Superfield effective action in six-dimensional supergauge theories

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Goals

Brief review of approach to study quantum effective action in $6D$ supersymmetric gauge theories. Demonstrating some of the wonders of extended supersymmetry in higher dimensions.

- Construction of the background field methods and deriving the manifestly supersymmetric and gauge invariant effective action for $6D, \mathcal{N} = (1, 0)$ gauge multiplet coupled to hypermultiplet
- Calculation of the one-loop off-shell divergences in vector multiplet and hypermultiplet sectors for arbitrary $\mathcal{N} = (1, 0)$ gauge theory
- Analysis of divergences in the $\mathcal{N} = (1, 1)$ SYM theory and prove that the one-loop off-shell divergence are completely absent
- Calculation of finite part of one-loop low-energy effective action
- Calculation of two-loop off-shell divergences in the $\mathcal{N} = (1, 1)$ SYM theory: gauge multiplet sector
- On-shell finiteness of the $\mathcal{N} = (1, 1)$ SYM theory at two loops in gauge multiplet sector
- Analysis of two-loop divergences in the $\mathcal{N} = (1, 1)$ SYM theory: hypermultiplet dependence

Based on a series of ten papers published during last five years in collaboration with E.A. Ivanov, B.M. Merzlikin and K.V. Stepanayatz in the journals JHEP, Nucl.Phys. B, Phys.Lett. B.

Modern interest to study higher dimensional supersymmetric field theories is associated with superstring theory.

Specific feature of the superstring theory is existence of so called D-branes which are the $D+1$ dimensional surfaces in the ten-dimensional space-time. In the low-energy limit the D-brane is associated with $D+1$-dimensional supersymmetric gauge theory. Therefore, study of low-energy limit of superstring theory can be related to (extended) supersymmetric field theory in various dimensions.

In this sense, $D3$-brane is associated with $D4$, $\mathcal{N}=4$ SYM theory, it seems it is a most symmetric quantum field theoretical model.

Basic Motivations. Study of quantum field models with large number of symmetries

- Explicit symmetries: gauge symmetry, global symmetries, supersymmetries.
- Quantization procedure with preservation of all explicit symmetries.
- Perturbation theory with preservation of all explicit symmetries.
- Hidden (on-shell) symmetries. Preservation of hidden symmetries.
- Divergences, renormalization and effective actions.
- Construction of the new extended supersymmetric invariants as the the quantum contributions to effective action

Some problems of higher dimensional supersymmetric gauge theories.

1. Describing the quantum structure of six-dimensional supersymmetric gauge theories dimensionally reduced from superstrings (initiated by N. Seiberg, E. Witten, 1996; N. Seiberg, 1997).

2. Description of the interacting multiple $M5$-branes.

- Hypothetic M-theory is characterized by two extended objects: $M2$-brane and $M5$-brane in eleven dimensional space.
- The field description of interacting multiple $M2$-branes is given by Bagger-Lambert-Gustavsson (BGL) theory which is $3D$, $\mathcal{N}=8$ supersymmetric gauge theory.
- Lagrangian description of the interacting multiple $M5$-branes is not constructed so far.

3. Problem of miraculous cancelation of some on-shell divergences in higher dimensional maximally supersymmetric gauge theories (theories with 16 supercharges). All these theories are non-renormalizable by power counting.

- Field limit of superstring amplitude shows that $6D, \mathcal{N} = (1,1)$ SYM theory is on-shell finite at one-loop (M.B. Green, J.H. Schwarz, L. Brink, 1982).
- Analysis based on on-shell supersymmetries, gauge invariance and field redefinitions (P.S. Howe, K.S. Stelle, 1984, 2003; G. Bossard, P.S. Howe, K.S. Stelle, 2009).
- Direct one-loop and two-loop component calculations (mainly in bosonic sector and mainly on-shell) (E.S. Fradkin, A.A. Tseytlin, 1983; N. Marcus, A. Sagnotti, 1984, 1985.)
- Direct calculations of scattering amplitudes in $6D, \mathcal{N} = (1,1)$ theory up to five loops and in $D8, 10$ theories up to four loops (L.V. Bork, D.I. Kazakov, M.V. Kompaniets, D.M. Tolkachev, D.E. Vlasenko, 2015).

  Results: On-shell divergences in maximally extended $6D$ SYM theory start at three loops. One-shell divergences in $8D$ and $10D$ SYM theories start at one loop.

  What will be off-shell?

Reminder of the notions

- Supersymmetric field models can be formulated in terms of conventional bosonic and fermionic fields. Component approach.
- In some cases the supersymmetric field models can be formulated in terms of superfields. A superfield depends on space-time coordinates x and some number of anicommuting (Grassmann) coordinates θ. The coefficients of expansion of superfield in anticommuting coordinates are the conventional bosonic and fermionic fields of supermultiplet.
- Advantage of component formulation: close relation with conventional field theory, convenient in classical field theory and to calculate the scattering amplitudes. Disadvantage: supersymmetry is not manifest.
- Advantage of superfield formulation: manifest supersymmetry, convenient in quantum field theory to study off-shell effects, simple proof of the non-renormalization theorems.
- Problem of unconstrained formulation. Solution for $4D, \mathcal{N} = 2$ and $6D, \mathcal{N} = (1, 0)$ gauge theories is given within harmonic superspace approach. Very simple prove that $4D$, $\mathcal{N} = 4$ SYM is completely finite quantum field theoretical model.

$6D$ superalgebra is described by two independent supercharges. The simplest representations corresponds to $\mathcal{N} = (1,0)$ and $\mathcal{N} = (0,1)$ supersymmetries. In this sense, the maximally extended rigid supergauge theory is the $\mathcal{N} = (1,1)$ SYM theory.

$6D, \mathcal{N} = (1,1)$ SYM theory possesses some properties close or analogous to $4D, \mathcal{N} = 4$ SYM theory.

- The $4D, \mathcal{N} = 4$ SYM theory is maximally extended rigid supergauge theory in four dimensions. The $6D, \mathcal{N} = (1,1)$ SYM theory is is maximally extended rigid supergauge theory in six dimensions
- The $6D, \mathcal{N} = (1,1)$ SYM theory is anomaly free as well as the $4D, \mathcal{N} = 4$ SYM theory and satisfies some non-renormaization theorems
- The $6D, \mathcal{N} = (1,1)$ SYM theory can be formulated in harmonic superspace as well as the $4D, \mathcal{N} = 4$ SYM theory
- In such a formulation, the $6D, \mathcal{N} = (1,1)$ SYM theory possesses the manifest $\mathcal{N} = (1,0)$ supersymmetry and additional hidden $\mathcal{N} = (0,1)$ supersymmetry analogous to $4D, \mathcal{N} = 4$ SYM theory where there is the manifest $\mathcal{N} = 2$ supersymmetry and additional hidden $\mathcal{N} = 2$ supersymmetry

Harmonic superspace

Basic references:

$4D$

A.Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky, E. Sokatchev, 1985.
A.Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, Harmonic Superspace, 2001.

General purpose: to formulate $\mathcal{N}=2$ models in terms of unconstrained $\mathcal{N}=2$ superfields. General idea: to use the parameters $u^{\pm i}(i=1,2)$ (harmonics) related to $SU(2)$ automorphism group of the $\mathcal{N}=2$ superalgebra and parameterizing the 2-sphere,
$u^{+i}u_i^- = 1$
It allows to introduce the $\mathcal{N}=2$ superfields with the same number of anticommuting coordinates as in case of the $\mathcal{N}=1$ supersymmetry. Prices for this are the extra bosonic variables, harmonics $u^{\pm i}$.

$6D$

P.S. Howe, K.S. Stelle, P.C. West, 1985.
B.M. Zupnik, 1986; 1999.
G. Bossard, E. Ivanov, A. Smilga, JHEP, 2015.

$(1,0)$ harmonic superspace

- $USp(2) \sim SU(2)$. The same harmonics $u^{\pm i}$, $i=1,2$ as in $4D,\mathcal{N}=2$ supersymmetry.
- Harmonic $6D,(1,0)$ superspace with coordinates $Z=(x^M,\theta^a_i,u^{\pm i})$, $a=1,2,3,4$.
- Analytic basis $Z_{(an)}=(x^M_{(an)},\theta^{\pm a},u^{\pm}_i)$,
  $x^M_{(an)}=x^M+\frac{i}{2}\theta^{-a}(\gamma^M)_{ab}\theta^{+b}, \quad \theta^{\pm a}=u^{\pm}_i\theta^{ai}$.
  The coordinates $\zeta=(x^M_{(an)},\theta^{+a},u^{\pm}_i)$ form a subspace closed under $(1,0)$ supersymmetry
- The harmonic derivatives

$$D^{++}=u^{+i}\frac{\partial}{\partial u^{-i}}+i\theta^{+}\ \partial\!\!\!/\theta^{+}+\theta^{+a}\frac{\partial}{\partial\theta^{-a}},$$

$$D^{--}=u^{-i}\frac{\partial}{\partial u^{+i}}+i\theta^{-}\ \partial\!\!\!/\theta^{-}+\theta^{-a}\frac{\partial}{\partial\theta^{+a}},$$

$$D^{0}=u^{+i}\frac{\partial}{\partial u^{+i}}-u^{-i}\frac{\partial}{\partial u^{-i}}+\theta^{+a}\frac{\partial}{\partial\theta^{+a}}-\theta^{-a}\frac{\partial}{\partial\theta^{-a}}$$

- Spinor derivatives in the analytic basis

$$D^+_a=\frac{\partial}{\partial\theta^{-a}}, \quad D^-_a=-\frac{\partial}{\partial\theta^{+a}}-2i\partial_{ab}\theta^{-b}, \quad \{D^+_a,D^-_b\}=2i\partial_{ab}$$

Hypermultiplet in harmonic superspace: off-shell Lagrangian formulation

- Off-shell hypermultiplet is described by the analytic superfield $q^+_A(\zeta, u)$, $D^+_a q^+_A(\zeta, u) = 0$, satisfying the reality condition $\widetilde{(q^{+A})} \equiv q^+_A = \varepsilon_{AB} q^{+B}$. Pauli-Gürsey indices $A, B = 1, 2$
- Off-shell hypermultiplet harmonic superfield contains infinite set of auxiliary fields which vanish on-shell due to the equations of motion

$$D^{++} q^+(\zeta, u) = 0$$

- The equations of motion follow from the action

$$S_{HYPER} = -\frac{1}{2} \int d\zeta^{(-4)} du \; q^{+A} D^{++} q^+_A$$

  Here $d\zeta^{(-4)} = d^6 x d^4 \theta^+$.
- On-shell the hypermultiplet is described by the scalar field $f^i(x)$ and the spinor field $\psi_a(x)$ satisfy the equations $\Box f^i = 0, \partial^{ab}\psi_b = 0$ (2 bosonic + 2 fermionic complex degrees of freedom.)

The $\mathcal{N}=(1,0)$ non-Abelian vector multiplet in $6D$, $\mathcal{N}=(1,0)$ harmonic superspace

- Harmonic covariant derivative

$$\nabla^{++} = D^{++} + iV^{++}$$

  Connection V^{++}, takes the values in the Lie algebra of the gauge group, this is an unconstrained analytic potential of the $6D, \mathcal{N}=(1,0)$ SYM theory.
- On-shell contents: $V^{++} = \theta^{+a}\theta^{+b}A_{ab} + 2(\theta^{+})_a\lambda^{-a}$, A_{ab} is a vector field, $\lambda^{-a} = \lambda^{ai}u^{-}_i, \lambda^{ai}$ is a spinor field.
- The superfield action of $6D, \mathcal{N}=(1,0)$ SYM theory is written in the form

$$S_{SYM} = \frac{1}{f^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\mathrm{tr}\int d^{14}z du_1 \dots du_n \frac{V^{++}(z,u_1)\dots V^{++}(z,u_n)}{(u^+_1 u^+_2)\dots(u^+_n u^+_1)}$$

  Here f is the dimensional coupling constant ($[f]=-1$)
- Gauge transformations

$$V^{++\prime} = -ie^{i\lambda}D^{++}e^{-i\lambda} + e^{i\lambda}V^{++}e^{-i\lambda}, \qquad q^{+\prime} = e^{i\lambda}q^{+}$$

Theory of $\mathcal{N}=(1,0)$ non-Abelian vector multiplet coupled to hypermultiplet

- Action

$$S[V^{++},q^{+}] = \frac{1}{f^2}\sum_{n=2}^{\infty}\frac{(-i)^n}{n}\mathsf{tr}\int d^{14}z\,du_1\ldots du_n\,\frac{V^{++}(z,u_1)\ldots V^{++}(z,u_n)}{(u_1^+u_2^+)\ldots(u_n^+u_1^+)}$$

$$-\int d\zeta^{-4}du\tilde{q}^{+}\nabla^{++}q^{+}$$

- Harmonic covariant derivative

$$\nabla^{++} = D^{++} + iV^{++}$$

- Equations of motion

$$\frac{1}{2f^2}F^{++} - i\tilde{q}^{+}q^{+} = 0\,, \qquad \nabla^{++}q^{+} = 0\,.$$

$$F^{++} = (D^{+})^4V^{--},\ D^{++}V^{--} - D^{--}V^{++} + i[V^{++},V^{--}] = 0$$

$\mathcal{N}=(1,1)$ **SYM theory**

$\mathcal{N}=(1,1)$ SYM theory can be formulated in terms of $\mathcal{N}=(1,0)$ harmonic superfields as the $\mathcal{N}=(1,0)$ vector multiplet coupled to hypermultiplet in adjoint representation. The theory is manifestly $\mathcal{N}=(1,0)$ supersymmetric and possesses the extra hidden $\mathcal{N}=(0,1)$ supersymmetry.

- Action

$$S[V^{++},q^+] = S_{SYM}[V^{++}] + S_{HYPER}[q^+,V^{++}]$$

- The action is manifestly $\mathcal{N}=(1,0)$ supersymmetric.
- The action is invariant under the transformations of extra hidden $\mathcal{N}=(0,1)$ supersymmetry

$$\delta V^{++} = \epsilon^+ q^+, \qquad \delta q^+ = -(D^+)^4(\epsilon^- V^{--})$$

where the transformation parameter $\epsilon^{\pm}_A = \epsilon_{aA}\theta^{\pm A}$.

General strategy

- We start with harmonic superfield formulations of vector multiplet coupled to hypermultiplet.
- Effective action is formulated in the framework of the harmonic superfield background field method. It provides manifest $\mathcal{N} = (1,0)$ supersymmetry and gauge invariance of effective action under the classical gauge transformations.
- Effective action can be calculated on the base of superfield proper-time technique. It provides preservation of manifest $\mathcal{N} = (1,0)$ supersymmetry and manifest gauge invariance at all steps of calculations.
- The effective action can also be calculated perturbatively on the base of Feynman diagrams in superspace (supergraph technique).
- One-loop analysis. We study the model, where the $\mathcal{N} = (1,0)$ vector multiplet interacts with hypermultiplet in the arbitrary representation of the gauge group. Then, we assume in the final result for one-loop divergences, that this representation is adjoint what corresponds to $\mathcal{N} = (1,1)$ SYM theory. Finite one-loop effective action without renormalization.
- Two-loop analysis. All the possible divergences can be listed, using the the superfield power counting and then they can be calculated in the framework of the background field method.

Aim: construction of gauge invariant effective action
Realization

- The superfields V^{++}, q^{+} are splitting into the sum of the background superfields V^{++}, Q^{+} and the quantum superfields v^{++}, q^{+}

$$V^{++} \to V^{++} + f v^{++}, \qquad q^{+} \to Q^{+} + q^{+}$$

- The action is expending in a power series in quantum fields. As a result, we obtain the initial action $S[V^{++}, q^{+}]$ as a functional $\tilde{S}[v^{++}, q^{+}; V^{++}, Q^{+}]$ of background superfields and quantum superfields.
- The gauge-fixing function are imposed only on quantum superfiled

$$\mathcal{F}^{(+4)}_{\tau} = D^{++} v^{++}_{\tau} = e^{-ib} (\nabla^{++} v^{++}) e^{ib} = e^{-ib} \mathcal{F}^{(+4)} e^{ib} \ ,$$

where $b(z)$ is a background-dependent gauge bridge superfield and τ means τ-frame. In the non-Abelian gauge theory, the gauge-fixing function is background-dependent.
- Faddev-Popov procedure is used. Ones obtain the effective action $\Gamma[V^{++}, Q^{+}]$ which is gauge invariant under the classical gauge transformations. Background field construction in the case under consideration is analogous to one in $4D, \mathcal{N}=2$ SYM theory (I.L.B, E.I. Buchbinder, S.M. Kuzenko, B.A. Ovrut, 1998).

Background field method

- The effective action $\Gamma[V^{++}, Q^{+}]$ is written in terms of path integral

$$e^{i\Gamma[V^{++},Q^{+}]} = Det^{1/2}\,\widehat{\Box} \int \mathcal{D}v^{++}\,\mathcal{D}q^{+}\,\mathcal{D}\mathbf{b}\,\mathcal{D}\mathbf{c}\,\mathcal{D}\varphi\; e^{iS_{quant}[v^{++},q^{+},\mathbf{b},\mathbf{c},\varphi,V^{++},Q^{+}]}$$

- The quantum action S_{quant} has the structure

$$S_{quant} = S[V^{++} + fv^{++}, Q^{+} + q^{+}] - S[V^{++}, Q^{+}] +$$

$$+S_{GF}[v^{++}, V^{++}] + S_{FP}[\mathbf{b}, \mathbf{c}, v^{++}, V^{++}] + S_{NK}[\varphi, V^{++}].$$

- Gauge fixing term $S_{GF}[v^{++}, V^{++}]$, Faddeed-Popov ghost action $S_{FP}[\mathbf{b}, \mathbf{c}, v^{++}, V^{++}]$, Nielsen-Kalosh ghost action $S_{NK}[\varphi, V^{++}]$
- Operator $\widehat{\Box}$

$$\widehat{\Box} = \eta^{MN}\nabla_M \nabla_N + W^{+a}\nabla^{-}_{a} + F^{++}\nabla^{--} - \frac{1}{2}(\nabla^{--}F^{++})$$

- All ghosts are the analytic superfields

Diagonalization of the quadratic part of action

Quadratic part of quantum action schematically is

$$S_2 + S_{GF} = -\frac{1}{2}\mathrm{tr}\int d\zeta^{-4} v^{++}\,\widehat{\Box}\,v^{++} - \frac{1}{2}\mathrm{tr}\int d\zeta^{-4} q^{+}\nabla^{++}q^{++} + \text{mixed terms},$$

where the mixed terms contain product of two quantum fields v^{++} and q^{+} and one background hypermulltiplet Q^{++}. That is

$$\text{mixed terms} \sim \mathrm{tr}\int d\zeta^{-4} Q^{+} v^{++} q^{+}.$$

To define the separate propagator for v^{++} and separate propagator for q^{+} we should eliminate somehow the mixed terms. It can be done by the different ways. We use the shift of quantum hypermultiplet schematically as

$$q^{+} \to q^{+} + \int d\zeta^{-4} G^{(1,1)} v^{++} Q^{+},$$

all indices are suppressed. Here $G^{(1,1)}$ is the Green function for the operator ∇^{++}. Such a transformation diagonalizes the expression $S_2 + S_{gh}$, however leads to the additional non-local in harmonics vertices. All these vertices contain cubic and quartic terms in v^{++} and will be needed starting from two loops. Besides, quadratic in v^{++} part of action will get the additional contribution quadratic in background hypermultiplet Q^{+}.

Illustration. One-loop approximation. Only quadratic in quantum fields and ghosts terms are taken into account in the path integral for effective action. It gives after some transformation the one-loop contribution $\Gamma^{(1)}[V^{++}, Q^{+}]$ to effective action in terms of formal functional determinants in analytic subspace of harmonic superspace

$$\Gamma^{(1)}[V^{++}, Q] = \frac{i}{2} Tr_{(2,2)} \ln[\delta^{(2,2)} \,\widehat{\Box}^{AB} - 2f^2 Q^{+\,m} (T^A G_{(1,1)} T^B)_m{}^n Q^+_n] -$$

$$-\frac{i}{2} Tr_{(4,0)} \ln \widehat{\Box} - i Tr \ln (\nabla^{++})^2_{\mathrm{Adj}} + \frac{i}{2} Tr \ln (\nabla^{++})^2_{\mathrm{Adj}} + i Tr \ln \nabla^{++}_{\mathrm{R}}$$

As usual, $TrlnO = lnDetO$, Tr means the functional trace in analytic subspace and matrix trace.
$(T^A)_m{}^n$ are are generators of the representation for the hypermultiplet.
The $G_{(1,1)}$ is the Green function for the operator ∇^{++}.
Index A numerates the generators, $V^{++} = V^{++A} T^A$. Operator $\widehat{\Box}$ acts on the components V^{++A} as $(\widehat{\Box} V^{++})^A = \widehat{\Box}^{AB} V^{++B}$
Adj and R mean that the corresponding operators are taken in the adjoint representation and in the representation for hypermultiplet.

Superfield Feynman diagrams (supergraphs)

- Perturbation theory can be given in terms of Feynman diagrams formulated in superspace
- Vector multiplet propagator

$$G^{(2,2)}(1|2) = -2\frac{(D_1^+)^4}{\Box_1}\delta^{14}(z_1 - z_2)\delta^{(-2,2)}(u_1, u_2)$$

- Hypermultiplet propagator

$$G^{(1,1)}(1|2) = \frac{(D_1^+)^4(D_2^+)^4}{\Box_1}\frac{\delta^{14}(z_1 - z_2)}{(u_1^+ u_2^+)^3}$$

- Ghost propagators have the analogous structure
- Superspace delta-function

$$\delta^{14}(z_1 - z_2) = \delta^6(x_1 - x_2)\delta^8(\theta_1 - \theta_2)$$

- The vertices are taken from the superfield action as usual

Superficial degree of divergence ω-total degree in momenta in loop integral.

- Consider the L loop supergraph G with P propagators, V vertices, N_Q external hypermultiplet legs, and the arbitrary number of vector multiplet external legs.
- One can prove that due to structure of the propagators and the Grassmann delta-functions in the propagators, any supergraph for effective action can be written through the integrals over full $\mathcal{N}=(1,0)$ superspace and contains only a single integral over $d^8\theta$ (non-renormalization theorem)!
- Mass dimensions: $[x]=-1$, $[p]=1$, $[\int d^6p]=6$, $[\theta]=-\frac{1}{2}$, $[\int d^8\theta]=4$, $[q^+]=1$, $[V^{++}]=0$.
- After summing all dimensions and using some identities, power counting gives $\omega(G)=2L-N_Q-\frac{1}{2}N_D$
- N_D is a number of spinor derivarives acting on external lines
- A number of space-time derivatives in the counterterms increases with L. The theory is multiplicatively non-renormalizable.
- Example: one loop approximation $\omega_{1-loop}(G)=2-N_Q$
- The possible divergences correspond to $\omega_{1-loop}=2$ and $\omega_{1-loop}=0$

Calculations of ω are analogous to ones in $4D, \mathcal{N}=2$ gauge theory (ILB, S.M. Kuzenko, B.A. Ovrut, 1998).

Possible candidate for one-loop divergences can be constructed on the based of dimensions, gauge invariance and $\mathcal{N}=(1,0)$ supersymmetry in the form (G. Bossard, E. Ivanov, A. Smilga, 2015)

$$\Gamma^{(1)}_{div} = \int d\zeta^{(-4)}\, du \Big[c_1 (F^{++A})^2 + i c_2 F^{++A} (\widetilde{q}^{+})^m (T^A)_m{}^n (q^+)_n + c_3 \Big((\widetilde{q}^{+})^m (q^+)_m \Big)^2$$

Where c_1, c_2, c_3 are arbitrary real constants.
We consider a possible form of one-loop counterterms on the base of power counting in quadratic approximation in fields and then restore a complete result of the base of gauge invariance.

- Let $N_Q = 0$, so that ω=2, and we use the dimensional regularization. The corresponding counterterm has to be quadratic in momenta and given by the full $\mathcal{N}=(1,0)$ superspace integral. The only admissible possibility is

$$\Gamma^{(1)}_1 \sim \int d^{14}z du V^{--} \Box V^{++}$$

  After some transformation it coincides with first term in $\Gamma^{(1)}_{div}$, with divergent coefficient c_1 proportional to $1/\varepsilon$, where $\varepsilon = d - 6$ is a regularization parameter.

- Let $N_Q = 2$, $N_D = 0$ so that $\omega = 0$ and we use the dimensional regularization. The corresponding counterterm has to be momentum independent and given by the full $\mathcal{N} = (1,0)$ superspace integral. The only admissible possibility is

$$\Gamma_2^{(1)} \sim \int d^{14}z du Q^+ V^{--} Q^+$$

  After some transformation it coincides with second term in $\Gamma^{(1)}_{div}$, with divergent coefficient c_2 proportional to $1/\varepsilon$, where $\varepsilon = d - 6$ is a regularization parameter.

- Let $N_Q = 4$, $N_D = 0$ so that $\omega = -2$ and the corresponding supergraph is convergent. It means that $c_3 = 0$ in $\Gamma^{(1)}_{div}$. As a result, all one-loop $(q^+)^4$ possible contributions to effective action are finite.

Calculating the one-loop divergences of superfield functional determinants is carried out in the framework of proper-time technique (superfield version of Schwinger-De Witt technique). Such technique allows us to preserve the manifest gauge invariance and manifest $\mathcal{N} = (1,0)$ supersymmetry at all steps of calculations.

General scheme of calculations

- Proper-time representation

$$TrlnO \sim Tr \int_0^\infty \frac{d(is)}{(is)^{1+\varepsilon}} e^{isO_1} \delta(1,2)|_{2=1}$$

- Here s is the proper-time parameter and ε is a parameter of dimensional regularization.
- Typically the $\delta(1,2)$ contains $\delta^8(\theta_1 - \theta_2)$, which vanishes at $\theta_1 = \theta_2$
- Typically the operator O contains some number of spinor derivatives D^+_a, D^-_a which act on the Grassmann delta-functions $\delta^8(\theta_1 - \theta_2)$ and can kill them. Non-zero result will be only if all these δ-functions are killed.
- Only these terms are taking into account which have the pole $\frac{1}{\varepsilon}$ after integration over proper-time.

Results of calculations

$$\Gamma^{(1)}_{div}[V^{++}, Q^{+}] = \frac{C_2 - T(R)}{3(4\pi)^3\varepsilon} tr \int d\zeta^{(-4)} du (F^{++})^2 -$$

$$-\frac{2if^2}{(4\pi)^3\varepsilon} \int d\zeta^{(-4)} du \widetilde{Q}^{+m} (C_2 \delta_m{}^n - C(R)_m{}^n) F^{++} Q^{+}{}_n.$$

- The quantities $C_2, T(R), C(R)$ are defined as follows

$$tr(T^A T^B) = T(R)\delta^{AB}$$

$$tr(T^A_{Adj} T^B_{Adj}) = f^{ACD} f^{BCD} = C_2 \delta^{AB}$$

$$(T^A T^A)_m{}^n = C(R)_m{}^n.$$

- Results of calculations correspond to analysis done on the base of power counting. The coefficients c_1, c_2 are found. The coefficient $c_3 = 0$ as we expected.
- In $\mathcal{N} = (1,1)$ SYM theory, the hypermultiplet is in the same representation as the vector multiplet. Then $C_2 = T(R) = C(R)$. Then $\Gamma^{(1)}_{div}[V^{++}, Q^{+}] = 0$!

Since $\mathcal{N}=(1,1)$ theory is finite at one-loop, the one-loop effective action does not require renormalization.

- Purpose. Construction of the $6D$, $\mathcal{N}=(1,1)$ analogue of the Heisenberg-Euler effective action.
- One-loop effective action is given by the functional determinants of some differential operators action in analytic superfields in $\mathcal{N}=(1,0)$ harmonic superspace.
- The $\mathcal{N}=(1,1)$ gauge multiplet are described by the $\mathcal{N}=(1,0)$ gauge superfield and hypermultiplet in the adjoint representation.
- Low-energy approximation. Effective action depends on background superfields satisfying the classical equations of motion and slowly varying in space-time.
- The gauge group is assumed to be $SU(N)$. The background superfields align in a fixed direction in the Cartan subalgebra of su(N). Such a choice of background corresponds to the spontaneous symmetry breaking $SU(N)\to SU(N-1)\times U(1)$. The background superfields form an Abelian $\mathcal{N}=(1,1)$ vector multiplet. In bosonic sector it contains real vector field A_M and four real scalars, which can be denoted φ and $\varphi^{(i,j)}$, $i,j=1,2$.

- The Abelian vector field and four scalars in six dimensions describe the bosonic world-volume degrees of freedom of a single $D5$-brane.
- Effective action is calculated in the framework of the superfield proper-time technique for the background superfields under the above assumptions. The calculations can be fulfilled. Leading low-energy contribution looks like

$$\Gamma^{(1)} \sim \int d\zeta^{(-4)} du \frac{(W^+)^4}{\Omega^2}.$$

Here $(W^+)^4 = -\frac{1}{24}\epsilon_{abcd}W^{+a}W^{+b}W^{+c}W^{+d}$, W^{+a} is $\mathcal{N}=(1,0)$ gauge superfield strength, $W^{+a} = -\frac{i}{6}\epsilon^{abcd}D^+_b D^+_c D^+_d V^{--}$ and Ω is the background hypermultiplet.

- Effective action in bosonic sector

$$\Gamma^{(1)}_{bos} \sim \int d^6x \frac{F^4}{\varphi^2},$$

where $F^4 = F_{MN}F^{MN}F_{PQ}F^{PQ} - 4F^{MN}F_{MR}F^{RS}F_{SN}$.

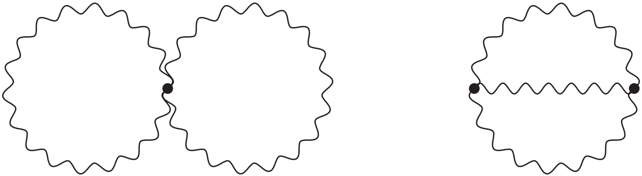
Procedure of calculations: gauge multiplet sector

- Two-loop divergences are calculated within background field method and proper-time technique like in one-loop case.
- We begin with only gauge multiplet background.
- Power counting shows that the only possible two-loop divergent contribution in the gauge superfield sector has the following structure

$$\Gamma^{(2)}_{\mathrm{div}}[V^{++}] = a \int d\zeta^{(-4)}\, du\, \mathrm{tr}\left(F^{++}\, \widehat{\Box}\, F^{++}\right)$$

  with some constant a, which diverges after removing a regularization.
- Within background field method, the two-loop contributions to superfield effective action are given by two-loop vacuum harmonic supergraphs with background field dependent lines.
- The background field dependent propagators (lines) are represented by proper-time integrals.
- Constant a in principle should have the following structure $a = \frac{d_1}{\varepsilon} + \frac{d_2}{\varepsilon^2}$ with arbitrary real parameters $d_1\, d_2$.

Two-loop supergraphs



Γ_{I} Γ_{II}

Figure: Two-loop Feynman supergraphs with gauge self-interactions vertices.

Γ_{III} Γ_{IV}

Figure: Two-loop Feynman supergraphs with hypermultiplet and ghosts vertices.

Procedure of calculations

- One can prove that the in the case under consideration the only two-loop divergent contribution comes from the '∞' supergraph.
- Contribution of this supergraph contains the product of two Green functions $G^{(2,2)}(z_1,u_1;z_2,u_2)$ at $z_1=z_2$.
- Divergent part of such Green function can be calculated and has the form $\sim \frac{1}{\varepsilon}F^{++}$. Therefore $G^{(2,2)}(z_1,u_1;z_2,u_2)|_{z_1=z_2} \sim \frac{1}{\varepsilon}F^{++} + g^{++}$ where g^{++} is some finite functional.
- It means that full two loop contribution of the '∞' supergraph looks like

$$b\int d\zeta^{(-4)}\,du\,(\frac{1}{\varepsilon}F^{++}+g^{++})\,\widehat{\square}\,(\frac{1}{\varepsilon}F^{++}+g^{++}).$$

  with some constant b. Therefore there are two types of contributions, one containing $\frac{1}{\varepsilon}$ and another one containing $\frac{1}{\varepsilon^2}$.
- The terms with simple pole $\frac{1}{\varepsilon}$ has the form $\sim \frac{1}{\varepsilon}F^{++}\,\widehat{\square}\,g^{++}$.
- However, the power counting tells us that the two loop divergence has the form $\sim F^{++}\,\widehat{\square}\,F^{++}$. Therefore, we must assume that $g^{++}=0$ or $g^{++}\sim F^{++}$.

Results of calculations in gauge multiplet sector

- Further we consider only the case $g^{++} = 0$.
- In this case, the divergent part of two-loop effective action has the form

$$\Gamma^{(2)}_{div} = \frac{8f^2}{(4\pi)^6 \varepsilon^2} (C_2)^2 \mathrm{tr} \int d\zeta^{(-4)}\, du\, F^{++} \,\widehat{\Box}\, F^{++},$$

  where $F^{++} = 0$ is the classical equation of motion in the case when the hypermultiplet is absent.
- Coefficient c_2 looks like

$$c_2 = \frac{8f^2}{(4\pi)^6 \varepsilon^2} (C_2)^2.$$

Field redefinition and cancelation of divergences

- Consider the off-shell transformation of the superfield V^{++} in the classical action $V^{++} \to V^{++} - a\,\widehat{\Box}\,F^{++}$.
- The corresponding transformation of the classical action is $\delta S = -a \int d\zeta^{(-4)}\,du\,\mathrm{tr}\,F^{++}\,\widehat{\Box}\,F^{++}$. That allows to cancel completely off-shell the two-loop divergence of the effective action in the gauge multiplet sector.
- Thus, one can state that the theory under consideration is off-shell finite at one- and two-loops (at least in gauge multiplet sector).

Hypermultiplet dependence of the two-loop divergences: indirect analysis.

- The hypermultiplet-dependent contribution to two-loop divergences can be obtained by the straightforward quantum computations of the two-loop effective action taking into account the hypermultiplet background.
- The general form of hypermultiplet dependent divergences can in principle be found without direct calculations, assuming the invariance of the effective action under the hidden $\mathcal{N}=(0,1)$ supersymmetry.
- The result has an extremely simple form

$$\Gamma^{(2)}_{\rm div}[V^{++}, q^{+}] = a \int d\zeta^{(-4)}\, du\, \mathrm{tr}\, E^{++}\, \widehat{\square}\, E^{++},$$

  where $E^{++} = F^{++} + \frac{i}{2}[q^{+A}, q^{+}_{A}]$ is the left hand side of classical equation of motion for vector multiplet superfield coupled to hypermultiplet.
- Two-loop divergences vanish on-shell as expected.

Summary

- The six-dimensional $\mathcal{N} = (1,0)$ supersymmetric theory of the non-Abelian vector multiplet coupled to hypermultiplet in the $6D$, $\mathcal{N} = (1,0)$ harmonic superspace was considered.
- Background field method in harmonic superspace was constructed .
- Manifestly supersymmetric and gauge invariant effective action, depending both on vector multiplet and hypermultiplet superfields, was formulated.
- Superficial degree of divergence is evaluated and structure of one- and two-loop counterterms was studied.
- An efficient manifestly gauge invariant and $\mathcal{N} = (1,0)$ supersymmetric technique to calculate the one- and two-loop contributions to effective action was developed. As an application of this technique, we found the one- and two-loop divergences of the theory under consideration.
- The same one-loop divergences have been calculated independently with help of $\mathcal{N} = (1,0)$ supergraphs.
- It is proved that $\mathcal{N} = (1,1)$ SYM theory is one-loop off-shell finite. There is no need to use the equations of motion to prove this property.
- One-loop low-energy effective action (some analogue of the $6D$, $\mathcal{N} = (1,1)$ Heisenberg-Euler effective action) was constructed.

Summary

- Two-loop divergences of the $6D, \mathcal{N} = (1,1)$ SYM theory were calculated in gauge multiplet sector. The hypermultiplet dependence of two-loop divergences were found on the base of hidden $\mathcal{N} = (0,1)$ supersymmetry.
- Two-loop divergences of the $6D, \mathcal{N} = (1,1)$ SYM theory were analyzed in gauge multiplet sector.
- The two-loop divergences in gauge multiplet sector are completely canceled by appropriate field redefinition in classical action.
- The hypermultiplet dependence of two-loop divergences were found on the base of hidden $\mathcal{N} = (0,1)$ supersymmetry.

Some open problems

- Some technical point. Direct calculation of the finite contribution g^{++}.
- Direct calculation of the two-loop divergences in the hypermultiplet sector on the base of background dependent supergraphs analysis (practically finished).
- Study of the finite contributions to effective action, construction of the low-energy effective Lagrangians.
- Study of the three-loop divergences.
- Construction of the higher covariant derivatives regularization in $6D, \mathcal{N} = (1,0)$ theories and its application to study effective action.
- Construction of the $6D, \mathcal{N} + (1,1)$ higher derivative theories and study finiteness of such theories.

THANK YOU VERY MUCH!