Hamiltonian BFV-BRST quantization for the systems with unfree gauge symmetry

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Gauge symmetry vs Unfree gauge symmetry

Assumptions implied by the second Noether theorem:

- the gauge parameters are arbitrary functions of space-time coordinates
- any on-shell vanishing local quantity reduces to a linear combination of the l.h.s. of Lagrangian equations and their derivatives

For the case of unfree gauge symmetry, these assumptions are violated:

- gauge parameters have to obey differential equations,
$$\delta_\varepsilon \phi^i = \Gamma^i_\alpha(\phi)\varepsilon^\alpha \,, \quad \Gamma^a_\alpha(\phi)\varepsilon^\alpha = 0 \tag{1}$$
- the completion functions τ_a exist such that vanish on-shell, while they cannot be expanded in the l.h.s. of Lagrangian equations with local coefficients
$$\exists\, \tau_a(\phi): \quad \tau_a(\phi) \approx 0 \,, \quad \tau_a(\phi) \neq K^i_a(\phi)\partial_i S \tag{2}$$

Modified Noether identities read
$$\Gamma^i_\alpha(\phi)\partial_i S(\phi) + \Gamma^a_\alpha(\phi)\tau_a(\phi) = 0 \tag{3}$$

Motivation

The examples of unfree gauge symmetry have been known for a long time:

- unimodular gravity ($\nabla_\mu \varepsilon^\mu = 0$)
- Maxwell-like higher spin models
- higher spins with transverse symmetry

The usual general formalism of gauge theory does not admit existence of constraints on gauge parameters and completion functions

Our aim is to extend the formalism to the class of theories with unfree gauge symmetry

- for Lagrangian formalism, see arXiv:1904.04038, arXiv:1907.03443
- for constrained Hamiltonian formalism, see arXiv:1911.11548, arXiv:2009.02848

Here we adjust the BFV-BRST Hamiltonian quantization method for the case of the unfree gauge symmetry

Example. Linearized unimodular gravity (LUG)

$$S = \frac{1}{4} \int d^4x \big(\partial_\mu h_{\nu\rho} \partial^\mu h^{\nu\rho} - 2 \partial_\mu h_{\nu\rho} \partial^\nu h^{\mu\rho} \big) \,, \quad \eta^{\mu\nu} h_{\mu\nu} = 0 \,, \tag{4}$$

where $\mu, \nu, \rho = 0, 1, 2, 3$, $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$
Gauge identities read

$$2\partial^\mu \frac{\delta S}{\delta h^{\mu\nu}} - \partial_\nu \tau \equiv 0 \,, \quad \tau = \frac{1}{2} \partial^\mu \partial^\nu h_{\mu\nu} \tag{5}$$

Once $\partial_\nu \tau \approx 0$, τ is a constant on-shell, so we have $\tau - \Lambda \approx 0$, where the specific value of the constant Λ is determined by the asymptotics of h

Unfree gauge symmetry transformations read

$$\begin{gathered} \delta_\varepsilon h_{\mu\nu} = \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu - \frac{1}{2} \eta_{\mu\nu} \partial_\rho \varepsilon^\rho \,, \\ \delta_\varepsilon S \equiv \int d^4x \, \partial_\mu \varepsilon^\mu \tau \end{gathered} \tag{6}$$

So, the action is invariant off-shell under the condition $\partial_\mu \varepsilon^\mu = 0$

Example. Constrained Hamiltonian formalism for LUG

$$S_H[h,\Pi,\lambda] = \int d^4x \big(\Pi^{ij}\dot{h}_{ij} - H - \lambda^i T_i\big)\,, \tag{7}$$

$$H = \Pi_{ij}\Pi^{ij} - \frac{1}{2}\Pi^2 + \frac{1}{4}\Big(2\partial^i h_{ij}\partial_k h^{kj} - \partial_i h \partial^i h - \partial_i h_{jk}\partial^i h^{jk}\Big)\,,\;\; T_i = -2\partial^j \Pi_{ji}\,, \tag{8}$$

where $i,j,k = 1,2,3$, $h = \eta^{ij}h_{ij}$, $\Pi = \eta_{ij}\Pi^{ij}$, $\lambda^i = h^{0i}$

Conservation of primary constraints T_i leads to the secondary constraint,

$$\dot{T}_i = \{T_i\,, H\} = -\partial_i \tau_0 = 0\,, \quad \tau_0 \equiv \partial^i\partial^j h_{ij} - \partial_i\partial^i h - \Lambda = 0 \tag{9}$$

The secondary constraint conserves by virtue of the primary ones:

$$\dot{\tau}_0 = \{\tau_0\,, H\} = -\partial^i T_i \tag{10}$$

Unfree gauge symmetry transformations read

$$\delta_\varepsilon h_{ij} = \partial_i\varepsilon_j + \partial_j\varepsilon_i\,,\;\; \delta_\varepsilon \Pi^{ij} = -\partial^i\partial^j\varepsilon^0 + \eta^{ij}\partial_k\partial^k\varepsilon^0\,,\;\; \delta_\varepsilon\lambda^i = \dot{\varepsilon}^i + \partial^i\varepsilon^0 \tag{11}$$

Variation oh the action reads

$$\delta_\varepsilon S_H \equiv \int d^4x \big((\dot{\varepsilon}^0 + \partial_i\varepsilon^i)\tau_0 - \partial_0(T_i\varepsilon^i + \tau_0\varepsilon^0)\big) \tag{12}$$

So, gauge parameters have to obey equation

$$\dot{\varepsilon}^0 + \partial_i\varepsilon^i = 0 \tag{13}$$

Modification of the FP-ansatz

The ghosts assigned to unfree gauge transformations are assumed to obey equations

$$\Gamma^a_\alpha(\phi)C^\alpha = 0\,, \quad \mathrm{gh}(C^\alpha) = 1\,, \quad \epsilon(C^\alpha) = 1\,, \tag{14}$$

where $\Gamma^a_\alpha(\phi)$ are operators of gauge parameter constraints
We impose independent gauges $\chi^I(\phi)$, the FP matrix is rectangular,

$$\frac{\delta_\varepsilon \chi^I}{\delta \varepsilon^\alpha} = \Gamma^i_\alpha(\phi)\partial_i \chi^I(\phi) \tag{15}$$

The FP ansatz for path integral is adjusted to the case of unfree gauge symmetry:

$$Z = \int [d\Phi] \exp\left\{\frac{i}{\hbar} S_{FP}(\phi)\right\}, \quad \Phi = \{\phi, \pi_I, C^\alpha, \bar{C}_I, \bar{C}_a\}\,, \tag{16}$$

$$\mathrm{gh}(\bar{C}_I) = \mathrm{gh}(\bar{C}_a) = 1\,, \quad \epsilon(\bar{C}_I) = \epsilon(\bar{C}_a) = 1\,, \quad \mathrm{gh}(\pi_I) = \epsilon(\pi_I) = 0\,, \tag{17}$$

where the FP action reads

$$S_{FP} = S(\phi) + \pi_I \chi^I(\phi) + \bar{C}_I \Gamma^i_\alpha(\phi)\partial_i \chi^I(\phi) C^\alpha + \bar{C}_a \Gamma^a_\alpha(\phi) C^\alpha \tag{18}$$

Minimal sector

Every first-class constraint is assigned with canonical pair of ghosts,

$$\{C^{\alpha_r}\,,\bar{P}_{\beta_r}\}=\delta^{\alpha_r}_{\beta_r}\,,\ \mathrm{gh}(C^{\alpha_r})=-\mathrm{gh}(\bar{P}_{\alpha_r})=1\,,\ \epsilon(C^{\alpha_r})=\epsilon(\bar{P}_{\alpha_r})=1 \tag{19}$$

Hamiltonian BFV-BRST generator of minimal sector begins with constraints,

$$Q_{\min}=\sum_{r=1}^{n}C^{\alpha_r}T_{\alpha_r}+\ldots\,,\quad \mathrm{gh}(Q_{\min})=1\,,\quad \epsilon(Q_{\min})=1\,, \tag{20}$$

where $\ldots$ stands for $\bar{P}$-depending terms iteratively defined by the equation

$$\{Q_{\min}\,,Q_{\min}\}=0 \tag{21}$$

The ghost extension of the Hamiltonian begins with original Hamiltonian H,

$$\mathcal{H}=H(q,p)+\ldots\,,\quad \mathrm{gh}(\mathcal{H})=0\,,\quad \epsilon(\mathcal{H})=0\,, \tag{22}$$

where $\bar{P}$-dependent terms are defined by the equation

$$\frac{\partial}{\partial t}Q_{\min}+\{Q_{\min}\,,\mathcal{H}\}=0 \tag{23}$$

Non-minimal sector

The number of gauge-fixing conditions coincides with the number of primary constraints, so the same number of non-minimal sector ghosts is introduced,

$$\{P^{\alpha_1}\,,\bar{C}_{\alpha_1}\} = \delta^{\alpha_1}_{\beta_1}\,,\ \mathrm{gh}(P^{\alpha_1}) = -\mathrm{gh}(\bar{C}_{\alpha_1}) = 1\,,\ \epsilon(P^{\alpha_1}) = \epsilon(\bar{C}_{\alpha_1}) = 1 \qquad (24)$$

The Lagrange multiplier canonical pairs are introduced for primary constraints T_{α_1} and gauge-fixing conditions χ^{α_1},

$$\{\lambda^{\alpha_1}\,,\pi_{\beta_1}\} = \delta^{\alpha_1}_{\beta_1}\,,\ \mathrm{gh}(\lambda_{\alpha_1}) = \mathrm{gh}(\pi_{\alpha_1}) = 0\,,\ \epsilon(\lambda^{\alpha_1}) = \epsilon(\pi_{\alpha_1}) = 0 \qquad (25)$$

Complete BSRT-charge reads

$$Q = Q_{\mathrm{min}} + \pi_{\alpha_1} P^{\alpha_1} \qquad (26)$$

Given the gauge conditions,

$$\dot{\lambda}^{\alpha_1} - \chi^{\alpha_1}(q,p) = 0\,, \qquad (27)$$

the gauge fermion is introduced,

$$\Psi = \bar{C}_{\alpha_1}\chi^{\alpha_1} + \lambda^{\alpha_1}\bar{P}_{\alpha_1}\,, \qquad (28)$$

and gauge-fixed Hamiltonian is defined by the usual rule,

$$H_\Psi = \mathcal{H} + \{Q\,,\Psi\} \qquad (29)$$

The gauge-fixed BFV-BRST action reads

$$S^{\Psi}_{\mathrm{BRST}} = \int dt \Big(p_i \dot{q}^i + \pi_{\alpha_1} \dot{\lambda}^{\alpha_1} + \sum_{r=1}^{n} \bar{P}_{\alpha_r} \dot{C}^{\alpha_r} + \bar{C}_{\alpha_1} \dot{P}^{\alpha_1} - H_{\Psi} \Big) \tag{30}$$

The transition amplitude reads

$$Z_{\Psi} = \int [D\Phi] \exp\left\{ \frac{i}{\hbar} S^{\Psi}_{\mathrm{BRST}} \right\}, \tag{31}$$

where $\Phi = \{q^i, p_i, \lambda^{\alpha_1}, \pi_{\alpha_1}, C^{\alpha_1}, \bar{P}_{\alpha_1}, C^{\alpha_2}, \bar{P}_{\alpha_2}, \dots, C^{\alpha_n}, \bar{P}_{\alpha_n}, P^{\alpha_1}, \bar{C}_{\alpha_1}\}$

Specifics of unfree gauge symmetry:

- the non-minimal sector is asymmetric with the minimal one unlike the usual BFV-formalism
- the secondary constraints, being a part of the BRST-generator Q, may be explicitly time-dependent, even though the original action does not involve time explicitly

Example. Hamiltonian BFV-BRST formalism for LUG

The BRST-charge reads

$$Q = -2C^i\partial^j\Pi_{ji} + C^0(\partial^i\partial^j h_{ji} - \partial_i\partial^i h - \Lambda) + \pi_i P^i \tag{32}$$

Impose three independent gauge conditions,

$$\dot{\lambda}^i - \chi^i = 0\,, \quad \chi^i = -\partial_j h^{ji} \tag{33}$$

Introduce gauge fermion $\Psi = \bar{C}_i\chi^i + \lambda^i\bar{P}_i$, and define gauge-fixed Hamiltonian,

$$H_\Psi = H - C^0\partial^i\bar{P}_i - C^i\partial_i\bar{P}_0 + \partial_j\bar{C}_i\partial^j C^i + \partial_j\bar{C}_i\partial^i C^j - \pi_i\partial_j h^{ji} + \lambda^i T_i - P^i\bar{P}_i \tag{34}$$

The partition function reads

$$Z_\Psi = \int [D\Phi]\exp\Big\{\frac{i}{\hbar}\int d^4x\big(\Pi^{ij}\dot{h}_{ij} - H - \lambda^i T_i + \pi_i(\dot{\lambda}^i + \partial_j h^{ji}) + \bar{P}_0(\dot{C}^0 + \partial_i C^i) + \bar{C}_i(\partial_j\partial^j C^i + \partial^i\partial_j C^j) + \bar{P}_i(\dot{C}^i + \partial^i C^0) + P^i(\bar{P}_i + \dot{\bar{C}}_i)\big)\Big\}\,, \tag{35}$$

where $\Phi = \{h_{ij}, \Pi^{ij}, \lambda^i, \pi_i, C^0, \bar{P}_0, C^i, \bar{P}_i, P^i, \bar{C}_i\}$
Integrating over $P^i, \bar{P}_i, \Pi^{ij}$, we get Lagrangian representation of Z,

$$Z_\Psi = \int [D\Phi']\exp\Big\{\frac{i}{\hbar}\int d^4x\big(\mathcal{L} + \pi_i\partial_\mu h^{\mu i} + \bar{C}_i\Box C^i + \bar{P}_0\partial_\mu C^\mu\big)\Big\}\,, \tag{36}$$

where $\Phi' = \{h_{\mu\nu}, \pi_i, \bar{P}_0, C^\mu, \bar{C}_i\}$, and $\mathcal{L}$ is the original Lagrangian

Summary

- We explain how the Hamiltonian BFV-BRST formalism is adjusted for the case of unfree gauge symmetry
- The main distinction is in the content of the non-minimal sector and gauge fixing procedure
- For the case when there are no higher-order ghosts vertices, we deduce from the phase-space integral the modified FP-quantization rules such that account for unfree gauge symmetry by imposing corresponding constraints on ghosts

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`Thank you for your attention!`