

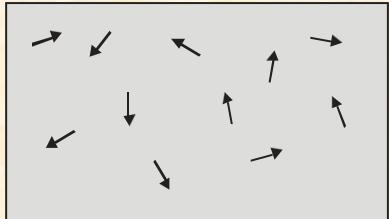
Spin distribution for the ‘t Hooft-Polyakov monopole in the geometric theory of defects

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Katanaev, Volovich Ann. Phys. 216(1992)1; ibid. 271(1999)203
Katanaev Theor.Math.Phys.135(2003)733; ibid. 138(2004)163
Phisics – Uspekhi 48(2005)675; Phys. Rev. D96(2017)84054
Katanaev. Universe 7(2021)256

Disclinations

Ferromagnets



$n^i(x)$ - unit vector field

n_0^i - fixed unit vector

$$n^i = n_0^j S_j^i(\omega)$$

$S_i^j \in \mathbb{SO}(3)$ - orthogonal matrix

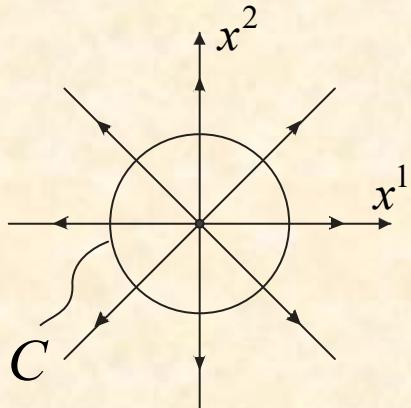
$\omega^{ij} = -\omega^{ji} \in \mathfrak{so}(3)$ - Lie algebra element (spin structure)

$$\omega_i = \frac{1}{2} \varepsilon_{ijk} \omega^{jk}$$

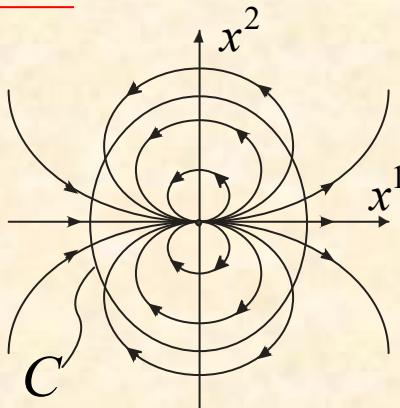
- rotational angle

ε_{ijk} - totally antisymmetric tensor ($\varepsilon_{123} = 1$)

Examples



$$\Theta = 2\pi$$



$$\Theta = 4\pi$$

$$\Omega^{ij} = \oint_C dx^\mu \partial_\mu \omega^{ij}$$

$\Theta_i = \varepsilon_{ijk} \Omega^{jk}$ - Frank vector
(total angle of rotation)

$$\Theta = \sqrt{\Theta^i \Theta_i}$$

Frank vector

$\omega^{ij}(x)$ - is not continuous !

$$\omega_{\mu}{}^{ij}(x) = \begin{cases} \partial_{\mu}\omega^{ij} & \text{- outside the cut} \\ \lim \partial_{\mu}\omega^{ij} & \text{- on the cut} \end{cases}$$

- SO(3)-connection
(continuous on the cut)

$$\Omega^{ij} = \oint dx^{\mu} \omega_{\mu}{}^{ij} = \iint dx^{\mu} \wedge dx^{\nu} (\partial_{\mu}\omega_{\nu}{}^{ij} - \partial_{\nu}\omega_{\mu}{}^{ij}) \quad \text{- the Frank vector}$$

$$R_{\mu\nu}{}^{ij} = \partial_{\mu}\omega_{\nu}{}^{ij} - \omega_{\mu}{}^{ik}\omega_{\nu k}{}^j - (\mu \leftrightarrow \nu) \quad \text{- curvature}$$

$$\boxed{\Omega^{ij} = \iint dx^{\mu} \wedge dx^{\nu} R_{\mu\nu}{}^{ij}} \quad \text{- definition of the Frank vector in the geometric theory}$$

Back to the spin structure: if $n \in \mathbb{R}^2$ then $\mathbb{SO}(3) \rightarrow \mathbb{SO}(2)$

Summary of the geometric approach (physical interpretation)

Media with dislocations and disclinations =

= \mathbb{R}^3 with a given Riemann-Cartan geometry

Independent variables $\begin{cases} e_\mu^i & \text{- triad field} \\ \omega_\mu^{ij} & \text{- SO(3)-connection} \end{cases}$

$$T_{\mu\nu}^i = \partial_\mu e_\nu^i - \omega_\mu^{ij} e_{\nu j} - (\mu \leftrightarrow \nu) \quad \text{- torsion} \quad (\text{surface density of the Burgers vector})$$

$$R_{\mu\nu}^{ij} = \partial_\mu \omega_\nu^{ij} - \omega_\mu^{ik} \omega_{\nu k}^j - (\mu \leftrightarrow \nu) \quad \text{- curvature} \quad (\text{surface density of the Frank vector})$$

Elastic deformations: $R_{\mu\nu}^{ij} = 0, \quad T_{\mu\nu}^i = 0$

Dislocations: $R_{\mu\nu}^{ij} = 0, \quad T_{\mu\nu}^i \neq 0$

Disclinations: $R_{\mu\nu}^{ij} \neq 0, \quad T_{\mu\nu}^i = 0$

Dislocations and disclinations: $R_{\mu\nu}^{ij} \neq 0, \quad T_{\mu\nu}^i \neq 0$

't Hooft-Polyakov monopole

$$(x^\alpha) \in \mathbb{R}^{1,3}, \quad \alpha = 0, 1, 2, 3 \quad i, j = 1, 2, 3$$

$$\eta_{\alpha\beta} := \text{diag}(+ - --) \quad \delta_{ij} := \text{diag}(+++)$$

- Lorentz metric
- metric in target space

$$L = -\frac{1}{4} F^{\alpha\beta i} F_{\alpha\beta i} + \frac{1}{2} \nabla^\alpha \varphi^i \nabla_\alpha \varphi_i - \frac{1}{4} \lambda (\varphi^2 - a^2)^2$$

- the geometric model

$$F_{\alpha\beta}{}^i := \partial_\alpha A_\beta{}^i - \partial_\beta A_\alpha{}^i + e A_\alpha{}^j A_\beta{}^k \epsilon_{jk}{}^i$$

$A_\alpha{}^i$ - $\mathbb{SU}(2)$ gauge field

$\varphi = (\varphi^i) \in \mathbb{R}^3$ - triplet of scalar fields in adjoint representation of $\mathbb{SU}(2)$ group

$\nabla_\alpha \varphi^i := \partial_\alpha \varphi^i + e A_\alpha{}^j \varphi^k \epsilon_{jk}{}^i$ - covariant derivative

$e \in \mathbb{R}$, $\lambda, a > 0$ - coupling constants

$$\mathbb{SO}(3) = \frac{\mathbb{SU}(2)}{\mathbb{Z}_2}$$

Static solutions

$$A_\alpha^i = 0, \quad \varphi^i = \text{const}, \quad \varphi^2 = a^2 \quad \text{- vacuum solution}$$

$$E = \frac{1}{4} F^{\mu\nu i} F_{\mu\nu i} + \frac{1}{2} \nabla^\mu \varphi^i \nabla_\mu \varphi_i + \frac{1}{4} \lambda (\varphi^2 - a^2)^2 \quad \text{- the energy}$$

(1+3) decomposition:

$$(x^\alpha) = (x^0, x^\mu) = (x^0, \mathbf{x}), \quad (A_\alpha^i) := (A_0^i, A_\mu^i), \quad \mu := 1, 2, 3$$

$$A_\alpha^i = A_\alpha^i(\mathbf{x}), \quad \varphi^i(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3 \quad \text{- static solutions}$$

$$A_0^i = 0 \quad \text{- additional requirement}$$

$$\nabla_\nu F^{\nu\mu}_i + e(\nabla^\mu \varphi^j) \varphi^k \varepsilon_{ijk} = 0$$

- equilibrium equations

$$-\nabla^\mu \nabla_\mu \varphi_i - \lambda(\varphi^2 - a^2) \varphi_i = 0$$

$$A_0^i \Big|_{r=\infty} = 0, \quad \varphi^2 \Big|_{r=\infty} = a^2$$

- boundary conditions

Spherically symmetric solutions

Assumption: $\mathbb{SU}(2)$ acts simultaneously in coordinate and target spaces

$$A_\mu^i = \frac{\epsilon_\mu^{ij} x_j}{er^2} (K - 1), \quad \varphi^i = \frac{x^i}{er^2} H \quad \text{- spherically symmetric ansatz}$$

$K(r), H(r)$ - unknown functions of radius r

$$r^2 K'' = K(K^2 + H^2 - 1)$$

- nonlinear system of equations

$$r^2 H'' = 2HK^2 + \lambda \left(\frac{H^2}{e^2} - a^2 r^2 \right)$$

The Bogomol'nyi-Prasad-Sommerfield (1975) solution (for $\lambda = 0$):

$$K = \frac{ear}{\operatorname{sh}(ear)}, \quad H = \frac{ear}{\operatorname{th}(ear)} - 1$$

Disclinations and dislocations

$\mathbb{SU}(2) \rightarrow \mathbb{SO}(3)$ acts simultaneously in coordinate and target spaces

Media without elastic stresses: \mathbb{R}^3 , $g_{\mu\nu} = \delta_{\mu\nu}$, $e_\mu^i = \delta_\mu^i$

$$E = \frac{1}{4} F^{\mu vi} F_{\mu vi} + \frac{1}{2} \nabla^\mu \varphi^i \nabla_\mu \varphi_i + \frac{1}{4} \lambda (\varphi^2 - a^2)^2$$

- the free energy

φ^i - sources for defects

$$\omega_\mu^{ij} = A_\mu^k \varepsilon_k^{ij} = (\delta_\mu^i x^j - \delta_\mu^j x^i) \frac{K-1}{er^2}$$

- spherically symmetric solution

$$R_{\mu\nu}^k = \varepsilon_{\mu\nu}^k \frac{K'}{er} - \frac{\varepsilon_{\mu\nu}^j x_j x^k}{er^3} \left(K' - \frac{K^2 - 1}{r} \right)$$

- continuous distribution
of disclinations and dislocations

$$T_{\mu\nu}^k = (\delta_\mu^k x_\nu - \delta_\nu^k x_\mu) \frac{K-1}{er^2}$$

Spin distribution

$n_0 = (n_{0i}) := (0, 0, 1)$ - unit vector at the origin along z axis

It is parallelly transported to a point x_1 along a path $x(t)$, $t \in [0, 1]$ $x(0) = 0$,

$$x(1) = x_1$$

$n_i(x_1) = S_i^j n_{0j} = P \exp \left(\int_0^1 dt \dot{x}^\mu \omega_\mu \right)_i^j n_{0j}$ - path ordered exponent

$S_i^j(x_0, x_1)$ is the rotational matrix

The first case $x(t) = (x_1^1 t, x_1^2 t, x_1^3 t) \Rightarrow \dot{x}^\mu = (x_1^1, x_1^2, x_1^3)$

- the ray from the origin to the point $x_1 = (x_1^1, x_1^2, x_1^3)$

$$\dot{x}^\mu \omega_\mu^{ij} = (x_1^j x_1^i - x_1^i x_1^j) \frac{K-1}{er^2} = 0 \Rightarrow S_i^j = \delta_i^j$$

The vector n_0 is not rotated

The second case $x(t) = (y^1 t, y^2 t, z) \Rightarrow \dot{x}^\mu = (y^1, y^2, 0)$

- the path from the origin to the point $x_1 := (y^1, y^2, z)$

Spin distribution 2

The integrand is $\dot{x}^\mu \omega_\mu^{ij}(t) = B^k(t) \varepsilon_k^{ij}$

where $B^1 = -y_2 z t \frac{K-1}{er^2}, \quad B^2 = y_1 z t \frac{K-1}{er^2}, \quad B^3 = 0$

The integrands commute $[\dot{x}^\mu \omega_\mu(t_1), \dot{x}^\nu \omega_\nu(t_2)] = 0 \quad \forall t_1, t_2$

The rotational matrix is $S_i^j = \exp(C^k \varepsilon_{ki}^j)$ where

$C^1 = -\frac{y_2 z}{e\rho^2} I, \quad C^2 = -\frac{y_1 z}{e\rho^2} I, \quad C^3 = 0$

$\rho^2 := y_1^2 + y_2^2 \quad I := \rho^2 \int_0^1 dt t \frac{K-1}{\rho^2 t^2 + z^2}$

Spin distribution 3

The rotational matrix for the Bogomol'nyi-Prasad-Sommerfield solution is

$$S = \begin{pmatrix} \frac{y_2^2}{\rho^2} + \frac{y_1^2}{\rho^2} \cos C & -\frac{y_1 y_2}{\rho^2} (1 - \cos C) & -\frac{y_1}{\rho} \sin C \\ -\frac{y_1 y_2}{\rho^2} (1 - \cos C) & \frac{y_1^2}{\rho^2} + \frac{y_2^2}{\rho^2} \cos C & -\frac{y_2}{\rho} \sin C \\ \frac{y_1}{\rho} \sin C & \frac{y_2}{\rho} \sin C & \cos C \end{pmatrix}$$

where $C^2 := C^k C_k = \frac{z^2}{e^2 \rho^2} I^2$

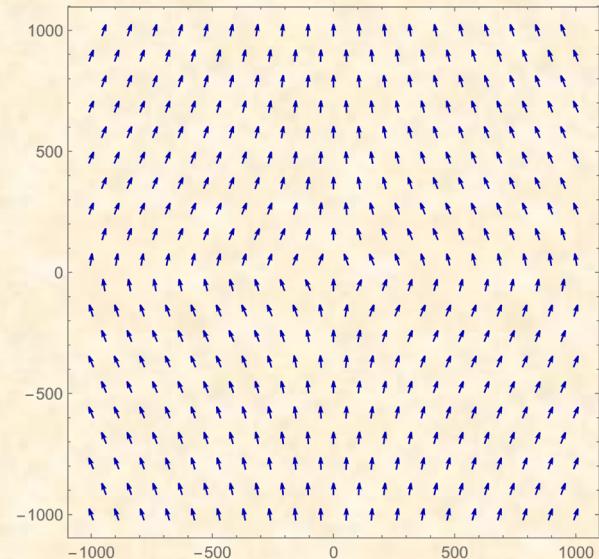
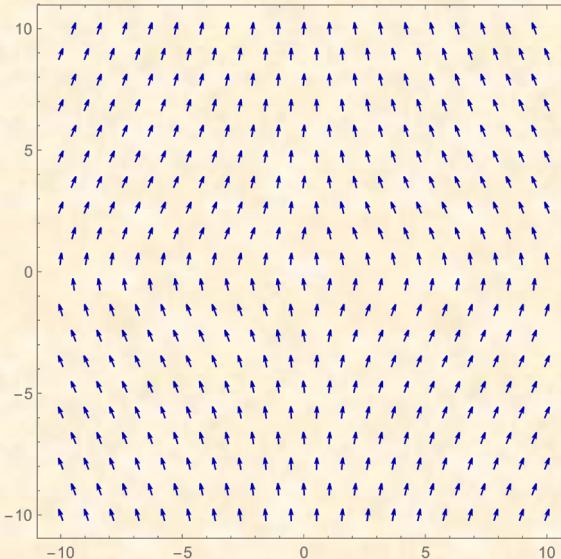
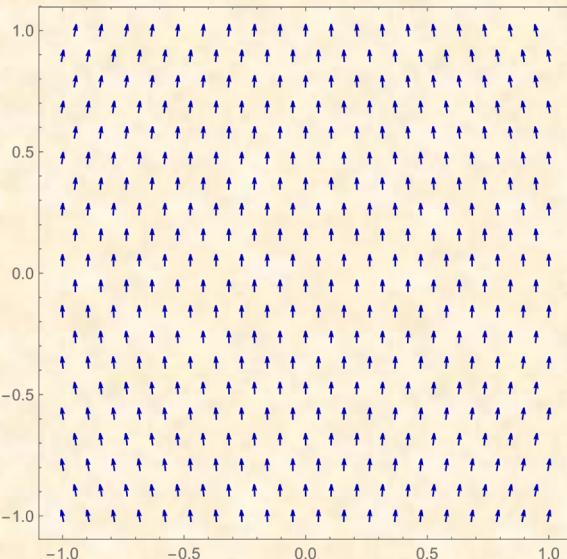
$$I = \ln \left(\frac{z}{\sqrt{\rho^2 + z^2}} \frac{\tanh \sqrt{\rho^2 + z^2}}{\tanh z} \right)$$

Spin distribution 4

The asymptotic: $S(0)=1 \quad S(\infty)=1$

There is rotational symmetry about z axis.

Therefore we can put $y_2 = 0$ to visualize the spin distribution.



The spin distribution is self-similar at large distances.

$$C \approx \frac{z}{ey_1} \ln \left(\frac{z}{\sqrt{y_1^2 + z^2}} \right)$$

Conclusion

- 1) All 't Hooft-Polyakov type solutions have straightforward physical interpretation in the geometric theory of defects.
- 2) They describe continuous distribution of dislocations and disclinations in elastic media.
- 3) The Bogomol'nyi-Prasad-Sommerfield solution is self-similar at large distances.