))\frac{\mu^{2\varepsilon}}{p^{2(1+2\varepsilon)}},$$

where

$$\begin{aligned} &G(p;\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) \\ &= \int \frac{d^dk_1 d^dk_2}{(2\pi)^{2d}} \frac{1}{k_1^{2\alpha_1}k_2^{2\alpha_2}(p-k_2)^{2\alpha_3}(p-k_1)^{2\alpha_4}(k_1-k_2)^{2\alpha_5}}. \end{aligned}$$

and

$$C_1 = \mathrm{C}\pi^2 + 24\mathrm{Cl}_4\left(\frac{\pi}{2}\right),$$

with $\mathrm{C} = \mathrm{Cl}_2(\pi/2)$ is Catalan's constant and $\mathrm{Cl_n}(\theta)$ is Clausen's function ($\mathrm{Cl_{2k}}(\theta) = \mathrm{Im}\ \mathrm{Li_{2k}}\left(\mathrm{e}^{\mathrm{i}\theta}\right)$). As can be understood from the above result, the required extension of the basis of transcendental constants includes polylogarithms with fourth-root of unity argument.

4.2 Results for the fermion self-energy

We now present our results for $\sigma_m(\xi)$ which are represented as

$$\sigma_m(\xi) = \sigma_m(0) + \xi\, \tilde{\sigma}_m(\xi)\,,$$

where we have explicitly separated the part independent from ξ which corresponds to the full result in the Landau gauge.

For the first two orders of the ε-expansion, we have

$$\sigma_1(0) = 0\,;$$
$$\sigma_2(0) = \pi\left[\frac{3\pi^2}{4} - 7 - ((1-3\,l_2)\pi^2 + 12)\varepsilon\right]\,;$$
$$\sigma_3(0) = \pi^{5/2}\left[\frac{43\pi^2}{4} - 105 + \varepsilon\left\{2(185 - 105\,l_2 + 137\zeta_3) - \frac{\pi^2}{6}(451 - 171\,l_2)\right\}\right]\,;$$
$$\sigma_4(0) = \pi^2\left[\left(\frac{43}{6}\pi^2 - 70\right)\frac{1}{\varepsilon} + \bar{\sigma}_4 + \frac{5954}{3} + \frac{173}{18}\pi^2 - \frac{513}{10}\pi^4\right]\,,$$

where $\bar{\sigma}_4$ contains the most complicated part

$$\bar{\sigma}_4 = 209\,l_2^4 + 5016\,a_4 + 4264\,\mathrm{Cl}_4(\pi/2) + \left(\frac{533}{3}\mathrm{C} - 930\,\mathrm{l}_2\right)\pi^2 + \frac{2078}{3}\zeta_3\,,$$

and

$$l_2 = \ln 2, \quad a_4 = \mathrm{Li}_4(1/2), \quad \zeta_\mathrm{n} = \mathrm{Li}_\mathrm{n}(1)\,,$$

where Li_n are polylogarithms.

With the same accuracy, we have for the coefficients $\tilde{\sigma}_m(\xi)$

$$\tilde{\sigma}_1(\xi) = -\frac{\pi^{3/2}}{2}(1-2(1-l_2)\varepsilon)\,;$$

$$\tilde{\sigma}_2(\xi) = \pi\,\xi\left[1-\frac{\pi^2}{4}-(4-(1-l_2)\pi^2)\varepsilon\right]\,;$$

$$\tilde{\sigma}_3(\xi) = \pi^{5/2}\,[\frac{3\pi^2}{4}-7+\left(1-\frac{\pi^2}{8}\right)\xi^2+\varepsilon\{-40-14l_2+\frac{\pi^2}{2}(4+9l_2)$$
$$+\left(2\,l_2-4+\frac{3\pi^2}{4}(1-l_2)\right)\xi^2\}]\,;$$

$$\tilde{\sigma}_4(\xi) = \pi^2[\left(70-\frac{43\pi^2}{6}\right)\frac{1}{\varepsilon}+\frac{520}{3}-\frac{\pi^2}{9}(881+42l_2)+\frac{129\pi^4}{27}-\frac{548}{3}\zeta_3$$
$$+\xi\left(28-\frac{33\pi^2}{4}+\frac{9\pi^4}{16}\right)+\xi^3\left(-\frac{4}{3}+\frac{3\pi^2}{4}-\frac{\pi^4}{16}\right)]\,.$$

From the above equations, we notice that

$$\sigma_4(\xi) = \pi^2 \left(\frac{43}{6}\pi^2 - 70 \right) \frac{(1-\xi)}{\varepsilon} + O(\varepsilon^0)\,,$$

$i.e.$, the total four-loop contribution is finite in the Feynman gauge.

4.3 Results for the fermion propagator

As in the case of $\sigma_m(\xi)$, it is convenient to present the results for $a_m(\xi)$ in the form

$$a_m(\xi) = a_m(0) + \xi\, \tilde{a}_m(\xi)\,,$$

where we have also explicitly separated the part independent from ξ which corresponds to the full result in the Landau gauge.

Since $\sigma_1(\xi) \sim \xi$, we see that $a_i(0) = \sigma_i(0)$ for $i \leq 3$. For $a_4(0)$, we have

$$\begin{aligned} a_4(0) &= \sigma_4(0) + \pi^2 \left(\frac{3\pi^2}{4} - 7 \right)^2 \\ &= \pi^2 \left[\left(\frac{43}{6}\pi^2 - 70 \right) \frac{1}{\varepsilon} + \bar{\sigma}_4 + \frac{6101}{3} - \frac{8}{9}\pi^2 - \frac{4059}{80}\pi^4 \right] . \end{aligned}$$

With the same accuracy, we have for the coefficients $\tilde{a}_m(\xi)$

$$\tilde{a}_1(\xi) = \tilde{\sigma}_1(\xi) = -\frac{\pi^{3/2}}{2}(1-2(1-l_2)\varepsilon)\,; \quad \tilde{a}_2(\xi) = \pi\,\xi(1-4\varepsilon)\,;$$

$$\tilde{a}_3(\xi) = \pi^{5/2}\,\varepsilon\left(\frac{43\pi^2}{4} - 105 + 2\xi^2\right)\,;$$

$$\tilde{a}_4(\xi) = \frac{\pi^2}{3}[\left(210 - \frac{43\pi^2}{2}\right)\frac{1}{\varepsilon} + 520 + \frac{2\pi^2}{3}(32 - 21\,l_2) - 548\zeta_3$$

$$+6\xi\left(7 - \frac{3\pi^2}{4}\right) - \xi^3]$$

From the above results, we see that the coefficients $\tilde{a}_m(\xi)$ $(m = 2,3,4)$ have simpler forms than the corresponding coefficients $\tilde{\sigma}_m(\xi)$. Moreover, we notice that

$$a_4(\xi) = \sigma_4(\xi) + O(\varepsilon^0) = \frac{2\pi^2}{3}\left(\frac{43\pi^2}{4} - 105\right)(1-\xi)\frac{1}{\varepsilon} + O(\varepsilon^0)\,,$$

$i.e.$, the total four-loop contribution is finite in the Feynman gauge.

5. LKF transformation

The LKF transformation relates the coefficients $a_k(\xi)$ and $a_m(\eta)$ as

$$a_k(\xi) = \sum_{m=0}^{k} (-2\Delta)^{k-m}\, a_m(\eta)\, \Phi(m,k,\varepsilon)\ \phi(k-m,\varepsilon),$$

where

$$\Phi(m,k,\varepsilon) = \frac{\Gamma(3/2 - m/2 - (m+1)\varepsilon)\Gamma(1 + k/2 + k\varepsilon)}{\Gamma(1 + m/2 + m\varepsilon)\Gamma(3/2 - k/2 - (k+1)\varepsilon)}$$

and

$$\phi(l,\varepsilon) = \frac{\Gamma^l(1/2 - \varepsilon)}{l!\,(1+2\varepsilon)^l \Gamma^l(1+\varepsilon)}\,.$$

5.1 Comparison with the perturbative results up to four loops

Consider $a_m(\xi)$ with $m \leq 4$. Keeping only the first two orders of the ε-expansion, we have:

$$
\begin{aligned}
a_0(\xi) &= a_0(\eta) = 1\,, \quad a_1(\xi) = a_1(\eta) - \frac{\pi}{2}\,\delta\,(1 + 2\varepsilon(l_2 - 1))\,a_0(\eta)\,, \\
a_2(\xi) &= a_2(\eta) - \frac{4}{\pi}\,\delta\,(1 - 2\varepsilon(l_2 + 1))\,a_1(\eta) + \delta^2\,(1 - 4\varepsilon)\,a_0(\eta)\,, \\
a_3(\xi) &= a_3(\eta) + 6\pi\varepsilon\,\delta\,a_2(\eta) - 12\varepsilon\,\delta^2\,a_1(\eta) + 2\pi\varepsilon\,\delta^3\,a_0(\eta)\,, \\
a_4(\xi) &= a_4(\eta) - \frac{2\delta}{3\pi\varepsilon}\,(1 + 2\varepsilon(3 - l_2))\,a_3(\eta) - 2\delta^2\,a_2(\eta) \\
&\quad + \frac{8\delta^3}{3\pi}\,a_1(\eta) - \frac{\delta^4}{3}\,a_0(\eta)\,,
\end{aligned}
$$

where $\delta = \sqrt{\pi}\Delta$.

Setting $\eta = 0$, $i.e.$, choosing the initial gauge as the Landau gauge, we can see that our results for $\tilde{a}_m(\xi)$ are completely determined by $a_l(\xi)$, $(l < m)$, $i.e.$, by the coefficients of lower orders in agreement with the properties of the LKF transformation.

Moreover, these results are in full agreement with the perturbative results presented in the previous section.

5.2 Beyond four-loops

The singularity of the four-loop coefficient $a_4(\xi)$ is $\sim (1-\xi)$, $i.e.$, the fermion propagator including up to four-loop corrections is finite in the Feynman gauge. This intriguing fact calls for a closer examination of higher order contributions and, as a first try, we will proceed by using the LKF transformation.

We therefore consider $a_5(\xi)$ and $a_6(\xi)$:

$$
\begin{aligned}
a_5(\xi) &= a_5(\eta) + \frac{45}{2}\pi\varepsilon\,\delta\,a_4(\eta) - \frac{15}{2}\,\delta^2\,a_3(\eta) - 15\pi\varepsilon\,\delta^3\,a_2(\eta) \\
&\quad +15\varepsilon\,\delta^4\,a_1(\eta) - \frac{3}{2}\pi\varepsilon\,\delta^5\,a_0(\eta)\,, \\
a_6(\xi) &= a_6(\eta) + \frac{4\delta}{5\pi\varepsilon}\,a_5(\eta) - 9\delta^2\,a_4(\eta) + \frac{2\delta^3}{\pi\varepsilon}\,a_3(\eta) + 3\delta^4\,a_2(\eta) \\
&\quad -\frac{12\delta^5}{5\pi}\,a_1(\eta) + \frac{\delta^6}{5}\,a_0(\eta)\,.
\end{aligned}
$$

We may then take the η-gauge as the Feynman gauge and consider $a_5(\xi)$ and $a_6(\xi)$ with accuracies $O(\varepsilon)$ and $O(\varepsilon^0)$, respectively. This yields:

$$
\begin{aligned}
a_5(\xi) &= a_5(1) - \frac{15}{2}\pi\,(\xi-1)^2\,a_3 + O(\varepsilon)\,, \\
a_6(\xi) &= a_6(1) + \frac{4(\xi-1)}{5\sqrt{\pi}\varepsilon}\,a_5(1) + \frac{2\sqrt{\pi}(\xi-1)^3}{\varepsilon}\,a_3 + O(\varepsilon^0)\,,
\end{aligned}
$$

where we took into account the fact that the finite part of a_3 is gauge-independent.

From these results, we see that the LKF transformation gives information about the ξ-dependence of $a_5(\xi)$ and $a_6(\xi)$, as expected. Some singularities may still be hidden in $a_6(1)$ and further understanding of the singular structure of $a_6(\xi)$ requires explicit 5- and 6-loop computations (at least in a specific gauge).

10. Summary II

We have examined the perturbative structure of the massless fermion propagator of quenched QED_3 up to four loops.

Our study was motivated by our recent publication where the gauge covariance of the fermion propagator of quenched QED_3 was studied using the LKF transformation in dimensional regularization ($d = 3 - 2\varepsilon$). This non-perturbative transformation revealed an interesting parity effect, whereby the contributions of odd orders, starting from the third one, to even orders are accompanied by singularities taking the form of poles, ε^{-1}, in dimensional regularization. In turn, even orders produce contributions to odd ones, starting from the third order, which are $\sim \varepsilon$.

Following arguments in favor of the IR (and ultraviolet) perturbative finiteness of massless quenched QED$_3$ (R.Jackiw, S.Templeton; 1981), (O.M. Del Cima, D.H.T.Franco, O.Piguet;2014), (N.Karthik, R.Narayanan;2017)

and therefore assuming the existence of a finite limit as $\varepsilon \to 0$, we concluded in the previous paper that, exactly in $d = 3$, all odd coefficients $a_{2t+1}(\xi)$ in perturbation theory, except a_1, should be exactly zero in any gauge.

This statement needed a check since analytical expressions for the fermion self-energy diagrams were known only at two-loop order.

This is what we have done in the present studying by computing the three- and four-loop corrections to the massless fermion propagator, $i.e.$, the coefficients $a_3(\xi)$ and $a_4(\xi)$, directly in the framework of perturbation theory. We found that $a_3(\xi)$ is finite and gauge-independent when $\varepsilon \to 0$. The coefficient $a_4(\xi)$ is, on the other hand, singular which violates the status of IR perturbative finiteness of massless quenched QED$_3$. The obtained singularity is such that all of its gauge-fixing dependent terms are entirely determined by lower order contributions in agreement with the properties of the LKF transformation.

In closing, let's note that the four-loop singularities were found to contribute to the coefficient $a_4(\xi)$ with a factor $\sim (1-\xi)$ and, thus, $a_4(\xi)$ is finite in the Feynman gauge. The reason for this intriguing effect is not clear at present and its elucidation requires additional research.