Free energy and entropy in Rindler and de Sitter space-times
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The method and setup

The effective action (S_{eff}) for the Gaussian theory is defined as:

$$Z = e^{i\mathcal{S}_{eff}} = \int d[\varphi] e^{iS[\varphi]} = \int d[\varphi] \, \exp\left[\frac{i}{2}\int d^d x\sqrt{g}\Big(\partial_\mu\varphi\partial_\mu\varphi - m^2\varphi^2\Big)\right] \qquad (1)$$

It is straightforward to see that

$$\frac{\partial}{\partial m^2}\log\int d[\varphi]e^{iS[\varphi]} = -\frac{i}{2}\int d^d x\sqrt{-g}\; G(x,x). \qquad (2)$$

This allows one to express the effective action via the Feynman propagator in the coincidence limit:

$$\mathcal{S}_{eff} = -\frac{1}{2}\lim_{M\to\infty}\int d^d x\sqrt{-g}\int_{M^2}^{m^2} d\bar{m}^2\; G(x,x). \qquad (3)$$

In the case of the finite temperature field theory, one has to use the thermal Feynman propagator:

$$G_\beta(x,y) = \frac{\mathrm{Tr}\left[e^{-\beta H}\; T\varphi(x)\varphi(y)\right]}{\mathrm{Tr}\; e^{-\beta H}}. \qquad (4)$$

The problem reduces to the construction of the scalar field propagator, then taking the coincidence limit, with an appropriate regularization.

Free energy in the Minkowskian coordinate

The thermal Feynman propagator in the Minkowskian coordinates is:

$$G_\beta(x,t) = \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \left[\frac{e^{i\omega_k |t| - i\vec{k}\vec{x}}}{2\omega_k} \frac{1}{e^{\beta\omega_k} - 1} + \frac{e^{-i\omega_k |t| + i\vec{k}\vec{x}}}{2\omega_k} \left(1 + \frac{1}{e^{\beta\omega_k} - 1} \right) \right]. \tag{5}$$

The effective action is:

$$\mathcal{S}^{\beta}_{eff} = -\frac{V_{d-1} T}{2} \int_{M^2}^{m^2} d\bar{m}^2 \int \frac{d^{d-1}k}{(2\pi)^{d-1}\,\omega_k} \left[\frac{1}{2} + \frac{1}{e^{\beta\omega_k} - 1} \right], \tag{6}$$

where T is the duration of time and V_{d-1} is the spatial volume.

- The $1/2$ term diverge, it is the standard UV divergence that do not depend on temperature.

$$i\mathcal{S}^{\beta}{}_{eff} \to -\beta F_\beta = -V_{d-1} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \log\left[1 - e^{-\beta\omega_k} \right]. \tag{7}$$

Free energy is:

- finite
- depend on the volume of space

Free energy in the Rindler chart

The metric of d dimensional Rindler chart is:

$$ds^2 = e^{2\xi\alpha}\Big(-d\eta^2 + d\xi^2\Big) + d\vec{x}^2, \tag{8}$$

in the following we will set the acceleration to one $\alpha = 1$.
The thermal Feynman propagator of the free massive scalar field is:

$$G_\beta\left(\eta_2, \xi_2, \vec{x}_2 | \eta_1, \xi_1, \vec{x}_1\right) = \tag{9}$$

$$= \int \frac{d^{d-2}k}{(2\pi)^{d-2}} \int_0^\infty \frac{d\omega}{\pi^2}\Bigg[e^{i\omega|\eta_2-\eta_1|-i\vec{k}\Delta\vec{x}}\sinh(\pi\omega)K_{i\omega}\left(\sqrt{m^2+k^2}e^{\xi_1}\right)K_{i\omega}\left(\sqrt{m^2+k^2}e^{\xi_2}\right)\frac{1}{e^{\beta\omega}-1}+$$
$$+e^{-i\omega|\eta_2-\eta_1|+i\vec{k}\Delta\vec{x}}\sinh(\pi\omega)K_{i\omega}\left(\sqrt{m^2+k^2}e^{\xi_1}\right)K_{i\omega}\left(\sqrt{m^2+k^2}e^{\xi_2}\right)\left(1+\frac{1}{e^{\beta\omega}-1}\right)\Bigg].$$

- Thermal propagator has an anomalous singularity at the horizon for non–canonical temperatures $\beta \neq \frac{2\pi}{\alpha}$ (Unruh temperature).

For $\beta = \pi$ one can represent the propagator in the following form:

$$G_\pi\Big(\eta_2, \xi_2, \vec{x}_2 | \eta_1, \xi_1, \vec{x}_1\Big) = \tag{10}$$

$$= G_{2\pi}\Big(e^{2\xi_1} + e^{2\xi_2} - 2e^{\xi_1+\xi_2}\cosh(\eta_1-\eta_2) + (\vec{x}_1-\vec{x}_2)^2\Big) +$$
$$+ G_{2\pi}\Big(e^{2\xi_1} + e^{2\xi_2} + 2e^{\xi_1+\xi_2}\cosh(\eta_1-\eta_2) + (\vec{x}_1-\vec{x}_2)^2\Big).$$

- First term is the standard Poincare invariant two-point function for the canonical temperature $\beta = \frac{2\pi}{\alpha}$.
- Second term is finite inside the Rindler wedge but becomes singular once both its points are taken to the horizon

The effective action is:

$$\mathcal{S}^{\beta}{}_{eff} = -TA_{d-2}\int_{M^2}^{m^2} d\bar{m}^2 \int \frac{d^{d-2}k}{(2\pi)^{d-2}} \int_0^{+\infty} \frac{d\omega}{\pi^2} \sinh(\pi\omega) \left[\frac{1}{2} + \frac{1}{e^{\beta\omega}-1}\right] \times \\ \times \int_{-\infty}^{\infty} d\xi e^{2\xi} K_{i\omega}\left(\sqrt{\bar{m}^2+k^2}e^{\xi}\right) K_{i\omega}\left(\sqrt{\bar{m}^2+k^2}e^{\xi}\right), \quad (11)$$

where A_{d-2} is the volume of the transverse $(d-2)$-dimensional flat space, and T is the duration of time.

This expression has several divergences.

- The first divergence (temperature independent) is coming from the $1/2$ term. This is the standard UV divergence.
- The second divergence (temperature dependent) is due to divergence of Green function on the horizon.

By using UV cut-off $\frac{1}{\epsilon}$:

$$iS^{\beta}{}_{eff} \to -\beta F_{\beta} = \frac{\pi^2}{3\beta}\frac{A_{d-2}}{\alpha}\int_{\epsilon^2}^{\infty}\frac{ds}{(4\pi s)^{\frac{d}{2}}}e^{-sm^2}. \tag{12}$$

By using cut-off of horizon $e^{\alpha\xi} = \delta$:

$$iS^{\beta}_{eff} \to -\beta F_{\beta} = \beta\frac{A_2\alpha^3}{2880\pi^2\delta^2}\left[\left(\frac{2\pi/\alpha}{\beta}\right)^4 + 10\left(\frac{2\pi/\alpha}{\beta}\right)^2\right] \tag{13}$$

There is an essential difference between the free energy of the scalar field in the Rindler and Minkowski:

- temperature dependence depends on the regularization procedure
- proportional to the "area" of the horizon A_{d-2}
- after subtracting the zero-point fluctuations, the free energy in Minkowski coordinates is finite in contrast to the Rindler one

Free energy in the de Sitter space time

The d-dimensional de Sitter space-time is the hyperboloid embedded in the $(d+1)$–dimensional ambient Minkowski space-time:

$$dS_d = \{X \in \mathbf{R}^{d,1},\ X_\alpha X^\alpha = -R^2\}, \quad \alpha = \overline{0, d}. \tag{14}$$

In what follows, we set the de Sitter radius to $R = 1$. The static patch of the de Sitter space-time is covered by the coordinates as follows:

$$\begin{cases} X^0 = \sqrt{1-r^2}\sinh t \\ X^1 = \sqrt{1-r^2}\cosh t \\ X^i = rz_i \quad 2 \le i \le d \end{cases}, \qquad t \in (-\infty, \infty),\ r \in (0,1). \tag{15}$$

Where z_i are the coordinates on the $(d-2)$–dimensional sphere. In these coordinates, the de Sitter metric takes the form:

$$ds^2 = -\left(1-r^2\right)dt^2 + \left(1-r^2\right)^{-1}dr^2 + r^2 d\Omega_{d-2}^2. \tag{16}$$

- Thermal propagator has an anomalous singularity at the horizon for non–canonical temperatures $\beta \neq \frac{2\pi}{H}$ (Gibbons-Hawking temperature) similarly to the Rindler space-time.

The effective action has the following form:

$$iS_{eff}^{\beta} \to -\beta F_{\beta} = \frac{\beta}{2^{d-1}\pi} \int_{\gamma} dy \frac{\frac{\pi y}{\beta} \coth\left(\frac{\pi y}{\beta}\right) - 1}{2y^2} \frac{e^{i\nu y}}{\sinh^{d-1}\left(\frac{|y|}{2}\right)}. \tag{17}$$

where contour $\gamma = (-\infty, -\epsilon) \cup (\epsilon, \infty)$ and $\nu = \sqrt{m^2 - \left(\frac{d-1}{2}\right)^2}$. Where is a difference between odd and even dimensions. In odd dimensions, one can use Cauchy's residue theorem to evaluate the integral.

$$F_{\beta} = \lim_{R\to\infty} \lim_{\epsilon\to 0} \left(I_{(-R,-\epsilon)\cup(\epsilon,R)} + I_{C_R} + I_{C_\epsilon} - I_{C_\epsilon}\right). \tag{18}$$

The sum of the first, second, and third terms define as F_{β}^{bulk} and the forth term as F_{β}^{hor}.

- The contour integrals in F_{β}^{bulk} give, via the Cauchy residue theorem by the two set of poles:

$$y = i\beta n, \quad n \in \mathbf{Z}^+ \quad \text{and} \quad y = i\frac{2\pi}{H}k \quad k \in \mathbf{Z}^+. \tag{19}$$

- The fourth term F_{β}^{hor} diverges in the limit $\epsilon \to 0$.

$$F_{\beta}^{hor} = -I_{C_\epsilon} = \sum_{k=1}^{\frac{d-1}{2}} \frac{a_{2k-1}(m,\beta)}{\epsilon^{2k-1}} + a_0(m,\beta) + O(\epsilon), \tag{20}$$

The finite term of F_β^{hor} is:

$$a_0(m,\beta) = \frac{(-1)^{\frac{d+1}{2}}}{3} \frac{m^{d-2}}{H\beta^2} \frac{A_{d-2}^{dS}}{2^d \pi^{\frac{d-2}{2}} \Gamma\left(\frac{d}{2}\right)}, \tag{21}$$

where $A_{d-2}^{dS} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} \frac{1}{H^{d-2}}$ is the surface area of the boundary of the static de Sitter space-time.

This expression is consistent with the finite contribution to the effective action in the Rindler:

$$F_\beta = \frac{(-1)^{\frac{d-1}{2}}}{3} \frac{m^{d-2}}{\alpha\beta^2} \frac{A_{d-2}}{2^d \pi^{\frac{d-2}{2}} \Gamma\left(\frac{d}{2}\right)}. \tag{22}$$

In the limit of large mass (i.e. limit of the week field), the main contribution to F_β^{bulk} comes from the the closest to the real axis pole, whose position depends on the temperature:

$$F_\beta^{bulk} \approx \begin{cases} -\frac{1}{2^{d-1}\beta\left[i\sin\left(\frac{\beta H}{2}\right)\right]^{d-1}} e^{-\beta m} & ,\text{if } \beta < \frac{2\pi}{H} \\ -\frac{(im/H)^{d-2}}{2^{d+1}H\pi^2 i(d-2)!}\left[\pi\frac{2\pi/H}{\beta}\cot\left(\pi\frac{2\pi/H}{\beta}\right)-1\right]e^{-\frac{2\pi}{H}m} & ,\text{if } \beta > \frac{2\pi}{H} \end{cases}. \quad (23)$$

Thus:

- large temperature limit — $F_\beta^{bulk} \sim e^{-\beta m}$
- low temperature limit — $F_\beta^{bulk} \sim e^{-\frac{2\pi}{H}m}$.

In the flat space limit $\frac{H}{m} \to 0$:

$$F_\beta^{bulk} \approx -\sum_{n=1}^{\infty} \frac{1}{2^{d-1}\beta n} \frac{e^{-\beta mn}}{\left[i \sin\left(\frac{n\beta H}{2}\right)\right]^{d-1}} \approx -\sum_{n=1}^{\infty} \frac{1}{\beta n} \frac{e^{-\beta mn}}{[in\beta H]^{d-1}}. \tag{24}$$

Then, if $\beta m \to 0$, one obtains that:

$$F_\beta^{bulk} \approx (-1)^{\frac{d+1}{2}} \frac{V_{d-1}^{sdS}}{\beta^d} \frac{\zeta(d)\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}}}. \tag{25}$$

- F_β^{hor} proportional to the area of the horizon and contains divergent terms (similar to the free energy in the Rindler chart)
- F_β^{bulk} proportional to the volume of the space-time and finite (similar to the free energy in the Minkowskian coordinates.)

The main bulk contribution to the free energy in the limit $\frac{m}{H} \to 0$:

$$F_{\beta}^{bulk} \approx -\frac{1}{2\beta}\int_{L}^{\infty}\frac{dy}{y}e^{-\frac{1}{d-1}\frac{m^2}{H^2}y} \approx \frac{1}{2\beta}\log\left(\frac{L}{d-1}\frac{m^2}{H^2}\right) \approx \frac{1}{\beta}\log\left(\sqrt{L}\frac{m}{H}\right). \qquad (26)$$

Free energy contains a logarithmic term — in any dimension.
Thus the logarithmic corrections to Bekenstein-Hawking entropy is:

$$\mathbf{S} \approx \frac{A_{d-2}^{dS}}{4} - \frac{1}{d-2}\log\left(A_{d-2}^{dS}\right). \qquad (27)$$

This logarithmic contribution becomes much larger than the classical entropy in the limit of a small de Sitter radius.