Recent results related to Feynman integrals calculus

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Setting the stage

Different problems solved with similar techniques

Phase-space integration
Numerically DRR, arbitrary d

Finite parts
of propagators
in $d = 3 - 2\varepsilon$

Feynman integrals

QCD in MOM scheme
differential equations in ε-form
six-root of unity GPLs

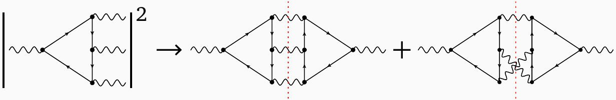
ϕ^3 renormalization
only divergencies
in $d = 6 - 2\varepsilon$

Results to be discused

1. Fully inclusive phase-space integrals needed for NNLO QCD crossection calculations for e^+e^- anihilation into hadrons
2. Four-loop fermion propagator in $d = 3 - 2\varepsilon$ QED
3. Four-loop QCD beta-function in MOM scheme and operators anomalous dimensions relevant for quark masses at three-loop order
4. Five-loop renormalization and critical exponents in φ^3 theory and its generalizations

Phase space integration

Main idea: treat integration over phase space as loop integrals with cut propagators



Fully inclusive massless phase-space integrals use cases:

- Total cross-section calculation, $R(s)$
- Boundary conditions for semi-inclusive observables, time-like splitting functions
- NNLO subtraction terms for $q\bar{q}$ with jets production in e^+e^- annihilation
- Boundary conditions for heavy quark pair production in e^+e^- annihilation

$1 \to 4$@tree, $1 \to 3$@1-loop and $1 \to 2$@2-loop are known [Gehrmann et al.'04]

Our goal: $1 \to 5$@tree, $1 \to 4$@1-loop, $1 \to 3$@2-loop and $1 \to 2$@3-loop

Details and results

- Final result: Solution of DRR in the form of fast convergent series for arbitrary d, analytical results in terms of MZVs for the interesting $d = 4-2\varepsilon$ case
- Intermidiate steps:
  - DRR system for all integrals corresponding to all possible cuts of four-loop propagators
  - Optical theorem relations connecting imaginary part of virual integral with cuts
  - Two-loop integrals for $1 \to 3$ to higher weight in $d = 6$ and $d = 8$, wo IR divergencies
  - Divergencies of **massive** one-loop $1 \to 4$ integrals in logairthmic dimension
- Fix arbitrary periodic functions in DRR solution with all available data
- Use `PSLQ` to reconstruct $d = 4-2\varepsilon$ result using MZV basis and high precision numbers

Summary: [Gituliar,Magerya,AP'JHEP18][Magerya,AP'JHEP19]

1. Integrals from all possible cuts available as fast convergent series for arbitrary d
2. For $d = 4-2\varepsilon$ numerical results successfully reconstructed with MZV upto weight 12

IR divergencies in $3-2\varepsilon$ dimensional QED

- Long standing question about IR finitnes of 3d QED [Jackiw,Templeton'81]
- Landau-Khalatnikov-Fradkin transformation predicts relations between different loops orders for fermion propagator self-energy [Gusynin,Kotikov,Teber'20]

Our goal: to calculate bare fermion propagator at 3- and 4-loop order in the $n_f \to 0$ limit

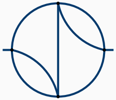
$$S_F(p,\xi) = \frac{iS(\xi)}{\hat{p}}, \quad S(\xi) = 1 + AS_1(\xi) + A^2 S_2(\xi) + A^3 S_3(\xi) + A^4 S_4(\xi) + \mathcal{O}(A_5)$$

LKF predictions:

1. If three-loop term $S_3(\xi) \neq 0$ then four-loop term $S_4(\xi)$ is divergent
2. From known result e.g. in Landau gauge $S_i(0)$ all ξ dependence can be reconstructed from lower loop results

Details and results

- Main difficulty to calculate 3- and 4-loop propagators in $d = 3 - 2\varepsilon$
- Numerical results with high precision available for arbitrary d
- For odd d ε-expansion contains MZVs only, but not the case for $d = 3 - 2\varepsilon$



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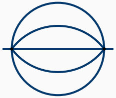


$$\cdot \left(-8\varepsilon\left(\mathscr{C}\pi^2 + 24\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\right) + O(\varepsilon^2)\right)$$

- Analytical expressions reconstructed using basis of fourth-root of unity GPLs

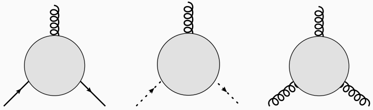
Summary: [AP, Gusynin,Kotikov,Teber'PRD20]

1. Quenched($n_f = 0$) 3d QED is divergent at 4-loop level and 3-loop part of fermion propagator is not zero $S_3(\xi) \neq 0$
2. All gauge dependent terms are in agreement with LKF predictions
3. We conjecture that all massless propagators in $d = 3 - 2\varepsilon$ are expressible through 4-th root of unity GPLs

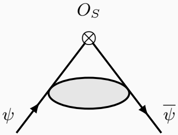
QCD renormalization in MOM scheme

- β_{QCD} known to 5-loop order in $\overline{\mathrm{MS}}$ scheme [Baikov,Chetyrkin,Kuhn'17]
- $\overline{\mathrm{MS}}$ scheme is simplest to perform calculations, but not unique
- MOM scheme $m = 0$, but off-shell legs provides matching with lattice data

Beta functions from renormalization of three-point vertices



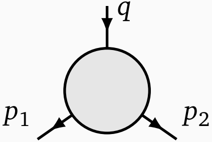
Mass MOM renormalization



- Most difficult to calculate 3-loop 3-pt integrals with all legs off-shell
- Four-loop beta-functions from conversion relation $a_{\mathrm{MOM}} = a_{\overline{\mathrm{MS}}}\left(1 + \sum_l X_l a^l_{\overline{\mathrm{MS}}}\right)$
- Quark masses from lattice data using three-loop O_S renormalization in MOM scheme

Details and results

DE system reducible to ε-form - solution using GPLs with fixed alphabet [Henn'13]



- Differential equations in q^2 variable to connect limits
- Symmetric point $p_1^2 = p_2^2 = q^2$, from arbitrary q^2 result
- 2-pt function limit $q^2 \to 0$, used to fix boundary conditions

All three-loop integrals expressible through 6-th root of unity GPLs upto weight 6 and even more restricted basis of HPLs with $e^{i\pi/3}$ argument [Kniehl,AP,Veretin'JHEP17]

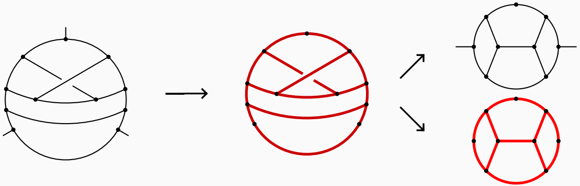
Summary: [Bednyakov,AP'PRD20a][Bednyakov,AP'PRD20b]

1. For the first time calculated three-loop 3-pt integrals in symmetric point
2. With 3-loop conversion factor derived 4-loop QCD beta-function in MOM scheme
3. NNNLO relation connecting $\overline{\mathrm{MS}}$ quark mass to MOM/RI mass from lattice data

Five-loop critical exponents in φ^3 theory

- Scalar φ^3 thery in $d = 6$ is a well known playground together with φ^4 in $d = 4$
- Has many generalizations interesting in practice:
  Potts model, Lee-Yang edge singularity, model with $O(n)$ symmetry [Fei,et al.'14]
- Existing four-loop calculations are due to known four-loop propagator integrals

TODO: Reduce divergencies of 5-loop vertices and propagators to 4-loop integrals only



Most difficult part: derivatives of propagators in external momenta generating numerators

Details and results

Method of soltion:

1. Apply $\mathscr{K}\mathscr{R}'G$ to each logarithmicaly divergent diagram in **massive** theory
2. Using IBP rewrite $\mathscr{K}G$ part through the set of master integrals wo spurious poles and easy to calculate, in our case: 4-pt functions with 16 additional 3-pt functions by hand
3. For 4-pt functions $\mathscr{K}G$ is easy to calculate with 4-loop integrals only
4. For 3-pt functions with IRR reduce problem to calculation of $\mathscr{K}\mathscr{R}'G$ for 4-loop propagator insertion into one-loop diagram

Summary: [Kompaniets,AP'to appear]

- $\mathscr{K}\mathscr{R}'$ for all φ^3 diagrams, can be used for other more complicated models
- Results for $O(n)$ model checked with existing large n expansion to $\mathscr{O}(1/n^3)$
- φ^3 theory results are in agreement with independent calculation [Schnetz et al.]

Conclusion

With the help of modern methods of Feynman integrals calculation we achieved following improvements in different QFT problems:

1. PS integrals for e^+e^- total crossection

   before: $1 \to 4$ after: $1 \to 5$

2. 3d QED fermion propagator

   before: 2-loop after: 4-loop

3. Integrals for symmetric point MOM scheme calculations

   before: 2-loop after: 3-loop

4. Scalar φ^3 theory in $d = 6$ renormalization

   before: 4-loop after: 5-loop

Thank you for attention!