WIGNER FUNCTION REPRESENTATION IN EIGENFUNCTION BASIS OF HARMONIC OSCILLATOR

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Definition of the Wigner function

$$W(x,p,t)=\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\exp\left(-i\frac{ps}{\hbar}\right)\left\langle x+\frac{s}{2}\middle|\hat{\rho}(t)\middle|x-\frac{s}{2}\right\rangle ds,$$

$\hat{\rho}(t)$ - DENSITY MATRIX

Calculating averages :

$$\langle\langle\mathcal{E}\rangle\rangle=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathcal{E}(x,p)W(x,p)\,dxdp,$$

DEFINITION :

$$W(x,p,t)=\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\exp\left(-i\frac{ps}{\hbar}\right)\left\langle x+\frac{s}{2}\right|\hat{\rho}(t)\left|x-\frac{s}{2}\right\rangle ds,$$

PROPERTIES :

$$W(x,p)=\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\Psi\left(x+\frac{s}{2}\right)\bar{\Psi}\left(x-\frac{s}{2}\right)\exp\left(-i\frac{ps}{\hbar}\right)ds,$$

$$W=\mathrm{Tr}\left[\hat{\rho}\hat{\mathcal{W}}\right],$$

$\hat{\mathcal{W}}$ - WEYL OPERATOR

For harmonic oscillator :

$$\Psi(x,t)=\sum_{n=0}^{+\infty}c_n(t)\Psi_n(x),$$

$$\Psi_n(x)=\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{m\omega x^2}{2\hbar}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right),\ n\in\mathbb{N}_0,$$

$$\left|\Psi(x,t)\right|^2=\sum_{n,k=0}^{+\infty}c_n\bar{c}_k\Psi_n\bar{\Psi}_k,\quad \int_{-\infty}^{+\infty}\left|\Psi\right|^2dx=\sum_{n=0}^{+\infty}\left|c_n\right|^2=1.$$

$$\mathcal{W}\overset{\det}{=}\left\{w_{n,k}(x,p)\right\}\overset{\det}{=}\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\bar{\Psi}_n\left(x-\frac{s}{2}\right)\Psi_k\left(x+\frac{s}{2}\right)\exp\left(-i\frac{ps}{\hbar}\right)ds,$$

Properties :

$$\mathcal{W} \overset{\text{det}}{=} \left\{ w_{n,k}(x,p) \right\} \overset{\text{det}}{=} \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} \bar{\Psi}_n \left(x - \frac{s}{2} \right) \Psi_k \left(x + \frac{s}{2} \right) \exp\left(-i \frac{ps}{\hbar} \right) ds,$$

$$w_{n,n}(x,p) = \frac{(-1)^n}{\pi\hbar} e^{-2\varepsilon(x,p)} L_n\left(4\varepsilon(x,p)\right),$$

- Harmonic oscillator Wigner function

$$\varepsilon(x,p) = \frac{1}{\hbar\omega} \left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \right),$$

- Harmonic oscillator energy

$$W(x,p) = \sum_{n,k=0}^{+\infty} \rho_{k,n} w_{n,k}(x,p) = \mathrm{Tr}\left[\rho \mathcal{W}(x,p) \right],$$

$$\rho_{k,n} = c_k \bar{c}_n$$

- Density matrix

Definition of polynomials P:

$$\mathcal{P}_{n,k}(z_1,z_2) \stackrel{\text{det}}{=} \frac{1}{\sqrt{2^{n+k}n!k!}} \sum_{s=0}^{\min(n,k)} \frac{1}{2^s s!} \frac{\partial^{2s}}{\partial z_1^s \partial z_2^s}\left[(2z_1)^n (2z_2)^k\right]. \quad n,k \in \mathbb{N}_0$$

$$z_1, z_2 \in \mathbb{C}$$

OR :

$$\mathcal{P}_{n,k}(z_1,z_2) = \sqrt{2^{n+k}n!k!} \sum_{s=0}^{\min(n,k)} \frac{z_1^{n-s} z_2^{k-s}}{2^s s!(k-s)!(n-s)!}.$$

Index permutation property :

$$\bar{\mathcal{P}}_{n+l,n}(-z,\bar{z}) = (-1)^l \mathcal{P}_{n,n+l}(-z,\bar{z}).$$

Theorem :

$$I=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho_2(x,y)\mathcal{P}_{n_1,k_1}(x,y)\mathcal{P}_{n_2,k_2}(x,y)\,dxdy,\quad \rho_2(x,y)=e^{-x^2-y^2}$$

$I=0$ AT n_1+n_2 AND k_1+k_2 HAVE DIFFERENT PARITY

$I>0$ AT n_1+n_2 AND k_1+k_2 EVEN OR ODD AT THE SAME TIME

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho_2(x,y)\mathcal{P}_{n_1,k_1}(x,y)\mathcal{P}_{n_2,k_2}(x,y)\,dxdy=\mathcal{N}^{(2)}_{n_1+n_2,k_1+k_2}\tilde{\delta}_{n_1+n_2,k_1+k_2},$$

$$\mathcal{N}^{(2)}_{n_1+n_2,k_1+k_2}=\pi\sqrt{n_1!n_2!k_1!k_2!}\underbrace{\sum_{s=0}^{\min(n_1,k_1)}\sum_{l=0}^{\min(n_2,k_2)}}_{s+l,n,k-even/odd}\frac{|k-(l+s)-1|!!}{s!(n_1-s)!(k_1-s)!}\frac{|n-(l+s)-1|!!}{l!(n_2-l)!(k_2-l)!}.$$

$$\tilde{\delta}_{n,k}=\begin{cases}1, & if\ \ n\ and\ k\ are\ even,\\ 1, & if\ \ n\ and\ k\ are\ odd,\\ 0, & if\ \ n\ is\ even\ and\ k\ is\ odd,\\ 0, & if\ \ n\ is\ odd\ and\ k\ is\ even.\end{cases}$$

Relations with Hermite polynomials :

$$\frac{1}{\sqrt{2^{n+k}\pi n!k!}}\int_{-\infty}^{+\infty} e^{-\varsigma^2} H_n(\varsigma+\varsigma_1) H_k(\varsigma+\varsigma_2)\, d\varsigma = \mathcal{P}_{n,k}(\varsigma_1,\varsigma_2). \quad n,k \in \mathbb{N}_0$$

Relations with Laguerre polynomials:

$$\frac{1}{2^n n!\sqrt{\pi}}\int e^{-\varsigma^2} H_n(\varsigma+\varsigma_1) H_n(\varsigma+\varsigma_2)\, d\varsigma = \mathcal{P}_{n,n}(\varsigma_1,\varsigma_2) = L_n(-2\varsigma_1\varsigma_2),$$

$$w_{n,k}(x,p) = \frac{(-1)^n}{\pi\hbar} e^{-\kappa^2 x^2 - \frac{p^2}{\hbar^2\kappa^2}} \mathcal{P}_{n,k}\left(-\kappa x - i\frac{p}{\hbar\kappa}, \kappa x - i\frac{p}{\hbar\kappa}\right),$$

$$\frac{(-1)^n}{\pi\hbar}\int_{-\infty}^{+\infty} e^{-\kappa^2 x^2 - \frac{p^2}{\hbar^2\kappa^2}} \mathcal{P}_{n,k}\left(-\kappa x - i\frac{p}{\hbar\kappa}, \kappa x - i\frac{p}{\hbar\kappa}\right) dp = \bar{\Psi}_n(x)\Psi_k(x).$$

Comment:

$$w_{n,k}(x,p)=\frac{(-1)^n}{\pi\hbar}e^{-\kappa^2x^2-\frac{p^2}{\hbar^2\kappa^2}}\mathcal{P}_{n,k}(-z,\bar{z}),\qquad z=\kappa x+i\frac{p}{\hbar\kappa}$$

$$w_{n,k}(x,p)=\frac{\sqrt{2^{n+k}n!k!}}{\pi\hbar}|z|^{2\min(n,k)}e^{-|z|^2}\sum_{s=0}^{\min(n,k)}\frac{(-1)^s}{2^s s!(k-s)!(n-s)!|z|^{2s}}\begin{cases}\bar{z}^{k-\min(n,k)}, k>n,\\ z^{n-\min(n,k)}, k<n,\end{cases}$$

$$w_{n,k}(x,p)=\frac{\sqrt{2^{n+k}n!k!}}{\pi\hbar}|z|^{n+k}e^{-|z|^2}\sum_{s=0}^{\min(n,k)}\frac{(-1)^s}{2^s s!(k-s)!(n-s)!|z|^{2s}}e^{i(n-k)\varphi},$$

$$|z|^2=\frac{2}{\hbar\omega}\left(\frac{p^2}{2m}+\frac{m\omega^2x^2}{2}\right)=2\varepsilon(x,p),$$

$$\varphi=\arg z=\mathrm{arctg}\left(\frac{p}{m\omega x}\right),$$

DEFINITION:

$$\Upsilon_{n,k}(x) \overset{\det}{=} x^{n+k}\sqrt{2^{n+k}n!k!}\sum_{s=0}^{\min(n,k)}\frac{(-1)^s}{2^s s!(k-s)!(n-s)!x^{2s}}.$$

$$\mathcal{P}_{n,k}(-z,\bar{z}) = (-1)^n \Upsilon_{n,k}(|z|)e^{i(n-k)\varphi},$$

$$\int_{-\infty}^{+\infty}\rho_1(x)\Upsilon_{n_1,k_1}(x)\Upsilon_{n_2,k_2}(x)dx = \mathcal{N}^{(1)}_{n_1+n_2,k_1+k_2}\tilde{\delta}_{n_1+n_2,k_1+k_2},$$

$$\mathcal{N}^{(1)}_{n_1+n_2,k_1+k_2} = \sqrt{\pi n_1!n_2!k_1!k_2!}\sum_{s=0}^{\min(n_1,k_1)}\sum_{l=0}^{\min(n_2,k_2)}\frac{(-1)^{s+l}|n_1+n_2+k_1+k_2-2(l+s)-1|!!}{s!l!(k_1-s)!(n_1-s)!(k_2-l)!(n_2-l)!},$$

$$\rho_1(x) = e^{-x^2}$$

GRAPH: $e^{-x^2}\Upsilon_{n,k}(x)/\sqrt{2^{n+k}n!k!}$

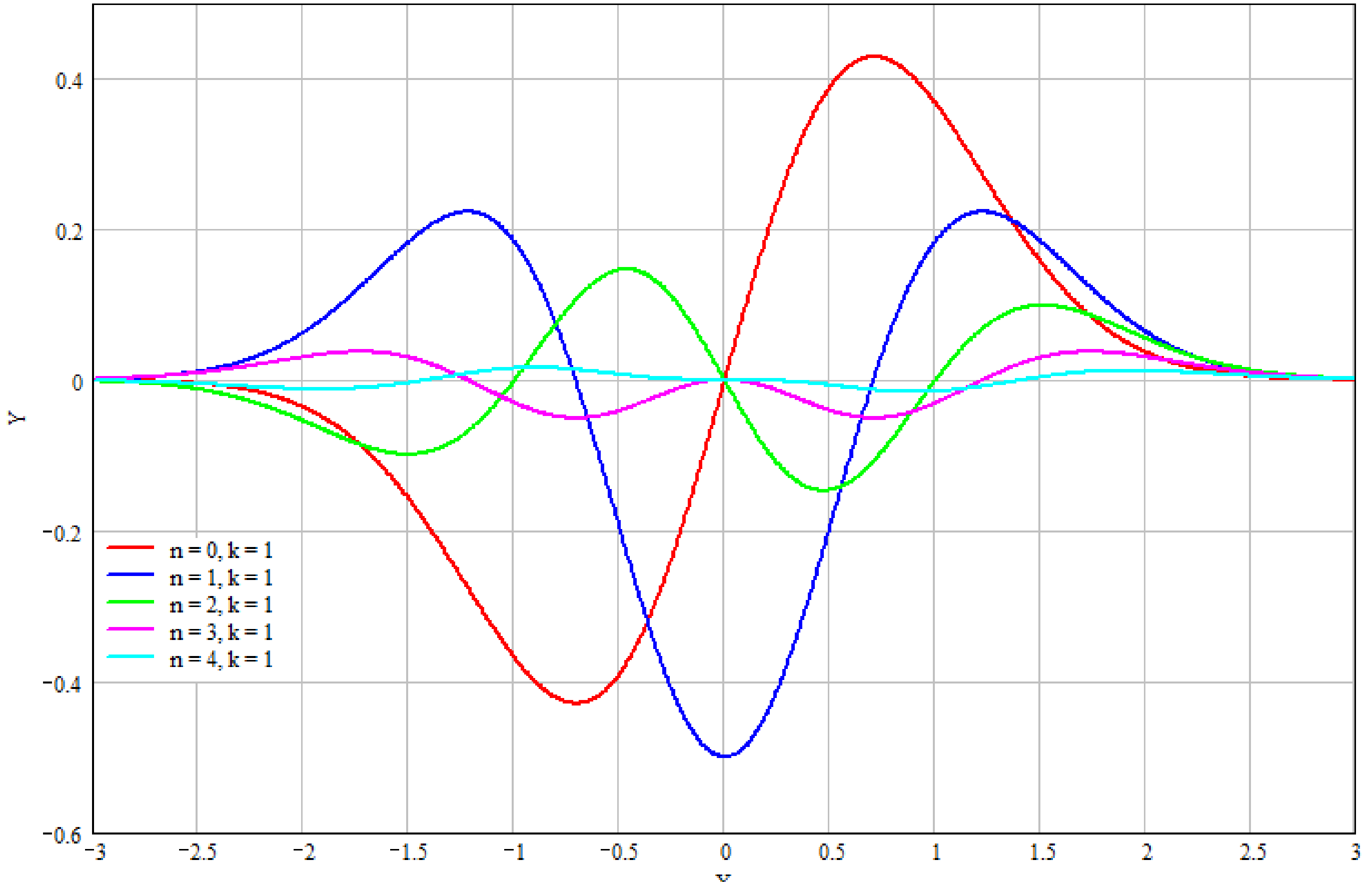


Fig. 1 Graph of the functions $e^{-x^2}\Upsilon_{n,k}(x)/\sqrt{2^{n+k}n!k!}$

VIA THE GAMMA POLYNOMIALS :

$$w_{n,k}\left(x,p\right)=\frac{1}{\pi\hbar}e^{-|z|^2}\Upsilon_{n,k}\left(|z|\right)e^{i(n-k)\varphi},$$

$$W\left(x,p\right)=\frac{e^{-2\varepsilon(x,p)}}{\pi\hbar}\sum_{n,k=0}^{+\infty}\left|\rho_{k,n}\right|\Upsilon_{n,k}\left(\sqrt{2\varepsilon\left(x,p\right)}\right)\vec{\mathrm{n}}_k^T\,\Omega^{(n,k)}\left(\varphi\right)\vec{\mathrm{n}}_n,$$

$$\Omega^{(n,k)}\left(\varphi\right)=\begin{pmatrix}\cos\left(\varpi_{n,k}\varphi\right) & \sin\left(\varpi_{n,k}\varphi\right)\\ -\sin\left(\varpi_{n,k}\varphi\right) & \cos\left(\varpi_{n,k}\varphi\right)\end{pmatrix},\quad \varpi_{n,k}=n-k.$$

$$\vec{\mathrm{n}}_k\stackrel{\det}{=}\begin{pmatrix}\cos\alpha_k\\ \sin\alpha_k\end{pmatrix}=\frac{1}{|c_k|}\begin{pmatrix}\mathrm{Re}\,c_k\\ \mathrm{Im}\,c_k\end{pmatrix},\qquad \alpha_k=\arg c_k,$$

"Basic" functions :

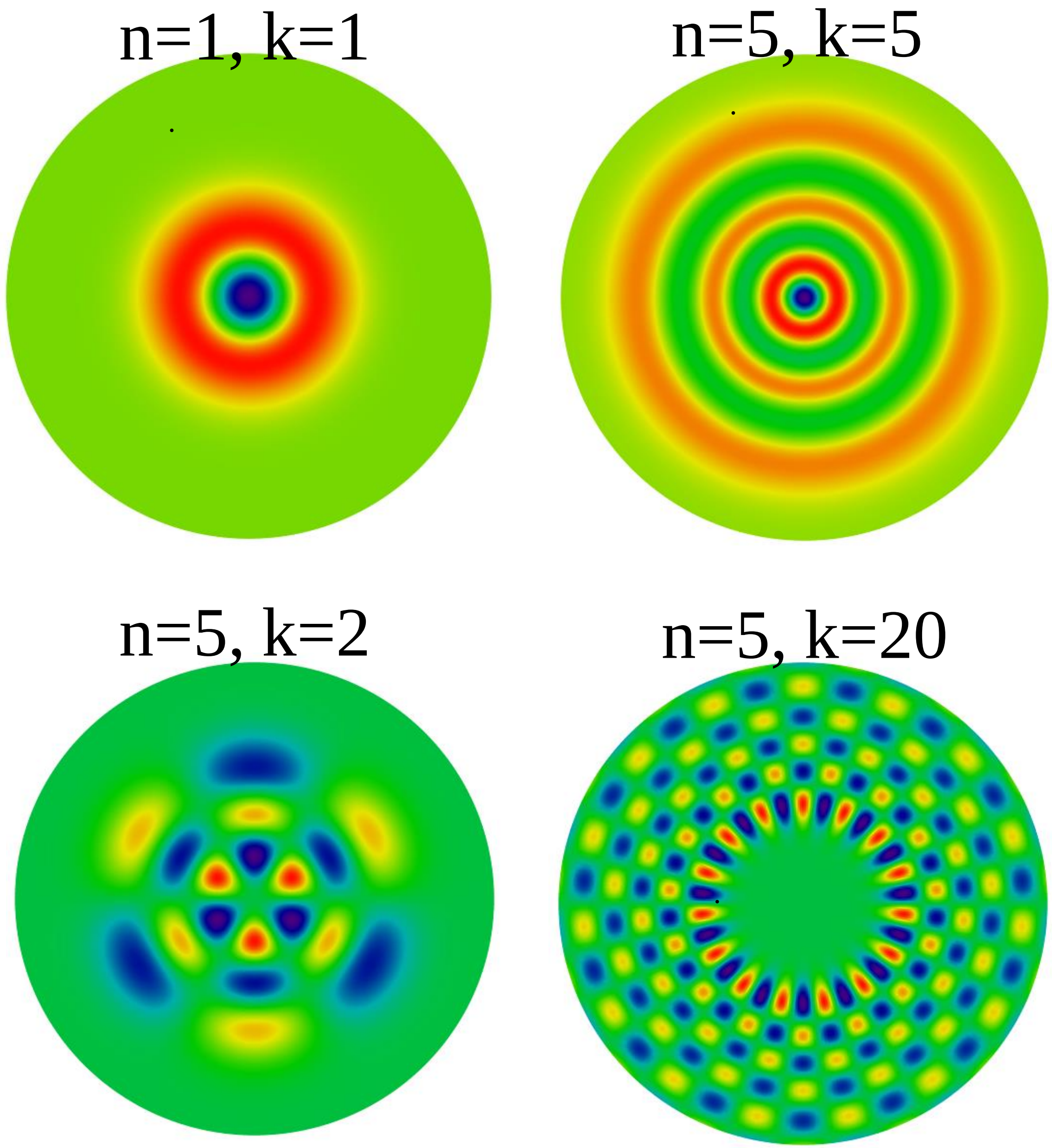
$$w_{n,k}(x,p) = \frac{1}{\pi\hbar} e^{-|z|^2} \Upsilon_{n,k}(|z|) e^{i(n-k)\varphi},$$

$$\mathrm{wc}_{n,k}(x,p) = e^{-2\varepsilon}\Upsilon_{n,k}\left(\sqrt{2\varepsilon}\right)\cos\left(\varpi_{n,k}\varphi\right), \ \mathrm{ws}_{n,k}(x,p) = e^{-2\varepsilon}\Upsilon_{n,k}\left(\sqrt{2\varepsilon}\right)\sin\left(\varpi_{n,k}\varphi\right).$$

$$z = \kappa x + i\frac{p}{\hbar\kappa}$$

$$|z|^2 = \frac{2}{\hbar\omega}\left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}\right) = 2\varepsilon(x,p),$$

$$\varphi = \arg z = \mathrm{arctg}\left(\frac{p}{m\omega x}\right),$$



"Basic" probability density functions

The average energy of an arbitrary quantum system with a potential in the form of a polynomial:

$$\mathcal{E}(x,p)=\frac{p^2}{2m}+U(x)=\hbar\omega\varepsilon(x,p)+\delta U(x),$$

$$\delta U(x)=U(x)-\frac{m\omega^2x^2}{2}.$$

$$\langle\langle\mathcal{E}\rangle\rangle=\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\mathcal{E}(x,p)W(x,p)\,dxdp,$$

$$\langle\langle\mathcal{E}\rangle\rangle=\sum_{n=0}^{+\infty}\left|\rho_{n,n}\right|\langle\langle\varepsilon_n\rangle\rangle+$$

$$+\sum_{n,k=0}^{+\infty}\left|\rho_{k,n}\right|\cos(\alpha_k-\alpha_n)\sqrt{2^{n+k}n!k!}\sum_{\frac{n-k+l}{2}\in\mathbb{Z},\, l\geq|n-k|}^{+\infty}a_l\left(\frac{\hbar}{4m\omega}\right)^{l/2}C_l^{\frac{n-k+l}{2}}\sum_{s=0}^{\min(n,k)}\frac{(-1)^s\left(\frac{k+n+l}{2}-s\right)!}{2^s s!(k-s)!(n-s)!},$$

Thanks for attention!