Analytic Properties of the Quark Density in QC_2D and the Sign Problem

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12.10.2022

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Our goal:

To find the best parametrization of the quark density
for its analytical continuation
from imaginary to real quark chemical potential

Outline

1. Simulation settings
2. Analytical continuation of the quark density
3. Cluster Expansion Model (CEM) vs Rational Fraction Model (RFM)
4. CEM and the fugacity expansion
5. Conclusions

Parameters of simulation

- Tree-level inproved Symanzik gauge action
- Staggered fermions with a diquark source (diquark coupling $\lambda = 0.00125$)

Sommer parameter $r_0 = 0.468$fm
Lattice spacing $a \approx 0.062$ fm
Lattice size $L \approx 1.74$ fm
$am_q = 0.0125$; $m_\pi \approx 800$ MeV

$N_c = 2$, $N_f = 2$
$N_s^3 \times N_t$ lattices: $N_s = 28$;
$N_t = 14$, 12
$T = 227$, 265 MeV

$$\theta = \frac{\mu_q}{T} = \frac{\mu_q' + \imath\mu_q''}{T} = \theta_R + \imath\theta_I$$

$$0 \le \theta_I \le \frac{\pi}{N_c}, \qquad 0 < \mu_q' < 600 \text{ MeV}$$

$$\text{We use } B = \frac{n_q V}{N_c} \text{ instead of } n_q$$

B is the baryon number in the lattice volume,

$$\begin{aligned} B(\theta) &= \frac{1}{N_c}\frac{\partial \ \ln Z_{GC}(\theta)}{\partial\theta} \\ &= \frac{N_f}{4N_c Z_{GC}}\int \mathcal{D}U e^{-S_G} (\det M)^{N_f/8} \mathrm{tr}\left[M^{-1}\frac{\partial M}{\partial\theta}\right] \ , \end{aligned}$$

where $M = D^\dagger(\mu_q)D(\mu_q) + \lambda^2$ and

$$Z_{GC}(\theta) = \int \mathcal{D}U e^{-S_G} (\det M)^{N_f/8} \tag{1}$$

is the Grand Canonical (GC) partition function.

Properties of the grand canonical partition function

$$Z_{GC}(\theta, T, V) = \sum_n \langle n| \exp\left(\frac{-\hat{H} + \mu\hat{Q}}{T}\right)|n\rangle \tag{2}$$

meets the fugacity expansion, that is the Laurent series in $\xi = e^\theta$:

$$Z_{GC}(\theta, T, V) = \sum_{k=-\infty}^{\infty} Z_C(kN_c, T, V)e^{kN_c\theta}, \tag{3}$$

it involves powers of ξ^{N_c} owing to the Roberge-Weiss symmetry

$$Z_{GC}(\theta_I, T, V) = Z_{GC}(\theta_I + 2\pi/N_c, T, V)\ , \tag{4}$$

$C-$parity $\implies Z_{GC}(\theta_I, T, V) = Z_{GC}(-\theta_I, T, V)$

- Problem:
  The baryon number $B(\theta)$ cannot be determined
  in lattice QCD at $\theta = \theta_R$ because of the sign problem.
- Solution:
  Find it at $\theta = \imath\theta_I$ and then employ analytical continuation in θ

- Problem in this way:
  Analytical continuation in θ depends on parametrization of $B(\theta)$
- Proposed solution:
  Test different parametrizations in the case of QC_2D,
  where $B(\theta)$ can be simulated at both $\theta = \theta_R$ and $\theta = \imath\theta_I$

Naive analytic continuation

Assuming that

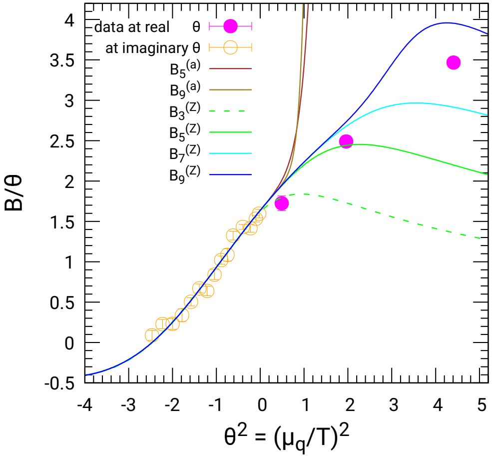
$$B(\theta)\Big|_{\theta_R=0} = \imath \sum_{n=1}^{\infty} a_n \sin\left(nN_c\theta_I\right), \tag{5}$$

we arrive at

$$B(\theta)\Big|_{\theta_I=0} = \sum_{n=1}^{\infty} a_n \sinh\left(nN_c\theta_R\right) \tag{6}$$

Limitatons:

- a_n are extracted from a fit over the segment $0 \le \theta_I \le \dfrac{\pi}{N_c}$
  $\Longrightarrow$ only a few of a_n can be determined.
- Series (6) converges only if $\lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = r$ exists
  and $|\theta_R| < \dfrac{-\ln r}{N_c}$



$T = 227\ \mathrm{MeV}$

$$B_J^{(a)} = \frac{2\sum_{n=1}^{J} nZ_n \sin(nN_c\theta_I)}{1 + 2\sum_{n=1}^{J} Z_n \cos(nN_c\theta_I)}$$

Rational Fraction Model (RFM)

[G. A. Almasi, B. Friman, K. Morita, P. M. Lo, and K. Redlich 2019]

$$B(\theta_I)\Big|_{\theta_R=0} = \sum_{k=1}^{\infty} a_k^{\mathrm{RFM}} \sin\left(kN_c\theta_I\right) \tag{7}$$

$$a_n^{\mathrm{RFM}} = (-1)^{n+1}\, d\, \frac{1+\dfrac{\pi^2(N_c^2-1)}{6}n^2}{n^3(1+n\kappa)}\,. \tag{8}$$

$a_n^{\mathrm{RFM}} \sim \dfrac{(-1)^k}{k^2}$ as $k\to\infty \implies$ nonanalytic behavior:

$$B(\theta) \sim \left(\theta_I - \frac{\pi}{N_c}\right)\ln\left(\frac{\pi}{N_c}-\theta_I\right) \qquad \text{as} \qquad \theta_I \to \frac{\pi}{N_c} \tag{9}$$

$$B_{RFM}(\theta) = d \left\{ \left(\frac{\pi^2 (N_c^2 - 1)}{6} + \kappa^2 \right) \left[\frac{\theta N_c}{2} - \right. \right. \tag{10}$$

$$- \left(\beta \left(\frac{1}{\kappa} \right) - \frac{\kappa}{2} \right) \sinh \left(\frac{\theta N_c}{\kappa} \right) + \frac{1}{2} \int_0^{\theta N_c} dt \, \tanh \frac{t}{2} \sinh \frac{\theta N_c - t}{\kappa} \right]$$

$$\left. + \frac{\pi^2}{12} \left(\theta N_c + \frac{(\theta N_c)^3}{\pi^2} \right) - \kappa \int_0^{\theta N_c} \ln \left(2 \cosh \frac{t}{2} \right) dt \right\}$$

where

$$\beta(z) = \frac{1}{2} \left(\psi \left(\frac{z+1}{2} \right) - \psi \left(\frac{z}{2} \right) \right), \qquad \psi(z) = \frac{1}{\Gamma(z)} \frac{d\Gamma(z)}{dz}$$

Cluster Expansion Model (CEM)

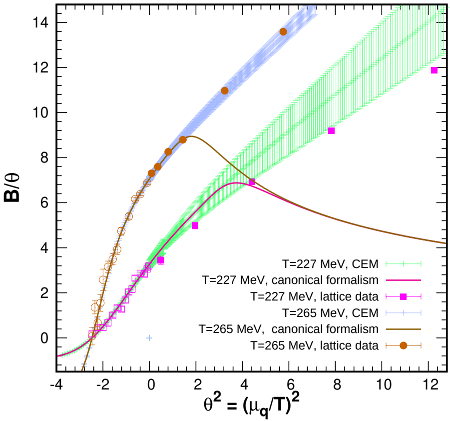
[V. Vovchenko, J. Steinheimer, O. Philipsen and H. Stoecker 2018]

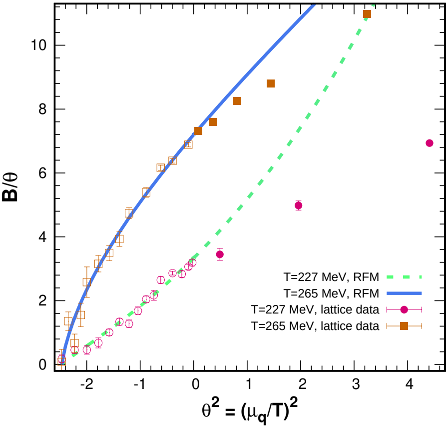
$$B(\theta_I)\Big|_{\theta_R=0} = \sum_{k=1}^{\infty} a_k^{\mathrm{CEM}} \sin\left(kN_c\theta_I\right) \qquad (11)$$

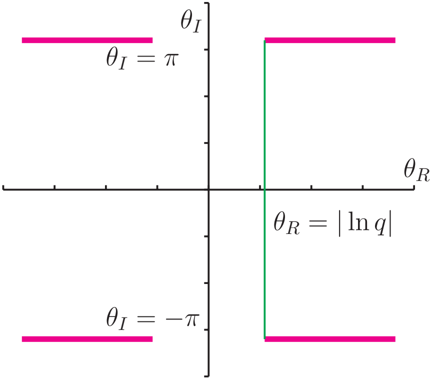
$$b_k \;=\; (-1)^{k+1}\,\frac{b\,q^{k-1}}{k}\left[1+\frac{6}{\pi^2(N_c^2-1)k^2}\right] \qquad (12)$$

$$B=\frac{b}{2q}\left\{\ln\frac{1+q\exp(\theta N_c)}{1+q\exp(-\theta N_c)}+\right.$$
$$\left.+\frac{6}{\pi^2(N_c^2-1)}\left[\mathrm{Li}_3\left(-qe^{-\theta N_c}\right)-\mathrm{Li}_3\left(-qe^{\theta N_c}\right)\right]\right\}\;. \qquad (13)$$

Comparison of the CEM and RFM with lattice data







$T < T_{RW}$. Cuts in θ plane of the CEM quark density.

Fugacity expansion

$$\frac{Z_{GC}(\theta, T, V)}{Z_C(0, T, V)} \;=\; 1 + \sum_{n=1}^{\infty} Z_n \left(e^{nN_c\theta} + e^{-nN_c\theta} \right) \tag{14}$$

provides a natural parametrization of $B(\theta)$,

$$B(\theta) = \frac{-1}{N_c} \frac{\partial (T \ln Z)}{\partial \mu_q} = \frac{2 \sum_{n=1}^{\infty} n Z_n \sinh(n N_c \theta)}{1 + 2 \sum_{n=1}^{\infty} Z_n \cosh(n N_c \theta)} \tag{15}$$

$$B(\theta_I)\Big|_{\theta_R=0} = \imath \sum_{n=1}^{\infty} a_n \sin\left(n N_c \theta_I\right) \tag{16}$$

$$\sum_{n=1}^{\infty} a_n \sin\left(n N_c \theta_I\right) = \frac{2 \sum_{n=1}^{\infty} n Z_n \sin(n N_c \theta_I)}{1 + 2 \sum_{n=1}^{\infty} Z_n \cos(n N_c \theta_I)} \tag{17}$$

Problem: Given a_n, find Z_n

$$\text{Trigonometric identities} \implies a_i = \sum_{j=1}^{\infty} W_{ij} Z_j \ , \tag{18}$$

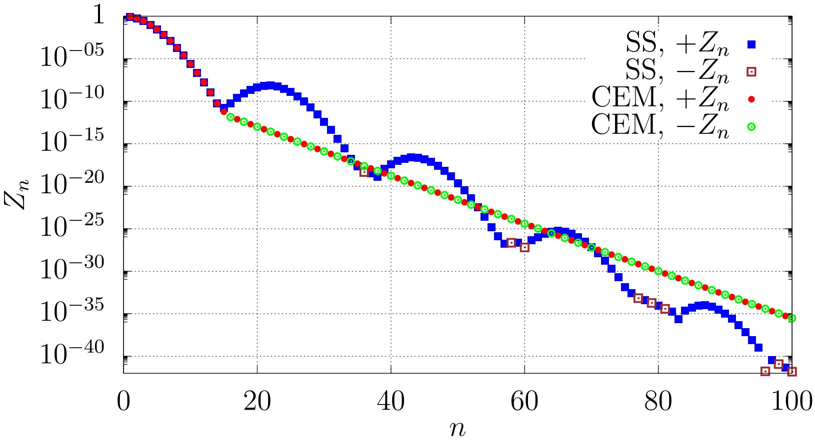
$$W_{jk} = 2j\delta_{jk} - a_{j+k} + a_{|j-k|} \cdot \mathrm{sign}(k-j) \qquad [\mathrm{sign}(0) = 0] \ . \tag{19}$$

$$\mathbf{Z} = \mathbf{W}^{-1}\mathbf{a} \, . \tag{20}$$

$$Z_{GC}(\theta_I) = \exp\left(N_c \sum_{n=1}^{N} \frac{a_n}{2n} \left(\cos(nN_c\theta_I) - 1 \right) \right) \tag{21}$$

The inverse of the fugacity expansion has the form

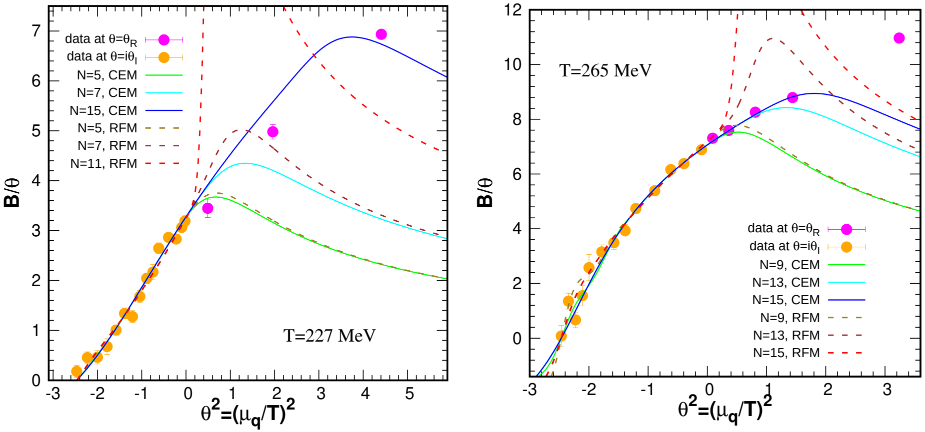
$$Z_C\,(n, T, V) = \int_0^{2\pi} \frac{d\theta_I}{2\pi} e^{-in\theta_I} Z_{GC}(\theta_I, T, V) \ , \tag{22}$$



SS - Z_n from the truncared Fourier series;

CEM - Z_n found using analytic formula; Empty symbols: $Z_n < 0$

Comparison of the fugacity expansions using CEM and RFM



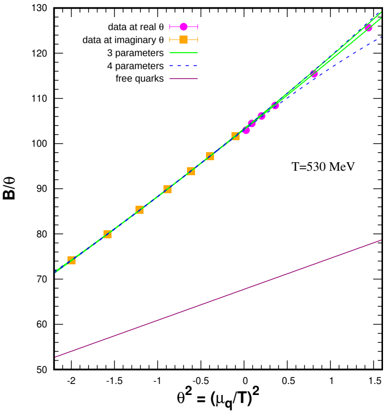
Phenomenological significance

Partial probabilities have the form $\mathcal{P}_n = \dfrac{Z_n e^{n\theta}}{1 + \sum_{j=1}^{\infty} Z_j}$

In view of C-Parity conservation, $Z_n = Z_{-n}$, one arrives at

$$'\mu_B = \frac{T}{2n} \ln \left(\frac{\mathcal{P}_n}{\mathcal{P}_{-n}} \right)$$

- One can measure (in principle) μ_B by measuring probabilities that the net baryon charge of the fireball equals n and $-n$.
- n-Independence of experimental μ_B measured by this method of provides a criterion of thermodynamical equilibrium.
- Z_n can also be extracted from experiment



$T > T_{RW}$. Functional dependence is the same as in the free case, however, the parameters differ from the free values.

Conclusions

We have studied the analytical continuation of the quark density in QC_2D at $T < T_{RW}$ using various parametrizations. It was found

theoretical framework of parametrization	Agreement of the respective analytical continuation with lattice data at real μ_q
truncated Fourier series	bad
CEM	excellent
RFM	poor
the grand canonical approach with the CEM	good at $\lvert\mu_q\rvert < 320 \div 390$ MeV

Problem of negative canonical partition functions $Z_C(n, T, V)$ calls for further work